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Applications of Minor-Summation Formula
I. Littlewood's Formulas

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The first and the third authors obtained a minor-summation formula of Pfaffian, which expresses a weighted sum of minors of any rectangular matrix in terms of a single Pfaffian. In this paper, as an application of this minor-summation formula, we give a new proof of Littlewood's formulas with additional parameters and generalize them for the classical groups.

INTRODUCTION

In our recent paper [IW1], we obtained a basic formula, which we call a *minor-summation formula of Pfaffian*. This formula expresses a weighted sum of maximal minors of an arbitrary rectangular matrix in terms of a single Pfaffian. Such a minor-summation formula has developed in the study of enumerative combinatorics of plane partitions. (See [I], [O1], and [St], for example.) Our minor-summation formula can be viewed as a

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Pfaffian version of the Cauchy−Binet formula and as a generalization of the relation $\mathrm{Pf}(TA^tT) = \det(T)\mathrm{Pf}(A)$ for square matrices. (See Section 2.)

The aims of this paper are to give a new elementary proof of Littlewood's formulas for Schur functions and to generalize them for the characters of the classical groups $Sp(2n, \mathbb{C})$ and $SO(N, \mathbb{C})$. The original Littlewood formulas [L] are the expansions of the products

$$\prod_{1 \le i < j \le n} (1 - x_i x_j)^{\pm 1} \qquad \text{and} \qquad \prod_{1 \le i \le j \le n} (1 - x_i x_j)^{\pm 1}$$

in terms of the Schur functions. Our generalizations for the classical groups provide the expansion formulas of the products of the form

$$\prod_{i=1}^{n} \left(x_i^k + x_i^{-k} \right)$$

in terms of the irreducible characters of $Sp(2n, \mathbb{C})$ and $SO(n, \mathbb{C})$. These Littlewood-type formulas might bring some information about the representation theory of classical groups.

This paper is organized as follows: We prepare some notations and review Weyl's character formula in Section 1. The minor-summation formula and its corollaries are presented in Section 2. Section 3 is devoted to the calculation of the subPfaffians, which appear as weights in the minor-summation formula. Littlewood's formulas for $GL(n, \mathbb{C})$ are derived in Section 4 and their generalizations for $Sp(2n, \mathbb{C})$ and $SO(N, \mathbb{C})$ are given in Section 5.

In this paper, we only deal with the Littlewood-type formulas. In a forthcoming paper, we will investigate several expansion formulas related to the dual pairs in the sense of R. Howe [H]. See [IW2] and [O2] for other applications.

1. NOTATIONS AND PRELIMINARIES

We will fix some notations concerning partitions and characters of the classical Lie algebras. And we collect some formulas for the irreducible characters.

Partitions

In this paper, we denote by $\mathbb{N}$ (resp. $\mathbb{Z}$) the set of non-negative integers (resp. the set of integers). Also, we use the notation $[i, j] = \{i, i+1, \ldots, j\}$ for $i, j \in \mathbb{Z}$ $(i \le j)$ and $[n] = [1, n]$.

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers with finite sum $|\lambda| = \sum_i \lambda_i$. The *length* $l(\lambda)$ of a partition λ is the number of non-zero terms of λ. If an integer i appears exactly m_i times as a part of λ, we write $\lambda = (1^{m_1} 2^{m_2} \dots)$. For example, (r^n) is the partition

$$\big(\underbrace{r, r, \dots, r}_{n \text{ times}}\big).$$

The conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of a partition λ is defined by $\lambda'_i = \#\{j : \lambda_j \geq i\}$.

For a partition λ, we denote by $r(\lambda)$ (resp. $c(\lambda)$) the number of rows (resp. columns) of odd length. We say that λ is *even* (resp. *transposed-even*) if $r(\lambda) = 0$ (resp. $c(\lambda) = 0$).

Given a partition λ, we put $p(\lambda) = \#\{i : \lambda_i \geq i\}$ and define

$$\alpha_j = \lambda_j - j, \qquad \beta_j = \lambda'_j - j \qquad \text{for } 1 \leq j \leq p(\lambda).$$

Then $\alpha_1 > \cdots > \alpha_{p(\lambda)} \geq 0$ and $\beta_1 > \cdots > \beta_{p(\lambda)} \geq 0$. We write $\lambda = (\alpha \mid \beta)$ and call this the *Frobenius notation* of λ.

For $r \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $\Gamma_{r,n}$ be the set of all partitions of the form $\lambda = (\beta_1 + r, \dots, \beta_p + r \mid \beta_1, \dots, \beta_p)$ with length $\leq n$. For example, $\Gamma_{2,2}$ consists of four partitions

$$\emptyset, \qquad (3) = (2 \mid 0), \qquad (4,1) = (3 \mid 1), \qquad (4,4) = (32 \mid 10).$$

A *half-partition* of length n is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative half-integers $\lambda_i \in \mathbb{N} + \frac{1}{2}$. Then we can write $\lambda = (\mu_1 + \frac{1}{2}, \dots, \mu_n + \frac{1}{2})$, where μ is a partition of length $\leq n$. If there is no confusion, we simply write $\lambda = \mu + \frac{1}{2}$.

If λ is a partition of length $\leq n$ (resp. a half-partition of length n), we associate to λ a subset $J(\lambda)$ of $\mathbb{N}$ (resp. $\mathbb{N} + \frac{1}{2}$) defined by

$$J(\lambda) = \{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n\}.$$

Then λ can be recovered from $J(\lambda) = \{j_1 < \cdots < j_n\}$ by putting $\lambda_i = j_{n+1-i} - n + i$.

Let T be an n-rowed matrix with columns indexed by a set I. Given an n-element subset J of I, we denote by T_J the $n \times n$ submatrix of T obtained by picking up the columns indexed by J. If $T = (t_{ij})_{i=1,\dots,n,\ j \in I}$, then $T_J = (t_{ij})_{i=1,\dots,n,\ j \in J}$. If $A = (a_{ij})_{i,\ j \in I}$ is a skew-symmetric matrix, then we write $A_J = (a_{ij})_{i,\ j \in J}$ by abuse of the notation.

Characters

We will consider the following four series of classical Lie algebras of types A, B, C, and D:

$$\begin{aligned}
\mathfrak{g}_{A(n)} &= \mathfrak{gl}(n,\mathbb{C}),\\
\mathfrak{g}_{B(n)} &= \mathfrak{so}(2n+1,\mathbb{C})\\
&= \left\{X \in \mathfrak{gl}(2n+1,\mathbb{C}) : J_{\mathfrak{so}(2n+1)}X + {}^tXJ_{\mathfrak{so}(2n+1)} = 0\right\},\\
\mathfrak{g}_{C(n)} &= \mathfrak{sp}(2n,\mathbb{C}) = \left\{X \in \mathfrak{gl}(2n,\mathbb{C}) : J_{\mathfrak{sp}(2n)}X + {}^tXJ_{\mathfrak{sp}(2n)} = 0\right\},\\
\mathfrak{g}_{D(n)} &= \mathfrak{so}(2n,\mathbb{C}) = \left\{X \in \mathfrak{gl}(2n,\mathbb{C}) : J_{\mathfrak{so}(2n)}X + {}^tXJ_{\mathfrak{so}(2n)} = 0\right\}.
\end{aligned}$$

Here $J_{\mathfrak{so}(N)}$ and $J_{\mathfrak{sp}(2n)}$ are the anti-diagonal matrices given by

$$J_{\mathfrak{so}(N)} = J_N, \qquad J_{\mathfrak{s}p(2n)} = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix},$$

where

$$J_k = \begin{pmatrix} & & & & 1\\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

See [W] for the representation theory of classical groups.

Let $\mathfrak{h}_{X(n)}$ be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}_{X(n)}$, where $X(n)$ represents $A(n)$, $B(n)$, $C(n)$, or $D(n)$. And let ε_i: $\mathfrak{h}_{X(n)} \to \mathbb{C}$ be the linear functional assigning the (i,i)-entry of $H \in \mathfrak{h}$ to H. Then we can take a simple system of roots as follows:

$$\begin{aligned}
\Pi_{A(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\},\\
\Pi_{B(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\},\\
\Pi_{C(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\},\\
\Pi_{D(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.
\end{aligned}$$

It is well known that the finite dimensional irreducible representations of $\mathfrak{g}_{X(n)}$ are parametrized by the dominant integral weights

$$P^+_{A(n)} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n : \lambda_i \in \mathbb{C}, \lambda_i - \lambda_{i+1} \in \mathbb{N}\},$$

$$P^+_{B(n)} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n : \lambda \text{ is a partition or a half-partition}\},$$

$$P^+_{C(n)} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n : \lambda \text{ is a partition}\},$$

$$P^+_{D(n)} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} \pm \lambda_n\varepsilon_n : \lambda \text{ is a partition or a half-partition}\}.$$

If $X(n) = A(n)$, $B(n)$, or $C(n)$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition or a half-partition, we denote by $\lambda_{X(n)}$ the (formal) irreducible character of $\mathfrak{g}_{X(n)}$ with highest weight $\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$. In the $D(n)$ case, we define $\lambda^{\pm}_{D(n)}$ to be the irreducible characters of $\mathfrak{so}(2n, \mathbb{C})$ with highest weights $\lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} \pm \lambda_n\varepsilon_n$, respectively. Note that $\lambda^+_{D(n)} = \lambda^-_{D(n)}$ if $l(\lambda) < n$. Here we regard a character as a Laurent polynomial in the variables $x_i^{\pm 1/2} = e^{\pm\varepsilon_i/2}$. For $\mathfrak{g}_{A(n)} = \mathfrak{gl}(n, \mathbb{C})$, the irreducible characters $\lambda_{A(n)}$ are often denoted by $s_\lambda(x_1, \ldots, x_n)$ and called the Schur functions.

We now recall Weyl's character formula. We introduce the n-rowed matrices

$$T^{X(n)} = \left(t^{X(n)}_{ik}\right)_{i=1,\ldots,n} \qquad (X = A, B, C, D+, D-, D)$$

with (i, k)-entries defined by

$$t^{A(n)}_{ik} = x_i^k \qquad \text{for } k \in \mathbb{N},$$

$$t^{B(n)}_{ik} = x_i^{k+1/2} - x_i^{-k-1/2} \qquad \text{for } k \in \tfrac{1}{2}\mathbb{N},$$

$$t^{C(n)}_{ik} = x_i^{k+1} - x_i^{-k-1} \qquad \text{for } k \in \mathbb{N},$$

$$t^{D^+(n)}_{ik} = x_i^k + x_i^{-k} \qquad \text{for } k \in \tfrac{1}{2}\mathbb{N},$$

$$t^{D^-(n)}_{ik} = x_i^k - x_i^{-k} \qquad \text{for } k \in \tfrac{1}{2}\mathbb{N},$$

and

$$t^{D(n)}_{ik} = \begin{cases} 1 & \text{if } k = 0 \\ x_i^k + x_i^{-k} & \text{if } k \geq 1. \end{cases}$$

Then Weyl's character formula can be written in the following form.

PROPOSITION 1.1. *For a partition or a half-partition* $\lambda = (\lambda_1, \dots, \lambda_n)$, *we have*

$$\lambda_{X(n)} = \frac{\det\left(T_{J(\lambda)}^{X(n)}\right)}{\det\left(T_{J(\emptyset)}^{X(n)}\right)} \qquad \text{for } X = A, B, C,$$

$$\lambda_{D(n)}^{\pm} = \frac{\det\left(T_{J(\lambda)}^{D^+(n)}\right) \pm \det\left(T_{J(\lambda)}^{D^-(n)}\right)}{\det\left(T_{J(\emptyset)}^{D(n)}\right)}.$$

The Weyl denominators

$$\Delta_{X(n)} = \det\left(T_{J(\emptyset)}^{X(n)}\right) \qquad \text{for } X = A, B, C, D$$

of Weyl's character formulas factorize as follows.

PROPOSITION 1.2.

$$\Delta_{A(n)} = \prod_{1 \le i < j \le n} (x_j - x_i),$$

$$\Delta_{B(n)} = (-1)^{n(n+1)/2} (x_1 \cdots x_n)^{-n+1/2} \times \prod_{i=1}^{n} (1 - x_i) \prod_{1 \le i < j \le n} (x_j - x_i)(1 - x_i x_j),$$

$$\Delta_{C(n)} = (-1)^{n(n+1)/2} (x_1 \cdots x_n)^{-n} \times \prod_{i=1}^{n} (1 - x_i^2) \prod_{1 \le i < j \le n} (x_j - x_i)(1 - x_i x_j),$$

$$\Delta_{D(n)} = (-1)^{n(n-1)/2} (x_1 \cdots x_n)^{-n+1} \prod_{1 \le i < j \le n} (x_j - x_i)(1 - x_i x_j).$$

The following lemma enables us to reduce the problems to the cases where the rank is even.

LEMMA 1.3. (1) *For a partition* λ *with length* $\le n + 1$, *we have*

$$s_\lambda(x_1, \dots, x_n, 0) = \begin{cases} s_\lambda(x_1, \dots, x_n) & \text{if } \lambda_{n+1} = 0 \\ 0 & \text{if } \lambda_{n+1} > 0. \end{cases}$$

(2) *Let* $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ *be a partition with length* $\le n + 1$ *or a half-partition of length* $n + 1$ *such that* $\lambda_1 \le m$. *Then* $(x_1 \cdots x_{n+1})^m$

$\cdot\lambda_{X(n)}(x_1,\ldots,x_{n+1})$ is a polynomial in the variables $x_1^{1/2},\ldots,x_{n+1}^{1/2}$ and satisfies

$$\left[x_1^m \cdots x_{n+1}^m \lambda_{B(n+1)}(x_1,\ldots,x_{n+1})\right]_{x_{n+1}=0} = \begin{cases} x_1^m \cdots x_n^m (\lambda_2,\ldots,\lambda_{n+1})_{B(n)}(x_1,\ldots,x_n) & \text{if } \lambda_1 = m \\ 0 & \text{if } \lambda_1 < m, \end{cases}$$

$$\left[x_1^m \cdots x_{n+1}^m \lambda_{C(n+1)}(x_1,\ldots,x_{n+1})\right]_{x_{n+1}=0} = \begin{cases} x_1^m \cdots x_n^m (\lambda_2,\ldots,\lambda_{n+1})_{C(n)}(x_1,\ldots,x_n) & \text{if } \lambda_1 = m \\ 0 & \text{if } \lambda_1 < m, \end{cases}$$

$$\left[x_1^m \cdots x_{n+1}^m \lambda^{\pm}_{D(n+1)}(x_1,\ldots,x_{n+1})\right]_{x_{n+1}=0} = \begin{cases} x_1^m \cdots x_n^m (\lambda_2,\ldots,\lambda_{n+1})^{\mp}_{D(n)}(x_1,\ldots,x_n) & \text{if } \lambda_1 = m \\ 0 & \text{if } \lambda_1 < m, \end{cases}$$

where $[f]_{x_{n+1}=0}$ indicates substituting $x_{n+1}=0$ into f.

Proof. It easily follows from Propositions 1.1 and 1.2, so we leave it to the readers. ∎

Another useful lemma derived from Weyl's character formula is the following relation among the irreducible characters.

LEMMA 1.4. (1) *If λ is a partition with length $\le n$, we have*

$$\left(\lambda + \frac{1}{2}\right)_{B(n)} = \prod_{i=1}^{n}\left(x_i^{1/2} + x_i^{-1/2}\right)\cdot\lambda_{C(n)}.$$

(2) *If λ is a partition with length $\le n$ or a half-partition of length n, we have*

$$\left(\lambda + \frac{1}{2}\right)^{+}_{D(n)} - \left(\lambda + \frac{1}{2}\right)^{-}_{D(n)} = \prod_{i=1}^{n}\left(x_i^{1/2} - x_i^{-1/2}\right)\cdot\lambda_{B(n)}.$$

2. MINOR-SUMMATION FORMULA

Our starting point is the following minor-summation formula.

THEOREM 2.1 [IW1, Theorem 1(1)]. *Assume that $n \le N$ are integers and n is even. Let $T=(t_{ik})_{1\le i\le n,1\le k\le N}$ be an $n\times N$ matrix and $A =$*

$(a_{kl})_{1\le k,l\le N}$ *be an* $N\times N$ *skew-symmetric matrix. Then we have*

$$\sum_{J\subset[N],\,\#J=n} \mathrm{Pf}(A_J)\det(T_J) = \mathrm{Pf}(TA^tT).$$

In this formula, the (i,j)-entry of the skew-symmetric matrix TA^tT is given by

$$(TA^tT)_{ij} = \sum_{k,l=1}^{N} a_{kl}t_{ik}t_{jl} = \sum_{1\le k<l\le N} a_{kl}\det\begin{pmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl}\end{pmatrix}.$$

We will apply this theorem to various skew-symmetric matrices A and the matrices $T^{X(n)}$ introduced in Section 1 to derive the Littlewood-type formulas for classical Lie algebras.

Theorem 2.1 implies several well-known formulas. For example, if $n=N$, then Theorem 2.1 says that

$$\mathrm{Pf}(TA^tT) = \det(T)\mathrm{Pf}(A) \tag{2.1}$$

for square matrices T and A. The following corollary is known as the Cauchy–Binet formula.

COROLLARY 2.2. *Let* $m\le n$ *be integers. Let* $X=(x_{ik})_{1\le i\le m,1\le k\le n}$ *and* $Y=(y_{ik})_{1\le i\le m,1\le k\le n}$ *be arbitrary matrices. Then we have*

$$\sum_{K\subset[n],\,\#K=m} \det(X_K)\det(Y_K) = \det(X\,{}^tY).$$

Proof. In Theorem 2.1, we take

$$A = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}, \qquad T = \begin{pmatrix} X & 0 \\ 0 & Y\end{pmatrix}.$$

Then it is easy to see that

$$\mathrm{Pf}(A_J) = \begin{cases} (-1)^{\binom{m}{2}} & \text{if } J=\{k_1<\cdots<k_m<k_1+n<\cdots<k_m+n\} \\ 0 & \text{otherwise}\end{cases}$$

and that, if $J=\{k_1<\cdots<k_m<k_1+n<\cdots<k_m+n\}$, then

$$\det(T_J) = \det(X_K)\det(Y_K),$$

where $K = \{k_1, \dots, k_m\}$. On the other hand, we have

$$\operatorname{Pf}(TA^tT) = \operatorname{Pf}\begin{pmatrix} 0 & X^tY \\ -Y^tX & 0 \end{pmatrix} = (-1)^{\binom{m}{2}} \det(X^tY).$$

The proof is completed by the minor-summation formula. ∎

By applying Theorem 2.1 to the matrices $T^{X(n)}$ and using Weyl's character formula (Proposition 1.1), we immediately see the following general formulas.

THEOREM 2.3. *Let n be an even integer and let $A = (a_{kl})$ be a skew-symmetric matrix with rows and columns indexed by $\mathbb{N}$ (resp. $\mathbb{N} + \frac{1}{2}$ in the $B(n)$ and $D(n)$ cases). Then we have*

$$\sum_{\lambda:\, l(\lambda) \le n} \operatorname{Pf}\left(A_{J(\lambda)}\right) \lambda_{A(n)}$$
$$= \frac{1}{\Delta_{A(n)}} \operatorname{Pf}\left(\sum_{k,l} a_{kl} x_i^k x_j^l\right)_{1 \le i,j \le n},$$

$$\sum_{\lambda:\, l(\lambda) \le n} \operatorname{Pf}\left(A_{J(\lambda)}\right) \lambda_{B(n)}$$
$$= \frac{1}{\Delta_{B(n)}} \operatorname{Pf}\left(\sum_{k,l} a_{kl} \left(x_i^{k+1/2} - x_i^{-k-1/2}\right)\left(x_j^{l+1/2} - x_j^{-l-1/2}\right)\right)_{1 \le i,j \le n},$$

$$\sum_{\lambda:\, l(\lambda) \le n} \operatorname{Pf}\left(A_{J(\lambda)}\right) \lambda_{C(n)}$$
$$= \frac{1}{\Delta_{C(n)}} \operatorname{Pr}\left(\sum_{k,l} a_{kl} \left(x_i^{k+1} - x_i^{-k-1}\right)\left(x_j^{l+1} - x_j^{-l-1}\right)\right)_{1 \le i,j \le n},$$

$$\sum_{\lambda:\, l(\lambda) \le n} \operatorname{Pf}\left(A_{J(\lambda)}\right) \left(\lambda^+_{D(n)} \pm \lambda^-_{D(n)}\right)$$
$$= \frac{1}{\Delta_{D(n)}} \operatorname{Pf}\left(\sum_{k,l} a_{kl} \left(x_i^k \pm x_i^{-k}\right)\left(x_j^l \pm x_j^{-l}\right)\right)_{1 \le i,j \le n}$$

where λ runs over all partitions of length $l(\lambda) \le n$ (resp. over all half-partitions of length n). ∎

Remark. The first formula for $\mathfrak{gl}(n, \mathbb{C})$ was obtained by T. Sundquist; see [Su].

3. SUBPFAFFIANS

When applying Theorem 2.3, we must calculate all the subPfaffians of the weight matrix A and evaluate the Pfaffian in the right hand side. This section is devoted to the calculation of subPfaffians, which will be used to derive the Littlewood-type formulas.

PROPOSITION 3.1. *Let $A = (a_{ij})_{i,j\geq 0}$ be the skew-symmetric matrix given by*

$$a_{ij} = s^i t^{j-1} u^{\mathrm{odd}(i)+\mathrm{even}(j)} v^{[i/2]+[(j-1)/2]} \qquad \text{for } 0 \leq i < j,$$

where $[x]$ is the largest integer not exceeding x and

$$\mathrm{odd}(k) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases} \qquad \mathrm{even}(k) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

That is,

$$A = \begin{pmatrix} 0 & 1 & tu & t^2v & t^3uv & t^4v^2 & t^5uv^2 & \cdots \\ & 0 & stu^2 & st^2uv & st^3u^2v & st^4uv^2 & st^5u^2v^2 & \cdots \\ & & 0 & s^2t^2v^2 & s^2t^3uv^2 & s^2t^4v^3 & s^2t^5uv^3 & \cdots \\ & & & 0 & s^3t^3u^2v^2 & s^3t^4uv^3 & s^3t^5u^2v^3 & \cdots \\ & & & & 0 & s^4t^4v^4 & s^4t^5uv^4 & \cdots \\ & & & & & 0 & s^5t^5u^2v^4 & \cdots \\ & & & & & & 0 & \cdots \\ & & & & & & & \ddots \end{pmatrix}.$$

Let J be an n-element subset of $\mathbb{N}$ and λ be the corresponding partition. Then the subPfaffian of A corresponding to J is given by

$$\mathrm{Pf}(A_J) = s^{e(\lambda)+m(m-1)} t^{o(\lambda)+m(m-1)} u^{r(\lambda)} v^{\sum_{i=1}^{2m}[\lambda_i/2]+m(m-1)}.$$

where $r(\lambda)$ denotes the number of rows of odd length in λ and

$$o(\lambda) = \sum_{i=1}^{m} \lambda_{2i-1}, \qquad e(\lambda) = \sum_{i=1}^{m} \lambda_{2i}.$$

Proof. If $J = \{j_1 < \cdots < j_{2m}\}$, then we have $j_k = \lambda_{2m+1-k} + k - 1$ and

$$\sum_{i=1}^{m} \lambda_{2i-1} + m(m-1) = \sum_{i=1}^{m} (j_{2i-1} - 1),$$

$$\sum_{i=1}^{m} \lambda_{2i} + m(m-1) = \sum_{i=1}^{m} j_{2i},$$

$$\sum_{i=1}^{2m} [\lambda_i/2] + m(m-1) = \sum_{i=1}^{m} [(j_{2i-1}-1)/2] + \sum_{i=1}^{m} [j_{2i}/2].$$

Also, since $\lambda_{2k} \equiv j_{2m+1-2k}$ and $\lambda_{2k-1} \equiv j_{2m+2-2k} - 1 \bmod 2$, we see that

$$\begin{aligned} r(\lambda) &= \mathrm{even}(\lambda_1) + \mathrm{even}(\lambda_2) + \cdots + \mathrm{even}(\lambda_{2m-1}) + \mathrm{even}(\lambda_{2m}) \\ &= \mathrm{even}(j_1) + \mathrm{odd}(j_2) + \cdots + \mathrm{even}(j_{2m-1}) + \mathrm{odd}(j_{2m}). \end{aligned}$$

Now the proposition can be obtained by putting

$$x_k = t^{j_k - 1} u^{\mathrm{even}(j_k - 1)} v^{[(j_k - 1)/2]} \quad \text{and} \quad y_k = t^{j_k} u^{\mathrm{odd}(j_k)} v^{[j_k/2]}$$

in the following lemma. ∎

LEMMA 3.2. *Let r be an even integer and $x_1, \ldots, x_r, y_1, \ldots, y_r$ be indeterminates. Then the Pfaffian of the skew-symmetric matrix with (i, j)-entry $x_i y_j$, $i < j$, is equal to*

$$\prod_{i=1}^{r/2} x_{2i-1} \cdot \prod_{i=1}^{r/2} y_{2i}.$$

Proof. See [IW1, Lemma 7]. ∎

PROPOSITION 3.3. *For an even integer $n = 2m$ and a non-negative integer r, let $A^{(n,r)} = (a_{ij})_{0 \le i, j \le 2n+r-1}$ and $B^{(n,r)} = (b_{ij})_{0 \le i, j \le 2n+r-1}$ be the $(2n+r) \times (2n+r)$ skew-symmetric matrices whose non-zero entries are given by*

$$a_{i,n-1-i} = a_{2n+r-1-i,n+r+i} = \begin{cases} 1 & \text{if } 0 \le i \le m-1, \\ -1 & \text{if } m \le i \le n-1, \end{cases}$$

$$b_{i,n+r+i} = b_{n-i-1,2n+r-1-i} = \begin{cases} 1 & \text{if } 0 \le i \le m-1, \\ -1 & \text{if } m \le i \le n-1. \end{cases}$$

That is, the matrices $A^{(n,r)}$ and $B^{(n,r)}$ are of the form

$$A^{(n,r)} = \begin{pmatrix} & J_m & 0 & & \\ -J_m & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \\ & & 0 & & -J_m \\ & & 0 & J_m & \end{pmatrix},$$

$$B^{(n,r)} = \begin{pmatrix} & & 0 & I_m & \\ & & 0 & & -I_m \\ 0 & 0 & 0 & 0 & 0 \\ -I_m & & 0 & & \\ & I_m & 0 & & \end{pmatrix},$$

where J_m is the anti-diagonal matrix with anti-diagonal entries 1 *and I_m is the identity matrix. Let J be a subset of* $[0, 2n + r - 1]$ *and λ be the corresponding partition. Then the subPfaffian of $A^{(n,r)} \pm B^{(n,r)}$ corresponding to J is given by*

$$\mathrm{Pf}\big((A^{(n,r)} \pm B^{(n,r)})_J\big) = \begin{cases} (-1)^{(|\lambda| - (r \pm 1)p(\lambda))/2} & \text{if } \lambda \in \Gamma_{r,n} \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We put $A^{\pm} = A^{(r,n)} \pm B^{(r,n)}$.

First, we show that $\mathrm{Pf}(A^{\pm}_J) = 0$ unless $\lambda \in \Gamma_{r,n}$. Suppose that $\mathrm{Pf}(A^{\pm}_J) \neq 0$. By the definition of the matrices $A^{(n,r)}$ and $B^{(n,r)}$, the columns of $A^{\pm}$ from the nth to the $(n + r - 1)$st are 0 vectors and, for each $0 \leq i \leq n - 1$, the ith column is proportional to the $(2n + r - 1 - i)$th column. Hence we see that the subset J contains exactly one element of each pair $\{j, 2n + r - 1 - j\}$.

Here we note that $p(\lambda) = \#\{j \in J: j \geq n\}$. The largest $p(\lambda)$ elements of J are $\lambda_1 + n - 1, \dots, \lambda_{p(\lambda)} + n - p(\lambda)$. From the above observation, we see that the smallest $p(\lambda)$ elements of the complement J^c in $[0, 2n + r - 1]$ are

$$n + r - \lambda_1, \dots, n + r + p(\lambda) - 1 - \lambda_{p(\lambda)}.$$

On the other hand, it follows from [M, I.(1.7)] that the smallest $p(\lambda)$ elements of J^c are

$$n - \lambda'_1, \dots, n + p(\lambda) - 1 - \lambda'_{p(\lambda)}.$$

Therefore we have

$$n + r + i - 1 - \lambda_i = n + i - 1 - \lambda'_i,$$

so that $\lambda_i = \lambda'_i$ for $1 \le i \le p$. This shows that $\lambda \in \Gamma_{r,n}$.

From now on, we assume that $\lambda = (\beta + r \mid \beta) \in \Gamma_{r,n}$. The matrix A^- is obtained from A^+ by multiplying by -1 the rows and the columns with index $\ge n + r$ and there are $p(\lambda)$ such rows of (or columns). Hence, by (2.1), we have

$$\mathrm{Pf}(A_J^-) = (-1)^{p(\lambda)} \mathrm{Pf}(A_J^+).$$

Therefore it is enough to prove

$$\mathrm{Pf}(A_J^+) = (-1)^{|\beta|},$$

because $|\lambda| = 2|\beta| + (r+1)p(\lambda)$. We will proceed by induction on $p(\lambda)$. If $p(\lambda) = 0$, i.e., $\lambda = \emptyset$, then we see that

$$\mathrm{Pf}\left(A^+_{J(\emptyset)}\right) = \mathrm{Pf}\begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix} = 1.$$

Suppose that $p(\lambda) > 0$ and $J = \{j_1 < \cdots < j_n\}$. If we put $k = 2n + r - 1 - j_n$ and $K = \{j_1, \dots, j_{n-1}, k\}$, then $k \le n - 1$ and $j_i = i$ for $1 \le i \le k - 1$. Hence A_J^+ is obtained from A_K^+ by permuting rows and columns by the same cyclic permutation $\sigma = (k, k+1, \dots, n)$. The inversion number of σ is equal to $n - k - 1 = \lambda_1 + r + 1 \equiv \beta_1 \pmod 2$. Hence we have

$$\mathrm{Pf}(A_J^+) = (-1)^{\beta_1} \mathrm{Pf}(A_K^+).$$

Let μ be the partition corresponding to K. Then it is easy to see that $\mu = (\beta_2 + r, \dots, \beta_p + r \mid \beta_2, \dots, \beta_p)$. By using the induction hypothesis, we have

$$\mathrm{Pf}(A_J^+) = (-1)^{\beta_1}(-1)^{\beta_2 + \cdots + \beta_p} = (-1)^{|\beta|}. \quad \blacksquare$$

4. LITTLEWOOD'S FORMULAS FOR $\mathfrak{g}_{A(n)}$

In this section we prove Littlewood's formulas and their generalizations for the Schur functions $s_\lambda(x_1, \dots, x_n) = \lambda_{A(n)}(x_1, \dots, x_n)$.

THEOREM 4.1. *Let $n = 2m$ be an even integer. Then we have*

$$\sum_{\lambda:\, l(\lambda)\le n} s^{e(\lambda)+m(m-1)} t^{o(\lambda)+m(m-1)} u^{r(\lambda)} v^{\sum_{i=1}^{2m}[\lambda_i/2]+m(m-1)} s_\lambda(x_1,\dots,x_n)$$

$$= \frac{1}{\Delta_{A(n)}} \frac{1}{\prod_{i=1}^{n}\left(1 - t^2 v x_i^2\right)}$$

$$\times \operatorname{Pf}\left(\frac{(x_j - x_i)\left\{\left(1 + t^2 v x_i x_j\right)\left(1 + stu^2 x_i x_j\right) + tu\left(1 + stx_i x_j\right)(x_i + x_j)\right\}}{1 - s^2 t^2 v^2 x_i^2 x_j^2} \right)_{1\le i,j\le n},$$

where $e(\lambda) = \sum_{i=1}^{m} \lambda_{2i}$ and $o(\lambda) = \sum_{i=1}^{m} \lambda_{2i-1}$.

Proof. Apply Theorem 2.3 to the skew-symmetric matrix A given in Proposition 3.1. Then a straightforward calculation gives us the (i, j)-entry of TA^tT:

$$\sum_{0\le k<l} s^k t^{l-1} u^{\mathrm{odd}(k)+\mathrm{even}(l)} v^{[k/2]+[(l-1)/2]} \det\begin{pmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{pmatrix}$$

$$= \frac{x_j - x_i}{\left(1 - t^2 v x_i^2\right)\left(1 - t^2 v x_j^2\right)}$$

$$\times \frac{\left(1 + t^2 v x_i x_j\right)\left(1 + stu^2 x_i x_j\right) + tu(x_i + x_j)\left(1 + stv x_i x_j\right)}{1 - s^2 t^2 v^2 x_i^2 x_j^2}.$$

Now the proof follows from Theorem 2.3 and Proposition 3.1. ∎

There are cases when the Pfaffian in the right hand side is factorized as a product. The following corollary is usually called Littlewood's formula.

COROLLARY 4.2 [L, p. 238; M, I, Ex. 5.7, 5.8].

$$(1) \qquad \sum_{\lambda:\, l(\lambda)\le n} t^{c(\lambda)} s_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{1 - tx_i} \prod_{1\le i<j\le n} \frac{1}{1 - x_i x_j},$$

$$(2) \qquad \sum_{\lambda:\, l(\lambda)\le n} u^{r(\lambda)} s_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1 + ux_i}{1 - x_i^2} \prod_{1\le i<j\le n} \frac{1}{1 - x_i x_j}.$$

Proof. Since $\lambda_{2i-1} - \lambda_{2i} = \#\{j\colon \lambda'_j = 2i - 1\}$, we have

$$o(\lambda) - e(\lambda) = c(\lambda).$$

Hence, if we put $s = t^{-1}$, $u = v = 1$ (resp. $s = t = v = 1$) in Theorem 4.1, then the summation side reduces to that of (1) (resp. (2)). On the other hand, the Pfaffian in the right hand side can be evaluated by using the Pfaffian form of Cauchy's determinant formula:

$$\mathrm{Pf}\left(\frac{x_j - x_i}{1 - x_i x_j}\right)_{1 \le i, j \le n} = \frac{\Delta_{A(n)}}{\prod_{1 \le i < j \le n}(1 - x_i x_j)}.$$

(See [Kn] or [St, Proposition 2.3(e)] for the proof). ∎

Now we apply Theorem 3.1 to the skew-symmetric matrices $A^{(n,r)} \pm B^{(n,r)}$ in Proposition 3.3 and derive another type of Littlewood's formulas. Then the Pfaffian in the right hand side is evaluated by the following lemma.

LEMMA 4.3. *If $n = 2m \in \mathbb{N}$ is even and $r \ge 0$, then we have*

$$\mathrm{Pf}\left(T^{A(n)}\left(A^{(r,n)} \pm B^{(r,n)}\right)^{t} T^{A(n)}\right) = \det\left(x_i^{j-1} \pm x_i^{2n+r-j}\right)_{1 \le i, j \le n}.$$

Proof. A direct calculation shows that the (i, j)-entry of $T^{A(n)}(A^{(r,n)} + B^{(r,n)})^t T^{A(n)}$ is equal to

$$\left(1 - x_i^{2m+r} x_j^{2m+r}\right)\frac{\left(x_j^m - x_i^m\right)^2}{x_j - x_i} \pm \left(x_j^{2m+r} - x_i^{2m+r}\right)\frac{\left(1 - x_i^m x_j^m\right)^2}{1 - x_i x_j}.$$

If we put $C = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$ and $S = (x_i^{j-1} \pm x_i^{2n+r-j})_{1 \le i, j \le n}$, then the (i, j)-entry of SC^tS is equal to the (i, j)-entry of $T^{A(n)}(A^{(r,n)} \pm B^{(r,n)t})T^{A(n)}$. Hence, we have

$$\begin{aligned} &\mathrm{Pf}\left(T^{A(n)}\left(A^{(r,n)} \pm B^{(r,n)}\right)^{t} T^{A(n)}\right) \\ &\qquad = \mathrm{Pf}(SC^tS) = \det(S)\mathrm{Pf}(C) = \det(S). \end{aligned}$$ ∎

Now we are in position to prove our generalization of Littlewood's formulas. Recall that $\Gamma_{r,n}$ is the set of all partitions λ with at most n parts and of the form

$$\lambda = (\beta_1 + r, \ldots, \beta_p + r \mid \beta_1, \ldots, \beta_p)$$

in the Frobenius notation.

THEOREM 4.4. (1) *If* $r \in \mathbb{N}$, *then we have*

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda|-(r-1)p(\lambda))/2} s_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n}(1-x_i)\prod_{i<j}(1-x_ix_j)\prod_{i=1}^{n}x_i^{r/2}\cdot\left(\frac{r}{2},\dots,\frac{r}{2}\right)_{B(n)}.$$

(2) *If* $r \geq 1$ *is an odd integer, then*

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda|-(r-1)p(\lambda))/2} s_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n}(1-x_i^2)\prod_{i<j}(1-x_ix_j)\prod_{i=1}^{n}x_i^{(r-1)/2}\cdot\left(\frac{r-1}{2},\dots,\frac{r-1}{2}\right)_{C(n)}.$$

(3) *If* $r \in \mathbb{N}$, *then we have*

$$\sum_{\lambda \in \Gamma_{r,n}^{\pm}} (-1)^{(|\lambda|-(r+1)p(\lambda))/2} s_\lambda(x_1,\dots,x_n) = \prod_{i<j}(1-x_ix_j)\prod_{i=1}^{n}x_i^{(r+1)/2}\cdot\left(\frac{r+1}{2},\dots,\frac{r+1}{2}\right)^{\pm}_{D(n)},$$

where $\Gamma_{r,n}^{\pm}$ *are subsets of* $\Gamma_{r,n}$ *defined by*

$$\Gamma_{r,n}^{+} = \{\lambda \in \Gamma_{r,n}: p(\lambda) \equiv n \bmod 2\},$$
$$\Gamma_{r,n}^{-} = \{\lambda \in \Gamma_{r,n}: p(\lambda) \equiv n-1 \bmod 2\}.$$

Remark. We can prove (3) for the case of $r=-1$ by modifying the matrix $A^{(r,n)} + B^{(r,n)}$. The original Littlewood formulas correspond to the cases of $r=0$ in (1), $r=1$ in (2), and $r=-1$ in (3). These cases are proved in [M, I, Ex. 5.9] by using Weyl's denominator formula (see also [KT]).

Proof. First we consider the case where n is even. By applying Theorem 2.3 to the matrices $A^{(r,n)} \pm B^{(r,n)}$ and by using Lemma 4.3 and the relation

$$\det\left(x_i^{j-1} \pm x_i^{2n+r-j}\right)_{1\le i,j\le n} = (-1)^{n(n\mp 1)/2}(x_1\cdots x_n)^{(2n+r-1)/2} \times \det\left(x_i^{(r+1)/2+j-1} \pm x_i^{-(r+1)/2-j+1}\right)_{1\le i,j\le n},$$

we obtain

$$\sum_{\lambda\in\Gamma_{r,n}}(-1)^{(|\lambda|-(r-1)p(\lambda))/2}\det T^{A(n)}_{J(\lambda)}$$

$$=(-1)^{n(n+1)/2}(x_1\cdots x_n)^{(2n+r-1)/2}\det T^{B(n)}_{J((r/2)^n)} \tag{4.1}$$

$$=(-1)^{n(n+1)/2}(x_1\cdots x_n)^{(2n+r-1)/2}\det T^{C(n)}_{J(((r-1)/2)^n)} \tag{4.2}$$

$$=(-1)^{n(n+1)/2}(x_1\cdots x_n)^{(2n+r-1)/2}\det T_{J(((r+1)/2)^n)},^{D^-(n)} \tag{4.3}$$

$$\sum_{\lambda\in\Gamma_{r,n}}(-1)^{(|\lambda|-(r+1)p(\lambda))/2}\det T^{A(n)}_{J(\lambda)}$$

$$=(-1)^{n(n-1)/2}(x_1\cdots x_n)^{(2n+r-1)/2}\det T^{D^+(n)}_{J(((r+1)/2)^n)}. \tag{4.4}$$

Now (1) (resp. (2)) follows from (4.1) (resp. (4.2)) and Proposition 1.2. By adding (4.3) and (4.4) and by subtracting (4.3) from (4.4), we see that

$$\sum_{\lambda\in\Gamma_{r,n}}(-1)^{(|\lambda|-(r+1)p(\lambda))/2}\left\{1\pm(-1)^{p(\lambda)}\right\}\det T^{A(n)}_{J(\lambda)}$$

$$=(-1)^{n(n-1)/2}(x_1\cdots x_n)^{(2n+r-1)/2}$$

$$\times\left\{\det T^{D^+(n)}_{J(((r+1)/2)^n)}\pm(-1)^n\det T^{D^-(n)}_{J(((r+1)/2)^n)}\right\}.$$

From this we obtain (3).

Next we prove (3) when n is odd. Since $n+1$ is even, we already know that

$$\sum_{\lambda\in\Gamma^{\pm}_{r,n+1}}(-1)^{(|\lambda|-(r+1)p(\lambda))/2}s_\lambda(x_1,\ldots,x_{n+1})$$

$$=\prod_{1\le i<j\le n+1}(1-x_ix_j)\prod_{i=1}^{n+1}x_i^{(r+1)/2}\cdot\left(\frac{r+1}{2},\ldots,\frac{r+1}{2}\right)^{\pm}_{D(n+1)}.$$

Substitute $x_{n+1}=0$ and use Lemma 1.3. Then, by noting $\Gamma^{\mp}_{r,n}=\{\lambda\in\Gamma_{r,n+1}:\lambda_{n+1}=0\}$, we obtain the desired formula. The other formulas are similarly proved. ∎

Remark. Here we proved Theorem 4.4 by using the minor-summation formula of Pfaffians. However, one can give another proof by applying the Cauchy–Binet formula to suitable matrices X with entries 0, ± 1, and $Y = T^{A(n)}$. The details are left to the readers.

5. LITTLEWOOD'S FORMULAS FOR $\mathfrak{g}_{B(n)}$, $\mathfrak{g}_{C(n)}$ AND $\mathfrak{g}_{D(n)}$

In this section we establish a generalization of Theorem 4.4 to B, C, D types. The proofs are similar to that of Theorem 4.4 and need a somewhat more complicated calculation. In order to treat all the types in a unified way, we introduce the n-rowed matrix $T^{\pm}(\alpha)$ for a half-integer α. The (i,k)-entry of $T^{\pm}(\alpha)$ is given by

$$t_{ik}^{\pm}(\alpha) = x_i^{k+\alpha} \pm x_i^{-k-\alpha} \qquad \text{for } k \in \tfrac{1}{2}\mathbb{N}.$$

Then we obtain the following lemma.

LEMMA 5.1. *Let $r \in \mathbb{N}$, $k \in \frac{1}{2}\mathbb{N}$ and let n be a positive even integer. Then we have*

$$\begin{aligned}
&\operatorname{Pf}\left(T^{+}(\alpha) A^{(r,n)} \pm B^{(r,n)}\right)^{t} T^{+}(\alpha))\\
&\quad = \prod_{i=1}^{n} \left(x_i^{n+\alpha+(r-1)/2} \pm x_i^{-n-\alpha-(r-1)/2}\right)\\
&\qquad \times \det\left(x_i^{j-n-(r+1)/2} \pm x_i^{-j+n+(r+1)/2}\right)_{1 \le i,j \le n},\\
&\operatorname{Pf}\left(T^{-}(\alpha)(A^{(r,n)} \pm B^{(r,n)})^{t} T^{-}(\alpha)\right)\\
&\quad = \prod_{i=1}^{n} \left(x_i^{n+\alpha+(r-1)/2} \mp x_i^{-n-\alpha-(r-1)/2}\right)\\
&\qquad \times \det\left(x_i^{j-n-(r+1)/2} \pm x_i^{-j+n+(r+1)/2}\right)_{1 \le i,j \le n}.
\end{aligned}$$

Proof. We put $N = 2n + r - 1$ and $A^{\pm} = A^{(n,r)} \pm B^{(n,r)} = (\alpha_{ij}^{\pm})_{0 \le i,j \le N}$. By the definition of $A^{(n,r)}$ and $B^{(n,r)}$, we have

$$\alpha_{kl}^{\pm} = \alpha_{N-k,N-l}^{\pm} = \pm \alpha_{k,N-l}^{\pm} = \pm \alpha_{N-k,l}^{\pm}.$$

Also, we note that

$$\begin{aligned}
&\left(x^{k+\alpha}+x^{-k-\alpha}\right)\left(y^{l+\alpha}+y^{-l-\alpha}\right)\\
&\qquad+\left(x^{N-k+\alpha}+x^{-N+k-\alpha}\right)\left(y^{N-l+\alpha}+y^{-N+l-\alpha}\right)\\
&\qquad\pm\left(x^{k+\alpha}+x^{-k-\alpha}\right)\left(y^{N-l+\alpha}+y^{-N+l-\alpha}\right)\\
&\qquad\pm\left(x^{N-k+\alpha}+x^{-N+k-\alpha}\right)\left(y^{l+\alpha}+y^{-l-\alpha}\right)\\
&\quad=x^{\alpha}y^{\alpha}\left(1\pm x^{-N-2\alpha}\right)\left(1\pm y^{-N-2\alpha}\right)\\
&\qquad\times\left(x^{k}y^{l}+x^{N-k}y^{N-l}\pm x^{N-k}y^{l}\pm x^{k}y^{N-l}\right).
\end{aligned}$$

Then the (i,j)-entry of $T^{+}(\alpha)A^{\pm t}T^{+}(\alpha)$ is given by

$$\begin{aligned}
&\left(T^{+}(\alpha)A^{\pm t}T^{+}(\alpha)\right)_{ij}\\
&\quad=\sum_{k,l=0}^{N}\alpha^{\pm}_{k,l}\left(x_i^{k+\alpha}+x_i^{-k-\alpha}\right)\left(x_j^{l+\alpha}+x_j^{-l-\alpha}\right)\\
&\quad=\frac{1}{4}x_i^{\alpha}x_j^{\alpha}\left(1\pm x_i^{-N-2\alpha}\right)\left(1\pm x_j^{-N-2\alpha}\right)\\
&\qquad\times\sum_{k,l=0}^{N}\alpha^{\pm}_{k,l}\left(x_i^{k}x_j^{l}+x_i^{N-k}x_j^{N-l}\pm x_i^{N-k}x_j^{l}\pm x_i^{k}x_j^{N-l}\right)\\
&\quad=x_i^{\alpha}x_j^{\alpha}\left(1\pm x_i^{-N-2\alpha}\right)\left(1\pm x_j^{-N-2\alpha}\right)\sum_{k,l=0}^{N}\alpha^{\pm}_{k,l}x_i^{k}x_j^{l}\\
&\quad=x_i^{\alpha}x_j^{\alpha}\left(1\pm x_i^{-N-2\alpha}\right)\left(1\pm x_j^{-N-2\alpha}\right)\left(T^{A(n)}A^{\pm t}T^{A(n)}\right)_{i,j}.
\end{aligned}$$

By using Lemma 4.3, we obtain the first formula. The second identity can be proved by the same argument. ∎

Now we can prove a generalization of Theorem 4.4, which we call Littlewood's formula of type B.

THEOREM 5.2. (1) *If* $r\in\mathbb{N}$ *and* $s\in\frac{1}{2}\mathbb{N}$, *then we have*

$$\begin{aligned}
&\sum_{\lambda\in\Gamma_{r,n}}(-1)^{(|\lambda|-(r+1)p(\lambda))/2}\left(\lambda+(s^n)\right)_{B(n)}\\
&\quad=(-1)^{n(n-1)/2}\prod_{i=1}^{n}\frac{x_i^{n+s+r/2}-x_i^{-n-s-r/2}}{x_i^{1/2}-x_i^{-1/2}}\\
&\qquad\times\left\{\left(\left(\frac{r+1}{2}\right)^{n}\right)^{+}_{D(n)}+\left(\left(\frac{r+1}{2}\right)^{n}\right)^{-}_{D(n)}\right\}.
\end{aligned}$$

(2) *If $r \in \mathbb{N}$ and $s \in \frac{1}{2}\mathbb{N}$, then we have*

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda|-(r-1)p(\lambda))/2} (\lambda + (s^n))_{B(n)}$$

$$= (-1)^{n(n+1)/2} \times \prod_{i=1}^{n} \left(x_i^{n+s+r/2} + x_i^{-n-s-r/2}\right) \left(\left(\frac{r}{2}\right)^n\right)_{B(n)}$$

$$= (-1)^{n(n+1)/2} \times \prod_{i=1}^{n} \left(x_i^{n+s+r/2} + x_i^{-n-s-r/2}\right)\left(x_i^{1/2} + x_i^{-1/2}\right) \left(\left(\frac{r-1}{2}\right)^n\right)_{C(n)}$$

$$= (-1)^{n(n+1)2} \times \prod_{i=1}^{n} \frac{x_i^{n+s+r/2} + x_i^{-n-s-r/2}}{x_i^{1/2} - x_i^{-1/2}} \left\{ \left(\left(\frac{r+1}{2}\right)^n\right)^{+}_{D(n)} - \left(\left(\frac{r+1}{2}\right)^n\right)^{-}_{D(n)} \right\}.$$

Here the second identity makes sense only when r is an odd integer.

Proof. First, we assume that n is even. Apply Theorem 2.3 to the matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & A^{(n,r)} \pm B^{(n,r)} \end{pmatrix}.$$

Then, by using Lemma 5.1, we have

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda|-(r\pm 1)p(\lambda))/2} \det T^{B(n)}_{J(\lambda+(s^n))}$$

$$= \prod_{i=1}^{n} \left(x_i^{n+s+r/2} \mp x_i^{-n-s-r/2}\right) \times \det\left(x_i^{j-n-(r+1)/2} \pm x_i^{-j+n+(r+1)/2}\right)_{i,j=1,\dots,n}$$

$$= \begin{cases} (-1)^{n(n-1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+r/2} - x_i^{-n-s-r/2}\right) \det T^{D(n)}_{J(((r+1)/2)^n)}, \\ (-1)^{n(n+1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+r/2} + x_i^{-n-s-r/2}\right) \det T^{B(n)}_{J((r/2)^n)}. \end{cases}$$

Dividing both sides by the Weyl denominator $\Delta_{B(n)} = \prod_{i=1}^{n}(x_i^{1/2} - x_i^{-1/2}) \cdot \Delta_{D(n)}$, we obtain (1) and the first equality in (2). The other equalities follow from Lemma 1.4.

Before proving the case where n is odd, we prepare the following lemma.

LEMMA 5.3. *If we put* $\Gamma'_{r,n+1} = \{\lambda \in \Gamma_{r,n+1} : \lambda_1 = n + r + 1\}$, *then the map given by*

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1}) \mapsto \bar{\lambda} = (\lambda_2 - 1, \dots, \lambda_{n+1} - 1)$$

provides a bijection from $\Gamma'_{r,n+1}$ *to* $\Gamma_{r,n}$. *And we have* $|\bar{\lambda}| = |\lambda| - 2n - r - 1$ *and* $p(\bar{\lambda}) = p(\lambda) - 1$.

Proof. If $\lambda \in \Gamma'_{r,n+1}$, then $l(\lambda) = \lambda_1 - r = n + 1$ and $\bar{\lambda} = (\lambda_2 - 1, \dots, \lambda_{n+1} - 1)$ is a partition. The Young diagram of $\bar{\lambda}$ is obtained from that of λ by removing the first row and the first column. Hence we have $\bar{\lambda} \in \Gamma_{r,n}$ and $p(\bar{\lambda}) = p(\lambda) - 1$. The inverse map sends $\mu \in \Gamma_{r,n}$ to $(n + r + 1, \mu_1 + 1, \dots, \mu_n + 1)$. ∎

Next, suppose that n is odd. We will prove the first equality in (2). (The other equalities can be proved in a similar way.) Since $n + 1$ is even, we already know that

$$\sum_{\lambda \in \Gamma_{r,n+1}} (-1)^{(|\lambda| - (r-1)p(\lambda))/2} \left(\lambda + (s^{n+1})\right)_{B(n+1)}$$
$$= (-1)^{(n+1)(n+2)/2} \prod_{i=1}^{n+1} \left(x_i^{n+1+s+r/2} + x_i^{-n-1-s-r/2}\right) \left(\left(\frac{r}{2}\right)^{n+1} \right)_{B(n+1)}.$$

We put $m = n + r + s + 1$. Multiply both sides by $(x_1 \cdots x_{n+1})^m$, substitute $x_{n+1} = 0$, and then divide by $(x_1 \cdots x_n)^m$. By Lemma 1.3, we obtain

$$\sum_{\lambda \in \Gamma'_{r,n+1}} (-1)^{(|\lambda| - (r-1)p(\lambda))/2} \left(\bar{\lambda} + ((s+1)^n)\right)_{B(n)}$$
$$= (-1)^{(n+1)(n+2)/2} \prod_{i=1}^{n} \left(x_i^{n+(s+1)+r/2} + x_i^{-n-(s+1)-r/2}\right) \left(\left(\frac{r}{2}\right)^{n} \right)_{B(n)}.$$

We can complete the proof by using Lemma 5.3 and the relation $|\lambda| - (r-1)p(\lambda) = |\bar{\lambda}| - (r-1)p(\bar{\lambda}) + 2n + 2$. ∎

Remark. Precisely speaking, we have not yet given a proof for the case where n is odd and $s = 0$ or $\frac{1}{2}$. However, one can give a direct proof for this case by using another minor-summation formula of Pfaffians [IW1, Theorem 1(2)] (or the Cauchy–Binet formula). The argument and the calculation needed in this case are almost the same as those in the case where n is even, so we omit the proof.

The Littlewood-type formula for $\mathfrak{g}_{C(n)}$ and $\mathfrak{g}_{D(n)}$ can be stated in the following form. We can deduce these theorems from Theorem 5.2 by using Lemma 1.4 or prove them by an argument similar to the proof of the $B(n)$ case.

THEOREM 5.4. (1) *If $r, s \in \mathbb{N}$, then we have*

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda|-(r+1)p(\lambda))/2} (\lambda + (s^n))_{C(n)} = (-1)^{n(n-1)/2} \prod_{i=1}^{n} \frac{x_i^{n+s+(r+1)/2} + x_i^{-n-s-(r+1)/2}}{x_i - x_i^{-1}} \times \left\{ \left(\left(\frac{r+1}{2} \right)^n \right)^{+}_{D(n)} + \left(\left(\frac{r+1}{2} \right)^n \right)^{-}_{D(n)} \right\}.$$

(2) *If $r, s \in \mathbb{N}$, then we have*

$$\begin{aligned} \sum_{\lambda \in \Gamma_{r,n}} & (-1)^{(|\lambda|-(r-1)p(\lambda))/2} (\lambda + (s^n))_{C(n)} \\ &= (-1)^{n(n+1)/2} \prod_{i=1}^{n} \frac{x_i^{n+s+(r+1)/2} + x_i^{-n-s-(r+1)/2}}{x_i^{1/2} + x_i^{-1/2}} \left(\left(\frac{r}{2} \right)^n \right)_{B(n)} \\ &= (-1)^{n(n+1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+(r+1)/2} + x_i^{-n-s-(r+1)/2} \right) \left(\left(\frac{r-1}{2} \right)^n \right)_{C(n)} \\ &= (-1)^{n(n+1)/2} \prod_{i=1}^{n} \frac{x_i^{n+s+(r+1)/2} + x_i^{-n-s-(r+1)/2}}{x_i - x_i^{-1}} \\ &\quad \times \left\{ \left(\left(\frac{r+1}{2} \right)^n \right)^{+}_{D(n)} - \left(\left(\frac{r+1}{2} \right)^n \right)^{-}_{D(n)} \right\}. \end{aligned}$$

Here the second identity makes sense only when r is odd.

THEOREM 5.5. (1) *If $r \in \mathbb{N}$ and $s \in \frac{1}{2}\mathbb{N}$, then we have*

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda|-(r+1)p(\lambda))/2} \left\{ (\lambda + (s^n))^{+}_{D(n)} \pm (\lambda + (s^n))^{-}_{D(n)} \right\} = (-1)^{n(n-1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+(r-1)/2} \pm x_i^{-n-s-(r-1)/2} \right) \times \left\{ \left(\left(\frac{r+1}{2} \right)^n \right)^{+}_{D(n)} + \left(\left(\frac{r+1}{2} \right)^n \right)^{-}_{D(n)} \right\}.$$

(2) *If* $r \in \mathbb{N}$ *and* $s \in \frac{1}{2}\mathbb{N}$, *then we have*

$$\sum_{\lambda \in \Gamma_{r,n}} (-1)^{(|\lambda| - (r-1)p(\lambda))/2} \left\{ (\lambda + (s^n))^{+}_{D(n)} \pm (\lambda + (s^n))^{-}_{D(n)} \right\}$$
$$= (-1)^{n(n+1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+(r-1)/2} \mp x_i^{-n-s-(r-1)/2} \right)$$
$$\times \left(x_i^{1/2} - x_i^{-1/2} \right) \left(\left(\frac{r}{2} \right)^n \right)_{B(n)}$$
$$= (-1)^{n(n+1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+(r-1)/2} \mp x_i^{-n-s-(r-1)/2} \right)$$
$$\times \left(x_i - x_i^{-1} \right) \left(\left(\frac{r-1}{2} \right)^n \right)_{C(n)}$$
$$= (-1)^{n(n+1)/2} \prod_{i=1}^{n} \left(x_i^{n+s+(r-1)/2} \mp x_i^{-n-s-(r-1)/2} \right)$$
$$\times \left\{ \left(\left(\frac{r+1}{2} \right)^n \right)^{+}_{D(n)} - \left(\left(\frac{r+1}{2} \right)^n \right)^{-}_{D(n)} \right\}.$$

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