Journal of Combinatorial Theory, Series A **88**, 136–157 (1999)
Article ID jcta.1999.2988, available online at http://www.idealibrary.com on IDEAL®

Applications of Minor-Summation Formula II. Pfaffians and Schur Polynomials

Masao Ishikawa[1]

Department of Mathematics, Faculty of Education, Tottori University, Koama, Tottori 680, Japan
E-mail: ishikawa@fed.tottori-u.ac.jp

and

Masato Wakayama[2]

Graduate School of Mathematics, Kyushu University, Hakozaki, Fukuoka 812, Japan
E-mail: wakayama@math.kyushu-u.ac.jp

Communicated by George Andrews

Received January 12, 1999

The purpose of this paper is, to establish, by extensive use of the minor summation formula of pfaffians exploited in (Ishikawa, Okada, and Wakayama, *J. Algebra* **183**, 193–216) certain new generating functions involving Schur polynomials which have a product representation. This generating function gives an extension of the Littlewood formula. During the course of the proof we develop some techniques for computing sub-Pfaffians of a given skew-symmetric matrix. After the proof we present an open problem which generalizes our formula. © 1999 Academic Press

Key Words: Pfaffian; generating function; Schur polynomial; character; partition.

1. INTRODUCTION

In this paper we establish certain new formulas concerning Schur polynomials with two parameters. The prototype of these formulas is the so-called Littlewood formula, which the reader can find in the book [Ma],

$$\sum_{\lambda} a^{c(\lambda)} s_{\lambda}(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{1}{1-ax_i} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-x_i x_j}, \tag{1.0}$$

[1] Partially supported by Grant-in-Aid for Scientific Research (C) 09640037 from the Ministry of Education, Science and Culture of Japan.
[2] Partially supported by Grant-in-Aid for Scientific Research (B) 08454010 from the Ministry of Education, Science and Culture of Japan.

0097-3165/99 $30.00

where the sum on the left is over all partitions λ and $c(\lambda)$ is the number of columns of odd length in λ. Here $s_\lambda = s_\lambda(x_1, x_2, ..., x_n)$ is the Schur polynomial of n variables corresponding to a partition λ. These Littlewood formulas have been generalized by several authors, e.g., [LP, YW]. Making extensive use of the minor summation formula of the Pfaffian developed in [IW1] we may extend these Littlewood formulas in various directions, e.g., [IOW, IW2, Ok, IW3]. In this paper we prove the following formulas algebraically by evaluating certain Pfaffians and by developing some techniques to compute sub-Pfaffians of a given skew-symmetric matrix.

THEOREM. *We have*

$$\sum_{\lambda} \varphi_\lambda^{2,0}(a,b)\, s_\lambda(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{1}{(1-ax_i)(1-bx_i)} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-x_i x_j}, \tag{1.1}$$

$$\sum_{\lambda} \varphi_\lambda^{1,1}(a,b)\, s_\lambda(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{1+bx_i}{1-ax_i} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-x_i x_j}, \tag{1.2}$$

where the functions $\varphi_\lambda^{2,0}(a,b)$ *and* $\varphi_\lambda^{1,1}(a,b)$ *of variables* a, b *are given by*

$$\varphi_\lambda^{2,0}(a,b) = a^{c(\lambda)} \prod_{k=1}^{\infty} [\lambda_k - \lambda_{k+1} + 1;\, a^{\varepsilon(k)} b], \tag{1.3}$$

$$\varphi_\lambda^{1,1}(a,b) = a^{c(\lambda)} \prod_{k=1}^{\infty} \{\lambda'_k - \lambda'_{k+1} + 1;\, a^{\varepsilon(\lambda'_k)}, b\}. \tag{1.4}$$

Here we put

$$[n; q] = \frac{1-q^n}{1-q}, \tag{1.5}$$

$$\{n; a, b\} = \begin{cases} \dfrac{1-b^{n+1}}{1-b^2} + ab\, \dfrac{1-b^{n-1}}{1-b^2} & \textit{if } n \textit{ is odd}, \\ (1+ab)\, \dfrac{1-b^n}{1-b^2} & \textit{if } n \textit{ is even}, \end{cases} \tag{1.6}$$

$$\varepsilon(k) = \begin{cases} -1 & \textit{if } k \textit{ is odd}, \\ 1 & \textit{if } k \textit{ is even}. \end{cases} \tag{1.7}$$

Since $\lambda_k - \lambda_{k+1} = 0$ for a sufficiently large number k the product representations in (1.3) and (1.4) are well-defined. In particular,

COROLLARY. *We have*

$$\sum_{\lambda} N(\lambda, q)\, s_\lambda(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{1}{(1-x_i)(1-qx_i)} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1-x_i x_j}, \tag{1.8}$$

$$\sum_{\lambda} N(\lambda', q)\, s_\lambda(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{1+qx_i}{1-x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1-x_i x_j}, \tag{1.9}$$

where we put $N(\lambda, q) = \prod_{i=1}^{\infty} [\lambda_i - \lambda_{i+1} + 1; q]$.

We briefly discuss a combinatorial proof of our main theorem in the last section and conclude that our proof gives an algebraic proof (i.e. an evaluation of Pfaffians) of the Pieri formula. It seems that our proof is still interesting as a method of evaluating Pfaffians even though it is possible to prove our theorem by a combinatorial method.

Naturally we may ask whether it is possible to generalize our theorem. We believe that the answer is yes, but we found that this problem is not as easy as we expected. In Section 5 we provide one conjecture which includes one more constant c. (Look at the conjecture in Section 5.) This conjecture looks very beautiful and mysterious to us, but we have not found any proof at this stage. We made sure that the coefficients of Shur functions coincide on both sides of the identity of the conjecture for smaller partitions. We used Maple V and calculated the identity for all partitions included in the 8 by 10 rectangle.

2. NOTATION AND GENERAL PRINCIPLE OF PROOFS

We fix some notation concerning partitions and symmetric polynomials.

Let us denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{Z}$ the set of integers. Let $[m]$ denote the subset $\{1, 2, ..., m\}$ of $\mathbb{N}$ for a positive integer m. A partition is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ...)$ of non-negative integers with a finite sum. Sometimes we use notation which indicates the number of times each integer occurs as a part: $\lambda = (1^{m_1} 2^{m_2} \cdots)$ means that exactly m_i of the parts of λ are equal to i. The partition $\lambda' = (\lambda'_1, \lambda'_2, ...)$ defined by $\lambda'_i = \#\{j : \lambda_j \geqslant i\}$ is called the conjugate partition of λ. The length $l(\lambda)$ of a partition λ is the number of non-zero terms of λ. For a partition λ, we denote by $c(\lambda)$ the number of columns of odd length in λ.

If λ is a partition of length $\leqslant n$, then we define the subset $J(\lambda)$ of $\mathbb{N}$ by

$$J(\lambda)=\{\lambda_1+n-1, \lambda_2+n-2, ..., \lambda_n\}. \tag{2.1}$$

Conversely, for a subset $J=\{j_1<\cdots<j_n\}$ of $\mathbb{N}$, let $\lambda(J)$ denote the partition defined by the equation

$$\lambda_i=j_{n+1-i}-n+i. \tag{2.2}$$

This clearly defines a one-to-one correspondence between n-element subsets of $\mathbb{N}$ and partitions of length $\leqslant n$.

For an n-row matrix T with columns indexed by I and an n-element subset J of I, we denote by T_J the $n\times n$ submatrix of T obtained by picking up the columns indexed by J. Namely, if $T=(T_{ij})_{i=1, ..., n,\ j\in I}$, then $T_J=(T_{ij})_{i=1, ..., n,\ j\in J}$.

A matrix $A=(a_{ij})_{i,\ j\in I}$ is said to be skew-symmetric if the entries of A satisfy $a_{ij}=-a_{ji}$. Given a skew-symmetric matrix $A=(a_{ij})_{i,\ j\in I}$ with the index set $I=\{i_1, ..., i_{2m}\}$ of even cardinality, we define the Pfaffian $\operatorname{pf}(A)$ of A by

$$\operatorname{pf}(A)=\sum_{\sigma\in S'_{2m}} \operatorname{sgn}\sigma \prod_{k=1}^{m} a_{i_{\sigma(2k-1)}i_{\sigma(2k)}}, \tag{2.3}$$

where

$$S'_{2m}=\{\sigma\in S_{2m} : \sigma(2k-1)<\sigma(j) \text{ for } 1\leqslant k\leqslant m, 2k-1<j\leqslant 2m\}. \tag{2.4}$$

For each $r\geqslant 0$ the rth complete symmetric polynomial h_r is defined by

$$h_r(x_1, x_2, ..., x_n)=\sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_r\leqslant n} x_{i_1}x_{i_2}\cdots x_{i_r}. \tag{2.5}$$

It is convenient to define h_r to be zero for $r<0$. Further, the Schur function (polynomial) in the variables $x_1, x_2, ..., x_n$ corresponding to the partition $\lambda=(\lambda_1, \lambda_2, ..., \lambda_n)$ is defined by

$$s_\lambda=s_\lambda(x_1, x_2, ..., x_n)=\det(x_i^{\lambda_j+n-j})/\det(x_i^{n-j}). \tag{2.6}$$

These Schur functions are, in fact, symmetric polynomials in the variables $x_1, x_2, ..., x_n$ and are known as the characters of the polynomial representations of the general linear group $GL(n, \mathbb{C})$. Recall now the minor summation formula of Pfaffians (in even cases).

LEMMA 2.1 [IW1]. *Assume that $n \leqslant N$ and n is even. Let $T=(t_{ik})$ be an $n \times N$ matrix and $A=(a_{kl})$ be an $N \times N$ skew-symmetric matrix. Then we have*

$$\sum_{I \subset [N],\ \#I=n} \operatorname{pf}(A_I^I) \det(T_I) = \operatorname{pf}(TA^tT), \tag{2.7}$$

where A_I^I denotes the $n \times n$ submatrix of A obtained by picking up the rows and columns indexed by the same index set I. Further, we note that the (i, j)-entry of the skew-symmetric matrix TA^tT is explicitly given by

$$(TA^tT)_{ij} = \sum_{1 \leqslant k < l \leqslant N} a_{kl} \begin{vmatrix} T_{ik} & T_{il} \\ T_{jk} & T_{jl} \end{vmatrix}, \qquad (1 \leqslant i, j \leqslant n). \tag{2.8}$$

In the preceding paper [IOW] we exploited machinery to establish identities on the irreducible characters of the classical groups. For instance, we take the matrix T as the special one; $T_{ij} = x_i^j$ $(i=1, 2, ..., n,\ j=0, 1, 2, ...)$ in the case of $\mathrm{GL}(n, \mathbb{C})$. Then the minor summation formula reads

$$\sum_{\lambda:\ l(\lambda) \leqslant n} \operatorname{pf}(A_{J(\lambda)})\, s_\lambda(x_1, ..., x_n) = \frac{1}{\prod_{i<j} (x_i - x_j)} \operatorname{pf}\left(\sum_{k, l} a_{kl} x_i^k x_j^l\right)_{i, j} \tag{2.9}$$

for any skew-symmetric matrix $A=(a_{kl})$ with rows and columns indexed by $\mathbb{N}$. Recall the identity which is a Pfaffian's counterpart of the Cauchy formula (see [Wy]):

$$\operatorname{pf}\left(\frac{x_j - x_i}{1 - x_i x_j}\right)_{i, j=1, ..., n} = \frac{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)}{\prod_{1 \leqslant i < j \leqslant n} (1 - x_i x_j)}, \tag{2.10}$$

for each even integer n (see [IW1, Ste]).

DEFINITION 2.1. Define $\beta_{k\ell}^{r,s} = \beta_{k\ell}^{r,s}(a_1, ..., a_r; b_1, ..., b_s)$ for $k, \ell \in \mathbb{N}$ via the equation

$$\sum_{k, \ell=0}^{\infty} \beta_{k\ell}^{r,s} x^k y^\ell = \frac{\prod_{k=1}^{s} (1 + b_k x)(1 + b_k y)}{\prod_{k=1}^{r} (1 - a_k x)(1 - a_k y)} \frac{y - x}{1 - xy}. \tag{2.11}$$

Further, we form a skew-symmetric matrix

$$B^{r,s} = B^{r,s}(a_1, ..., a_r; b_1, ..., b_s) = (\beta_{k\ell}^{r,s}(a_1, ..., a_r; b_1, ..., b_s))_{0 \leqslant k, 0 \leqslant \ell}. \tag{2.12}$$

DEFINITION 2.2. We put

$$\varphi_\lambda^{r,s}(a_1, ..., a_r, b_1, ..., b_s) = \mathrm{pf}(B^{r,s}(a_1, ..., a_r; b_1, ..., b_s)_{J(\lambda)}). \qquad (2.13)$$

It is clear that this $\varphi_\lambda^{r,s}(a_1, ..., a_r, b_1, ..., b_s)$ is a symmetric polynomial with variables $a_1, a_2, ..., a_r$ or $b_1, b_2, ..., b_r$ respectively, since one can easily see that $\beta_{k\ell}^{r,s}$ is as well. Then the following lemma follows immediately from (2.11) and (2.13).

LEMMA 2.2. *The identity*

$$\sum_\lambda \varphi_\lambda^{r,s}(a_1, ..., a_r, b_1, ..., b_s)\, s_\lambda(x_1, ..., x_n) = \prod_{i=1}^n \frac{\prod_{k=1}^s (1+b_k x_i)}{\prod_{l=1}^r (1-a_l x_i)} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-x_i x_j} \qquad (2.14)$$

holds, where the sum is over all partitions λ.

In the formula (2.14) above, we face the problem of evaluating all the sub-Pfaffians $\varphi_\lambda^{r,s}(a_1, ..., a_r, b_1, ..., b_s)$ in an explicit form. We call this problem the (r, s) case problem in this paper. We settle the (2, 0) and (1, 1) cases stated in the main theorem. We shall give also a conjecture for the (3,0) case in Section 5. Though we have some evidence which leads us to expect that there are explicit formulas for further cases, we have not obtained that result.

We close this section by quoting a useful formula. Although the formula itself is known (see, e.g. [Ste]), we give here another proof by Lemma 2.2.

LEMMA 2.3. *Let A and B be $m \times m$ skew-symmetric matrices. Put $s = [m/2]$, the integer part of $m/2$. Then*

$$\mathrm{pf}(A+B) = \sum_{t=0}^{s} \sum_{\mathbf{i} \in I_{2t}^m} (-1)^{|\mathbf{i}|-t}\, \mathrm{pf}(A_{\mathbf{i}})\, \mathrm{pf}(B_{\mathbf{i}^c}), \qquad (2.15)$$

where we denote by $\mathbf{i}^c$ the complementary set of $\mathbf{i}$ in $[m]$ which is arranged in increasing order, and $|\mathbf{i}| = i_1 + \cdots + i_{2t}$ for $\mathbf{i} = (i_1, ..., i_{2t})$. In particular, we have the expansion formula for a Pfaffian with respect to any column (row): For any i, j we have

$$\delta_{ij}\, \mathrm{pf}(A) = \sum_{k=1}^{m} (-1)^{k+j-1} a_{kj}\, \mathrm{pf}(A^{ki}), \qquad (2.16)$$

$$\delta_{ij}\, \mathrm{pf}(A) = \sum_{k=1}^{m} (-1)^{i+k-1} a_{ik}\, \mathrm{pf}(A^{jk}), \qquad (2.17)$$

where A^{ij} represents the $(m-2)\times(m-2)$ skew-symmetric matrix which is obtained from A by removing both the (i, j)-th rows and the (i, j)-th columns for $1 \leqslant i \neq j \leqslant m$.

Proof. Let I_m be an identity matrix of degree m. It is clear that

$$(I_m \quad I_m)\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_m \\ I_m \end{pmatrix} = A + B.$$

Hence by the minor summation formula we see that

$$\begin{aligned}\operatorname{pf}(A+B) &= \operatorname{pf}\left((I_m \quad I_m)\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}{}^t (I_m \quad I_m)\right) \\ &= \sum_{\mathbf{k} \in I_m^{2m}} \operatorname{pf}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{\mathbf{k}} \det(I_m \quad I_m)_{\mathbf{k}}.\end{aligned}$$

The only index $\mathbf{k}$ in I_m^{2m} for which $\det(I_m I_m)_{\mathbf{k}}$ does not vanish is of the form $\mathbf{k} = (\mathbf{i}, (m, m, ..., m) + \mathbf{i}^c)$ for $\mathbf{i} \in I_s^m$ and in this case we have $\det(I_m I_m)_{\mathbf{k}} = (-1)^{\sigma(\mathbf{i}, \mathbf{i}^c)}$, where $\sigma(\mathbf{i}, \mathbf{i}^c)$ means the number of inversions of $\mathbf{i}$ via $\mathbf{i}^c$. Further, if s is even then

$$\operatorname{pf}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{\mathbf{k}} = \operatorname{pf}\begin{pmatrix} A_{\mathbf{i}} & 0 \\ 0 & B_{\mathbf{i}^c} \end{pmatrix} = \operatorname{pf}(A_{\mathbf{i}}) \operatorname{pf}(B_{\mathbf{i}^c}).$$

This Pfaffian obviously vanishes in the case s is odd. Hence we see

$$\begin{aligned}\operatorname{pf}(A+B) &= \sum_{\mathbf{k} \in I_m^{2m}} \sum_{\mathbf{k} = (\mathbf{i}, (m, m, ..., m) + \mathbf{i}^c)} \operatorname{pf}(A_{\mathbf{i}}) \operatorname{pf}(B_{\mathbf{i}^c})(-1)^{\sigma(\mathbf{i}, \mathbf{i}^c)} \\ &= \sum_{t=0}^{[m/2]} \sum_{\mathbf{i} \in I_{2t}^m} (-1)^{|\mathbf{i}| - t} \operatorname{pf}(A_{\mathbf{i}}) \operatorname{pf}(B_{\mathbf{i}^c}),\end{aligned}$$

because $\sigma(\mathbf{i}, \mathbf{i}^c) = |\mathbf{i}| - t$ for $\mathbf{i} \in I_{2t}^m$.

The latter assertion can be proved by applying the previous result to the following form of the decomposition of a skew-symmetric matrix A with respect to the ith row and column:

$$A = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix} + \begin{pmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{pmatrix}.$$

This completes the proof. ∎

3. PROOF OF THE FIRST FORMULA

The following lemma is shown by a simple calculation.

LEMMA 3.1. *Let* $\beta_{ij}^{2,0}$ *be as in Definition* 2.1. *Then*

$$\beta_{ij}^{2,0} = a^{j-i-1}[i+1; ab][j-i; a^{-1}b]$$
$$= \frac{1-a^{i+1}b^{i+1}}{1-ab} \frac{a^{j-i}-b^{j-i}}{a-b}. \tag{3.1}$$

Let $B^{2,0}$ be a skew-symmetric matrix whose entries are $\beta_{ij}^{2,0}$ as in Definition 2.1. The proof of the first identity of the main theorem is due to an evaluation of the sub-Pfaffians of $B^{2,0}$.

PROPOSITION 3.2. *We have*

$$\text{pf}(B_{J(\lambda)}^{2,0}) = a^{c(\lambda)} \prod_{k=1}^{\infty} [\lambda_k - \lambda_{k+1} + 1; a^{\varepsilon(k)}b]. \tag{3.2}$$

Proof. We proceed by induction on even integers n. When $n=2$, it is easy to see that (3.2) derives directly from Lemma 3.1. So we assume $n \geqslant 4$. Using the formula (2.16) we expand $\text{pf}(B_{J(\lambda)})$ with respect to the first row and column. We have thus

$$\text{pf}(B_{J(\lambda)}) = \sum_{k=1}^{n-1} (-1)^{k-1} \text{pf}(B_{J(\lambda)-\{j_1, j_{n+1-k}\}}) B_{j_1, j_{n+1-k}}. \tag{3.4}$$

We put $m_i = \lambda_i - \lambda_{i+1} + 1$. We define $\mu_{k,l}$ to be the partition corresponding to the set

$$J(\lambda) - \{j_{n+1-l}, j_{n+1-k}\} = \{j_1, ..., \hat{j}_{n+1-l}, ..., \hat{j}_{n+1-k}, ..., j_n\}$$

for $k<l$. Then a straightforward computation shows $\mu_{k,l} = (\lambda_1+2, ..., \lambda_{k-1}+2, \lambda_{k+1}+1, ..., \lambda_{l-1}+1, \lambda_{l+1}, ..., \lambda_n)$. Also, from the fact that $c(\lambda) = \sum_{i=1}^{\infty} (\lambda_{2i-1} - \lambda_{2i})$ we note that

$$c(\mu_{k,l}) = \sum_{i=k}^{l-1} m_i \varepsilon(i) + c(\lambda) + 1. \tag{3.5}$$

Let $v_k = \mu_{1,k} = (\lambda_1 + 2, ..., \lambda_{k-1} + 2, \lambda_{k+1} + 1, ..., \lambda_{n-1} + 1)$. By our induction hypothesis for $l(v_k) < n$ we have

$$\operatorname{pf}(B_{J(\lambda) - \{j_1, j_{n+1-k}\}}) = \begin{cases} a^{c(\mu_1)} \prod_{i=2}^{n-1} [m_i; a^{\varepsilon(i)} b][m_{n-1} + m_n; ab] & \text{if} \quad k = 1, \\ a^{c(\mu_k)} \prod_{i=1}^{k-2} [m_i; a^{\varepsilon(i)} b][m_{k-1} + m_k; a^{\varepsilon(k-1)} b] & \\ \quad \times \prod_{i=k+1}^{n-2} [m_i; a^{\varepsilon(i-1)} b][m_{n-1} + m_n; ab] & \text{if} \quad 2 \leqslant k \leqslant n-2, \\ a^{c(\mu_{n-1})} \prod_{i=1}^{n-3} [m_i; a^{\varepsilon(i)} b][m_{n-2} + m_{n-1} + m_n; ab] & \text{if} \quad k = n-1. \end{cases} \tag{3.6}$$

By substituting (3.5) and (3.6) and $B_{j_1 j_{n+1-k}} = a^{\lambda_k - \lambda_n + n - k - 1}[m_n; ab]$ $[\sum_{i=k}^{n-1} m_i; a^{-1} b]$ into (3.4), we obtain

$$\begin{aligned} \operatorname{pf}(B_{J(\lambda)}) = {} & a^{c(\lambda) + \sum_{i=1}^{n-1} m_i(\varepsilon(i) + 1)} \prod_{i=2}^{n-2} [m_i; a^{\varepsilon(i-1)} b][m_{n-1} + m_n; ab] \\ & \times [m_n; ab] \left[\sum_{i=1}^{n-1} m_i; a^{-1} b \right] + \sum_{k=2}^{n-2} (-1)^{-1} a^{c(\lambda) + \sum_{i=k}^{n-1} m_i(\varepsilon(i) + 1)} \\ & \times \prod_{i=1}^{k-2} [m_i; a^{\varepsilon(i)} b] \prod_{i=k+1}^{n-2} [m_i; a^{\varepsilon(i-1)} b] \times [m_{k-1} + m_k; a^{\varepsilon(k-1)} b] \\ & \times [m_{n-1} + m_n; ab][m_n; ab] \left[\sum_{i=k}^{n-1} m_i; a^{-1} b \right] \\ & + a^{c(\lambda)} \prod_{i=1}^{n-3} [m_i; a^{\varepsilon(i)} b][m_{n-1}; a^{-1} b][m_n; ab] \\ & \times [m_{n-2} + m_{n-1} + m_n; ab]. \end{aligned} \tag{3.7}$$

We claim that (3.7) is equal to $a^{c(\lambda)} \prod_{k=1}^{n} [m_i; a^{\varepsilon(i)}b]$. Put

$$P_n = a^{\sum_{i=1}^{n-1} m_i(\varepsilon(i)+1)} \frac{\prod_{i=2}^{n-2} [m_i; a^{\varepsilon(i-1)}b][\sum_{i=1}^{n-1} m_i; a^{-1}b]}{\prod_{i=1}^{n-1} [m_i; a^{\varepsilon(i)}b][m_{n-1}+m_n; ab]}$$
$$+ \sum_{k=2}^{n-2} (-1)^{k-1} a^{\sum_{i=k}^{n-1} m_i(\varepsilon(i)+1)}$$
$$\times \frac{\left[\begin{array}{c}\prod_{i=k+1}^{n-2} [m_i; a^{\varepsilon(i-1)}b][m_{k-1}+m_k; a^{\varepsilon(k-1)}b] \\ \times [\sum_{i=k}^{n-1} m_i; a^{-1}b][m_{n-1}+m_n; ab]\end{array}\right]}{\prod_{i=k-1}^{n-1} [m_i; a^{\varepsilon(i)}b]}$$
$$+ \frac{[m_{n-2}+m_{n-1}+m_n; ab]}{[m_{n-2}; ab]}. \qquad (3$$

Then it is enough to show that $P_n = 1$ for all even integers n with $n \geqslant 4$. We prove this by induction on even integers n. When $n=4$, we can show by direct calculation that $P_4 = 1$. Suppose that this holds for an even integer $n \geqslant 4$. Then by the expression (3.8) we obtain

$$P_{n+2} = a^{\sum_{i=1}^{n-1} m_i(\varepsilon(i)+1)+2m_n} \frac{\prod_{i=2}^{n} [m_i; a^{\varepsilon(i-1)}b][\sum_{i=1}^{n+1} m_i; a^{-1}b]}{\prod_{i=1}^{n+1} [m_i; a^{\varepsilon(i)}b]}$$
$$\times [m_{n+1}+m_{n+2}; ab] + \sum_{k=2}^{n} (-1)^{k-1} a^{\sum_{i=k}^{n-1} m_i(\varepsilon(i)+1)+2m_n}$$
$$\times \frac{\prod_{i=k+1}^{n} [m_i; a^{\varepsilon(i-1)}b][m_{k-1}+m_k; a^{\varepsilon(k-1)}b][\sum_{i=k}^{n+1} m_i; a^{-1}b]}{\prod_{i=k-1}^{n+1} [m_i; a^{\varepsilon(i)}b]}$$
$$\times [m_{n+1}+m_{n+2}; ab] + \frac{[m_n+m_{n+1}+m_{n+2}; ab]}{[m_{n+2}; ab]},$$

Replace $[\sum_{i=k}^{n+1} m_i; a^{-1}b]$ in the first and second terms of this formula by the following expression

$$\left[\sum_{i=k}^{n-1} m_i; a^{-1}b\right] + a^{-\sum_{i=k}^{n-1} m_i} b^{\sum_{i=k}^{n-1} m_i} [m_n+m_{n+1}; a^{-1}b],$$

then we obtain

$$P_{n+2} = A + B + \frac{[m_n+m_{n+1}+m_{n+2}; ab]}{[m_{n+2}; ab]}. \qquad (3.9)$$

Here A and B are respectively given by

$$A = a^{\sum_{i=1}^{n-1} m_i(\varepsilon(i)+1)+2m_n}[m_{n+1}+m_{n+2}; ab] \times \frac{\prod_{i=2}^{n}[m_i; a^{\varepsilon(i-1)}b][\sum_{i=1}^{n-1} m_i; a^{-1}b]}{\prod_{i=1}^{n+1}[m_i; a^{\varepsilon(i)}b]}$$
$$+ \sum_{k=2}^{n} (-1)^{k-1} a^{\sum_{i=k}^{n-1} m_i(\varepsilon(i)+1)+2m_n}[m_{n+1}+m_{n+2}; ab] \times \frac{\prod_{i=k+1}^{n}[m_i; a^{\varepsilon(i-1)}b][m_{k-1}+m_k; a^{\varepsilon(k-1)}b][\sum_{i=k}^{n-1} m_i; a^{-1}b]}{\prod_{i=k-1}^{n+1}[m_i; a^{\varepsilon(i)}b]},$$

$$B = a^{\sum_{i=1}^{n-1} m_i\varepsilon(i)+2m_n} b^{\sum_{i=1}^{n-1} m_i}[m_n+m_{n+1}; a^{-1}b] \times [m_{n+1}+m_{n+2}; ab] \frac{\prod_{i=2}^{n}[m_i; a^{\varepsilon(i-1)}b]}{\prod_{i=1}^{n+1}[m_i; a^{\varepsilon(i)}b]}$$
$$+ \sum_{k=2}^{n} (-1)^{k-1} a^{\sum_{i=k}^{n-1} m_i\varepsilon(i)+2m_n} b^{\sum_{i=k}^{n-1} m_i}[m_n+m_{n+1}; a^{-1}b] \times [m_{n+1}+m_{n+2}; ab] \times \frac{\prod_{i=k+1}^{n}[m_i; a^{\varepsilon(i-1)}b][m_{k-1}+m_k; a^{\varepsilon(k-1)}b]}{\prod_{i=k-1}^{n+1}[m_i; a^{\varepsilon(i)}b]}.$$

Substituting

$$[m_{k-1}+m_k; a^{\varepsilon(k-1)}b] = [m_{k-1}; a^{\varepsilon(k-1)}b] + a^{\varepsilon(k-1)\,m_{k-1}} b^{m_{k-1}}[m_k; a^{\varepsilon(k-1)}b]$$

into B yields

$$B = a^{\sum_{i=1}^{n-1} m_i\varepsilon(i)+2m_n} b^{\sum_{i=1}^{n-1} m_i} \times [m_n+m_{n+1}; a^{-1}b][m_{n+1}+m_{n+2}; ab] \frac{\prod_{i=2}^{n}[m_i; a^{\varepsilon(i-1)}b]}{\prod_{i=1}^{n+1}[m_i; a^{\varepsilon(i)}b]}$$
$$+ \sum_{k=2}^{n} (-1)^{k-1} a^{\sum_{i=k}^{n-1} m_i\varepsilon(i)+2m_n} b^{\sum_{i=k}^{n-1} m_i}[m_n+m_{n+1}; a^{-1}b] \times [m_{n+1}+m_{n+2}; ab] \times \frac{\prod_{i=k+1}^{n}[m_i; a^{\varepsilon(i-1)}b]}{\prod_{i=k}^{n+1}[m_i; a^{\varepsilon(i)}b]}$$
$$+ \sum_{k=2}^{n} (-1)^{k-1} a^{\sum_{i=k-1}^{n-1} m_i\varepsilon(i)+2m_n} b^{\sum_{i=k-1}^{n-1} m_i}[m_n+m_{n+1}; a^{-1}b] \times [m_{n+1}+m_{n+2}; ab] \times \frac{\prod_{i=k}^{n}[m_i; a^{\varepsilon(i-1)}b]}{\prod_{i=k-1}^{n+1}[m_i; a^{\varepsilon(i)}b]}$$
$$= -a^{2m_n} \frac{[m_n+m_{n+1}; a^{-1}b][m_{n+1}+m_{n+2}; ab]}{[m_n; ab][m_{n+1}; ab]}. \tag{3.10}$$

In the meanwhile, our induction hypothesis $P_n=1$ applied to A yields

$$A=a^{2m_n}\frac{[m_{n-1};ab][m_n;a^{-1}b][m_{n+1}+m_{n+2};ab]}{[m_n;ab][m_{n+1};a^{-1}b][m_{n-1}+m_n;ab]}$$
$$\times\left(a^{\sum_{i=1}^{n-1}m_i(\varepsilon(i)+1)}\frac{\left[\begin{array}{l}\prod_{i=2}^{n-2}[m_i;a^{\varepsilon(i-1)}b][m_{n-1}+m_n;ab]\\ [\sum_{i=1}^{n-1}m_i;a^{-1}b]\end{array}\right]}{\prod_{i=1}^{n-1}[m_i;a^{\varepsilon(i)}b]}\right.$$
$$+\sum_{k=2}^{n-2}(-1)^{k-1}a^{\sum_{i=k}^{n-1}m_i(\varepsilon(i)+1)}[m_{k-1}+m_k;a^{\varepsilon(k-1)}b]$$
$$\left.\times[m_{n-1}+m_n;ab]\times\frac{\prod_{i=k+1}^{n-2}[m_i;a^{\varepsilon(i-1)}b][\sum_{i=k}^{n-1}m_i;a^{-1}b]}{\prod_{i=k-1}^{n-1}[m_i;a^{\varepsilon(i)}b]}\right)$$
$$+a^{2m_n}\frac{[m_n;a^{-1}b][m_{n-2}+m_{n-1};ab][m_{n+1}+m_{n+2};ab]}{[m_{n-2};ab][m_n;ab][m_{n+1};a^{-1}b]}$$
$$=a^{2m_n}\frac{[m_{n-1};ab][m_n;a^{-1}b][m_{n+1}+m_{n+2};ab]}{[m_n;ab][m_{n+1};a^{-1}b][m_{n-1}+m_n;ab]}$$
$$\times\left(1-\frac{[m_{n-2}+m_{n-1}+m_n;ab]}{[m_{n-2};ab]}\right)$$
$$+a^{2m_n}\frac{[m_n;a^{-1}b][m_{n-2}+m_{n-1};ab][m_{n+1}+m_{n+2};ab]}{[m_{n-2};ab][m_n;ab][m_{n+1};a^{-1}b]}.$$

Owing to $[xq][y;q]-[x+y+z;q][y;q]+[x+y;q][y+zq]=[x;q]$ $[y+z;q]$, it follows that

$$A=a^{2m_n}\frac{[m_n;a^{-1}b][m_{n+1}+m_{n+2};ab]}{[m_n;ab][m_{n+1};a^{-1}b]}. \tag{3.11}$$

Combining (3.10) and (3.11) with (3.9), we obtain

$$P_{n+2}=a^{2m_n}\frac{[m_n;a^{-1}b][m_{n+1}+m_{n+2};ab]}{[m_n;ab][m_{n+1};a^{-1}b]}$$
$$-a^{2m_n}\frac{[m_n+m_{n+1};a^{-1}b][m_{n+1}+m_{n+2};ab]}{[m_n;ab][m_{n+1};ab]}$$
$$+\frac{[m_n+m_{n+1}+m_{n+2};ab]}{[m_{n+2};ab]},$$

and this is indeed shown to be 1 by a simple calculation. This completes the proof. ∎

4. PROOF OF THE SECOND FORMULA

The following lemma is also easily verified.

LEMMA 4.1. *Let $\beta_{ij}^{1,1}$ be as in Definition* 2.1. *Then $\beta_{ij}^{1,1}$ determined by*

$$\beta_{ij}^{1,1}=\begin{cases}1 & \text{if } i=0,\quad j=1\\ a^{j-1}(1-a^{-1}b) & \text{if } i=0,\quad j\geqslant 1,\\ 1+ab+b^2 & \text{if } i\geqslant 1,\quad j\geqslant i+1,\\ a^{j-i-1}(1+a^{-1}b)(1+ab) & \text{if } i\geqslant 1,\quad j\geqslant i+2.\end{cases} \tag{4.1}$$

Let $B^{1,1}$ be a skew-symmetric matrix whose entries are $\beta_{ij}^{1,1}$. We obtain the following evaluation of the sub-Phaffians of $B^{1,1}$ which proves the second identity of the main theorem.

PROPOSITION 4.2. *We have*

$$\operatorname{pf}(B_{J(\lambda)})=a^{c(\lambda)}\prod_{k=1}^{\infty}\{\lambda'_k-\lambda'_{k+1}+1;\,a^{\varepsilon(\lambda'_k)},b\}. \tag{4.2}$$

LEMMA 4.3. *Let A be a skew-symmetric matrix of even degree of the form*

$$A=\left[\begin{array}{c|c} B & D\\ \hline -{}^tD & C\end{array}\right].$$

Suppose that the rank of the submatrix D is less than or equal to 1. *Then*

(1) *If B is a $2m\times 2m$ matrix, C is $2n\times 2n$, and D is $2m\times 2n$, then we have*

$$\operatorname{pf}(A)=\operatorname{pf}(B)\operatorname{pf}(C). \tag{4.3}$$

(2) *If B is a $(2m-1)\times(2m-1)$ matrix, C is $(2n+1)\times(2n+1)$, and D is $(2m-1)\times(2n+1)$, and we assume that D is of the form $D=\langle\boldsymbol{\alpha},\mathbf{d}\rangle={}^t\boldsymbol{\alpha}\mathbf{d}$ for some two vectors $\boldsymbol{\alpha}={}^t(\alpha_1,\alpha_2,\ldots,\alpha_{2n+1})$ and $\mathbf{d}={}^t(d_1,d_2,\ldots,d_{2n+1})$, then*

$$\operatorname{pf}(A)=\operatorname{pf}\left(\left[\begin{array}{c|c} B & \mathbf{d}\\ \hline -{}^t\mathbf{d} & 0\end{array}\right]\right)\sum_{j=1}^{2n+1}(-1)^{j-1}\alpha_j\operatorname{pf}(C_{j^c}). \tag{4.4}$$

Here C_{j^c} indicates the submatrix of degree $2\ell-2$ of C which is obtained by removing the jth row and jth column from C.

Proof. In both cases we write A in the form

$$A=\left[\begin{array}{c|c} B & D \\ \hline -{}^tD & O \end{array}\right]+\left[\begin{array}{c|c} O & O \\ \hline O & C \end{array}\right]$$

and apply Lemma 2.3. Then we have

$$\operatorname{pf}(A)=\sum_{k=0}^{m+n}\sum_{\substack{J\subset[2m+2n]\\ \sharp J=2k}}(-1)^{s(J)-k}\operatorname{pf}\left(\left[\begin{array}{c|c} B & D \\ \hline -{}^tD & O \end{array}\right]_J\right)\operatorname{pf}\left(\left[\begin{array}{c|c} O & O \\ \hline O & C \end{array}\right]_{J^c}\right).$$

First we consider the case (1). Put $I_1=[2m]$ and $I_2=\{2m+1, ..., 2m+2n\}$. If $J\not\supset I_1$, then there exists some $j\in I_1$ such that $j\in J^c$, and this implies that

$$\operatorname{pf}\left(\left[\begin{array}{c|c} O & O \\ \hline O & C \end{array}\right]_{J^c}\right)$$

vanishes. Thus we can assume $J\supset I_1$. Further, if $\sharp(J\cap I_2)\geqslant 2$, then

$$\operatorname{pf}\left(\left[\begin{array}{c|c} B & D \\ \hline -{}^tD & O \end{array}\right]_J\right)$$

vanishes since the rank of D is $\leqslant 1$. Consequently, only the term with index set $J=I_1$ remains non-zero in the above sum and this proves (4.3). Next we consider the case (2). Put $I_1=[2m-1]$ and $I_2=\{2m, ..., 2m+2n\}$. In the same manner as that used for we can show that if the product of Pfaffians above does not vanish then it is necessary to hold the condition $J\supset I_1$ and $\sharp(J\cup I_2)=1$. Thus we have

$$\begin{aligned}\operatorname{pf}(A)&=\sum_{\substack{J=[2m-1]\cup\{2m-1+j\}\\ 1\leqslant j\leqslant 2n+1}}(-1)^{s(I)-m}\operatorname{pf}\left(\left[\begin{array}{c|c} B & \alpha_j\mathbf{d} \\ \hline -\alpha_j{}^t\mathbf{d} & O \end{array}\right]\right)\operatorname{pf}(C_{j^c})\\ &=\operatorname{pf}\left(\left[\begin{array}{c|c} B & \mathbf{d} \\ \hline -{}^t\mathbf{d} & O \end{array}\right]\right)\sum_{j=1}^{2n+1}(-1)^{j-1}\alpha_j\operatorname{pf}(C_{j^c}).\end{aligned}$$

This proves (4.4). ∎

Proof of Proposition 4.2. We shall proceed by induction on even integers n. When $n=2$, it is easy to see that (4.2) holds from Lemma 4.1. Assume $n\geqslant 4$. Put

$$m_i=\lambda'_i-\lambda'_{i+1}+1.$$

We separate the proof into two cases, that is, the case $\lambda_{n-2} > \lambda_{n-1}$ and the case $\lambda_{n-2} = \lambda_{n-1}$, and use Lemma 4.3(1). We first consider the case $\lambda_{n-2} > \lambda_{n-1}$. In this case, since $j_2 + 1 < j_3$, the (i, k)-entry of $B^{1,1}_{J(\lambda)}$ is given by $a^{j_i - j_k - 1}(1 + a^{-1}b)(1 + ab)$ for $1 \leqslant i \leqslant 2$ and $3 \leqslant k \leqslant n$. Thus we can directly apply Lemma 4.3(1) to $B^{1,1}_{J(\lambda)}$ to obtain

$$\mathrm{pf}(B^{1,1}_{J(\lambda)}) = \beta^{1,1}_{j_1 j_2}\, \mathrm{pf}(B^{1,1}_{J(\lambda) - \{j_1, j_2\}}). \tag{4.5}$$

Let μ be the partition defined by $\mu = \lambda(J(\lambda) - \{j_1, j_2\}) = \mu_{n-1,n}$ and put $l = \lambda_{n-2}$. Then

$$\mu'_i = \begin{cases} n-2 & \text{for} \quad 1 \leqslant i \leqslant l+2, \\ \lambda'_{i-2} & \text{for} \quad i > l+2. \end{cases} \tag{4.6}$$

By Lemma 4.1,

$$\beta^{1,1}_{j_1 j_2} = \beta^{1,1}_{\lambda_n, \lambda_{n-1}+1} = \begin{cases} a^{\lambda_{n-1} - \lambda_n}(1 + a^{-1}b)(1 + ab) & (\lambda_{n-1} > \lambda_n), \\ 1 + ab + b^2 & (\lambda_{n-1} = \lambda_n), \end{cases} \tag{4.7}$$

and by (3.5) we have $c(\mu) = c(\lambda) - (\lambda_{n-1} - \lambda_n)$. On the one hand, if $\lambda_{n-1} > \lambda_n$ then, by our induction hypothesis, we have

$$\begin{aligned} \mathrm{pf}(B_{J(\lambda)}) &= a^{\lambda_{n-1} - \lambda_n}(1 + a^{-1}b)(1 + ab) \\ &\quad \times a^{c(\lambda) - (\lambda_{n-1} - \lambda_n)} \prod_{i=l}^{\infty} \{m_i; a^{\varepsilon(\lambda'_i)}, b\}. \end{aligned} \qquad (4$$

Since $\lambda_{n-2} > \lambda_{n-1} > \lambda_n$, we have

$$\lambda'_i = \begin{cases} n & \text{if} \quad 1 \leqslant i \leqslant \lambda_n, \\ n-1 & \text{if} \quad \lambda_n < i \leqslant \lambda_{n-1}, \\ n-2 & \text{if} \quad \lambda_{n-1} \leqslant i \leqslant l = \lambda_{n-2}. \end{cases}$$

This fact and (4.8) prove (4.2) in this special case. On the other hand, if $\lambda_{n-1} = \lambda_n$, then, by our induction hypothesis, we observe that

$$\mathrm{pf}(B_{J(\lambda)}) = (1 + ab + b^2) \times a^{c(\lambda)} \prod_{i=l}^{\infty} \{m_i; a^{\varepsilon(\lambda'_i)}, b\}. \tag{4.9}$$

Since $\lambda_{n-2} > \lambda_{n-1} = \lambda_n$, we have

$$\lambda'_i = \begin{cases} n & \text{if} \quad 1 \leqslant i \leqslant \lambda_n = \lambda_{n-1}, \\ n-2 & \text{if} \quad \lambda_{n-1} < i \leqslant l = \lambda_{n-2}. \end{cases}$$

Hence this fact and (4.9) also prove (4.2) in this case.

Next assume $\lambda_{n-2}=\lambda_{n-1}$, i.e., $j_3=j_2+1$. In this case, since $B_{j_2 j_3}=1+ab+b^2=(1+a^{-1}b)(1+ab)-a^{-1}b$, we use Lemma 2.3 and Lemma 4.3(1) to obtain

$$\mathrm{pf}(B^{1,1}_{J(\lambda)})=\beta^{1,1}_{j_1 j_2}\,\mathrm{pf}(B^{1,1}_{J(\lambda)-\{j_1,j_2\}})-a^{-1}b\,\mathrm{pf}(B^{1,1}_{J(\lambda)-\{j_2,j_3\}}). \tag{4.10}$$

Let μ and l be as before. We define the partition ν to be $\nu=\lambda(J(\lambda)-\{j_2,j_3\})=\mu_{n-2,n-1}$. By (3.5), $c(\nu)=c(\lambda)+2+\lambda_{n-2}-\lambda_{n-1}=c(\lambda)+2$. In view of (4.7) we have to separate our proof into two sub-cases. First we consider the sub-case $\lambda_{n-1}>\lambda_n$. Note that μ'_i is as in (4.6) and

$$\nu'_i=\begin{cases} n-2 & \text{for} \quad 1\leqslant i\leqslant \lambda_n,\\ n-3 & \text{for} \quad \lambda_n<i\leqslant \lambda_{n-3}+2,\\ \lambda'_{i-2} & \text{for} \quad i>\lambda_{n-3}+2. \end{cases} \tag{4.11}$$

From out induction hypothesis we have

$$\begin{aligned}\mathrm{pf}(B_{J(\lambda)})={}&a^{\lambda_{n-1}-\lambda_n}(1+a^{-1}b)(1+ab)\\ &\times a^{c(\lambda)-(\lambda_{n-1}-\lambda_n)}\{m_l-1;a,b\}\prod_{i=l+1}^{\infty}\{m_i;a^{\varepsilon(\lambda'_i)},b\}\\ &-a^{-1}b\times a^{c(\lambda)+2}(1+ab)\{m_l-2;a,b\}\prod_{i=l+1}^{\infty}\{m_i;a^{\varepsilon(\lambda'_i)},b\}\end{aligned} \tag{4.12}$$

By the recursion formula $\{m;a^{-1},b\}=(1+a^{-1}b)\{m-1;a,b\}-ab\{m-2;a^{-1},b\}$, Eq. (4.12) is equal to

$$\mathrm{pf}(B_{J(\lambda)})=a^{c(\lambda)}(1+ab)\{m_l;a,b\}\prod_{i=l+1}^{\infty}\{m_i;a^{\varepsilon(\lambda'_i)},b\}. \tag{4.13}$$

Since $\lambda_{n-2}=\lambda_{n-1}>\lambda_n$, we have

$$\lambda'_i=\begin{cases} n & \text{if} \quad 1\leqslant i\leqslant \lambda_n,\\ n-1 & \text{if} \quad \lambda_n<i\leqslant l=\lambda_{n-2}, \end{cases}$$

and from this we see that (4.2) holds in this case.

Finally, we consider the sub-case $\lambda_{n-1}=\lambda_n$. In this case

$$\nu'_i=\begin{cases} n-2 & \text{for} \quad 1\leqslant i\leqslant l,\\ n-3 & \text{for} \quad l<i\leqslant \lambda_{n-3}+2,\\ \lambda'_{i-2} & \text{for} \quad i>\lambda_{n-3}+2. \end{cases} \tag{4.14}$$

From our induction hypothesis we have

$$\begin{aligned}\mathrm{pf}(B_{J(\lambda)}) = (1+ab+b^2) \times a^{c(\lambda)}\{\lambda'_l - \lambda'_{l+1}; a, b\} \\ \times \{m_l - 2; a, b\} \prod_{i=l+1}^{\infty} \{m_i; a^{\varepsilon(\lambda'_i)}, b\} \\ - a^{-1}b \times a^{c(\lambda)+2}\{m_l - 3; a, b\} \prod_{i=l+1}^{\infty} \{m_i; a^{\varepsilon(\lambda'_i)}, b\}. \qquad (4.15)\end{aligned}$$

Use the recursion formula $\{m; a, b\} = (1+ab+b^2)\{m-2; a, b\} - ab(1+ab)\{m-3; a^{-1}, b\}$ to see that (4.15) is equal to

$$\mathrm{pf}(B_{J(\lambda)}) = a^{c(\lambda)}\{m_l; a, b\} \prod_{i=l+1}^{\infty} \{m_i; a^{\varepsilon(\lambda'_i)}, b\}. \qquad (4.16)$$

Since $\lambda_{n-2} = \lambda_{n-1} = \lambda_n$, we have $\lambda'_i = n$ for $1 \leqslant i \leqslant l$. This shows that (4.2) holds and hence the proof is complete. ∎

5. ONE CONJECTURE

Define the symmetric functions $P_r(a, b, c)$ and $Q_r(a, b, c)$ by

$$P_r(a, b, c) = \sum_{k=0}^{r} \frac{a^{k+1} - b^{k+1}}{a-b} \frac{1 - a^{r-k+1}b^{r-k+1}}{1-ab} c^k, \qquad (5.1)$$

$$Q_r(a, b, c) = \sum_{k=0}^{r} h_{r-k}(a, b, c)\, a^k b^k c^k, \qquad (5.2)$$

where h_r is a rth complete symmetric polynomial. For convention we define $P_r = Q_r = 0$ if $r < 0$. A composition is a sequence $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n, ...)$ of integers containing finitely many non-zero terms. We denote by $\mathbb{Z}^\infty$ the set of all compositions. For a partition $\lambda = (\lambda_1, \lambda_2, ...)$ we define the composition $\gamma = \Gamma(\lambda)$ by

$$\gamma_i = \lambda_i - \lambda_{i+1}. \qquad (5.3)$$

For each integer i we define $D_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ by

$$D_i(\gamma_1, ..., \gamma_i, ..., \gamma_n, ...) = (\gamma_1, ..., \gamma_i - 1, ..., \gamma_n, ...).$$

For a composition $\gamma = (\gamma_1, ..., \gamma_n, ...)$ we define the symmetric function $F_\gamma(a, b, c)$ by

$$F_\gamma(a, b, c) = h_{\gamma_1}(a, b, c) \prod_{k=1}^{\infty} P_{\gamma_{2k}}(a, b, c) \prod_{k=1}^{\infty} Q_{\gamma_{2k+1}}(a, b, c). \qquad (5.4)$$

For each composition γ, since $P_{\gamma_{2k}}(a, b, c)$, $Q_{\gamma_{2k+1}}(a, b, c)$ are identically equal to 1 for a sufficiently large number k, the product in (5.4) is well-defined. According to their definitions of h_r, P_r, and Q_r, if there is a negative γ_j for some j then $F_\gamma = 0$. Further, we define the operation of D_i to F_γ by

$$D_i F_\gamma = F_{D_i \gamma}.$$

We expect that the following conjecture would hold for the type (3, 0) case.

Conjecture.

$$\varphi_\lambda^{3,0}(a, b, c) = \prod_{k=1}^{\infty} (1 - abc\, D_k\, D_{k+1})\, F_{\Gamma(\lambda)}(a, b, c). \tag{5.5}$$

A s a consequence we have

$$\sum_\lambda \left\{ \prod_{k=1}^{\infty} (1 - abc\, D_k\, D_{k+1})\, F_{\Gamma(\lambda)}(a, b, c) \right\} s_\lambda(x_1, ..., x_n)$$
$$= \prod_{i=1}^{n} \frac{1}{(1 - ax_i)(1 - bx_i)(1 - cx_i)} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}. \tag{5.6}$$

To settle the (4, 0) case problem or more generally the $(r, 0)$ case problem is still a problem. Further, the reader can challenge the special (r, s) cases or general (r, s) case.

Remark. Recall the Schur–Weyl duality for the pair $(\mathrm{GL}_n, \mathrm{GL}_m))$

$$\sum_\lambda s_\lambda(x_1, ..., x_n)\, s_\lambda(y_1, ..., y_m) = \prod_{i, j} \frac{1}{1 - x_i y_j},$$

and the Littlewood–Richardson rule

$$s_\mu(x)\, s_\nu(x) = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda(x)$$

(see, e.g., [Ma]). Combining these identities with the Littlewood formula (1.0) for $a = 0$, we see that $\varphi_\lambda^{n,0}$ is expressed by

$$\varphi_\lambda^{n,0}(a_1, ..., a_n) = \sum_{\substack{\mu, \nu \\ \nu' : \text{ even partition}}} c_{\mu\nu}^{\lambda} s_\mu(a_1, ..., a_n), \tag{5.7}$$

while we cannot expect to obtain a desirable product expression by this formula.

6. CONCLUDING REMARKS

We shall give another explanation (proof) of our formulas briefly based on the Pieri formula (a special case of the Littlewood–Richardson rule for an irreducible decomposition of tensor products) and the Littlewood formula.

By the Pieri formula

$$s_\lambda(x)\, s_{(r)}(x) = \sum_{\substack{\mu \\ \mu-\lambda:\ \text{horizontal } r\text{-strip}}} s_\mu(x). \tag{6.1}$$

Here a horizontal r strip means a skew diagram θ which consists of at most one square in each column such that $|\theta| = r$ (see [Ma, I, Sect. 1]). Recall the Littlewood formula

$$\sum_\lambda a^{c(\lambda)} s_\lambda(x_1, x_2, ..., x_n) = \prod_{i=1}^n \frac{1}{1-ax_i} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-x_i x_j}.$$

Multiply both sides of the above formula by

$$\sum_{r=0}^\infty s_{(r)}(x_1, x_2, ..., x_n)\, b^r = \prod_{i=1}^n \frac{1}{1-bx_i}$$

to see that

$$\sum_\mu \sum_{\substack{\lambda,\, r \geqslant 0 \\ \mu-\lambda: \\ \text{horizontal } r\text{-strip}}} a^{c(\lambda)} b^r s_\mu(x_1, x_2, ..., x_n)$$

$$= \prod_{i=1}^n \frac{1}{(1-ax_i)(1-bx_i)} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-x_i x_j}.$$

Denote $\mu = (\mu_1, \mu_2, ...)$ and $\lambda = (\lambda_1, \lambda_2, ...)$. If $\mu - \lambda$ is a horizontal strip then

$$c(\lambda) = c(\mu) - k_1 + k_2 - k_3 + k_4 - \cdots, \tag{6.2}$$

where $k_i = \mu_i - \lambda_i$ and satisfies $\mu_i - \mu_{i+1} \geqslant k_i \geqslant 0$. Hence

$$\sum_{\substack{\lambda,\, r \geqslant 0 \\ \mu - \lambda: \\ \text{horizontal } r\text{-strip}}} a^{c(\lambda)} b^r = a^{c(\mu)} \sum_{k_1=0}^{\mu_1-\mu_2} (a^{-1}b)^{k_1} \sum_{k_2=0}^{\mu_2-\mu_3} (ab)^{k_2} \sum_{k_3=0}^{\mu_3-\mu_4} (a^{-1}b)^{k_3} \cdots$$

$$= a^{c(\mu)} \frac{1-(a^{-1}b)^{\mu_1-\mu_2+1}}{1-a^{-1}b} \frac{1-(ab)^{\mu_2-\mu_3+1}}{1-ab}$$

$$\times \frac{1-(a^{-1}b)^{\mu_3-\mu_4+1}}{1-a^{-1}b} \cdots$$

$$= a^{c(\mu)} \prod_{k=1}^{\infty} [\mu_k - \mu_{k+1} + 1;\, a^{\varepsilon(k)} b].$$

This proves the formula (1.1).

The formula (1.2) can be proved quite similarly. In fact, in place of (6.1) we employ the formula

$$s_\lambda(x)\, s_{(1^r)}(x) = \sum_{\substack{\mu \\ \mu - \lambda:\ \text{vertical } r\text{-strip}}} s_\mu(x). \tag{6.3}$$

Note that if $\mu - \lambda$ is a vertical strip then we have

$$c(\lambda) = c(\mu) + \sum_{i \geqslant 0} \frac{1-(-1)^{j_i}}{2}, \tag{6.4}$$

where $j_i = \mu'_i - \lambda'_i$ and satisfies $\mu'_i - \mu'_{i+1} \geqslant j_i \geqslant 0$. Hence an elementary manipulation shows

$$\sum_{\substack{\lambda,\, r \geqslant 0 \\ \mu - \lambda:\ \text{vertical } r\text{-strip}}} a^{c(\lambda)} b^r = a^{c(\mu)} \prod_{k=1}^{\infty} \{\mu'_k - \mu'_{k+1} + 1;\, a^{\varepsilon(\mu'_k)}, b\}.$$

This implies the formula (1.2).

We close this paper by listing some remarks.

Remarks. (1) Although it is seemingly less obvious from the definition of $\varphi_\lambda^{2,0}(a, b)$ in (1.3), it is clear that $\varphi_\lambda^{2,0}(a, b)$ is symmetric with respect to the variables a and b by the formula (1.1). In fact, since $c(\lambda)=\sum_{i=1}^{\infty}(\lambda_{2i-1}-\lambda_{2i})$, as we saw in the computation above, an elementary calculation shows that

$$\varphi_{\lambda}^{2,0}(a,b):=\operatorname{pf}\left(\left(\frac{1-(ab)^{i+1}}{1-ab}\frac{a^{j-i}-b^{j-i}}{a-b}\right)_{J(\lambda)}\right)$$
$$=\prod_{j:\,\text{even}}\frac{1-(ab)^{\lambda_j-\lambda_{j+1}+1}}{1-ab}\prod_{k:\,\text{odd}}\frac{a^{\lambda_k-\lambda_{k+1}+1}-b^{\lambda_k-\lambda_{k+1}+1}}{a-b},$$

(see Proposition 3.2). Moreover, by this expression, if we put $a=b^{-1}=e^{i\theta}$ and $t=\cos\theta$, then in particular, in contrast to Theorem 3.1 in [IW2], we see that

$$\sum_{\lambda}\prod_{k=1}^{\infty}(\lambda_{2k}-\lambda_{2k+1}+1)\,U_{\lambda_{2k-1}-\lambda_{2k}}(t)\,s_\lambda(x)$$
$$=\prod_{i=1}^{n}\frac{1}{1-2tx_i+x_i^2}\prod_{1\leqslant i<j\leqslant n}\frac{1}{1-x_ix_j},$$

where $U_n(t)=(\sin(n+1)\,\theta)/\sin\theta$ represents the nth Chebyshev polynomial of the second kind.

(2) As well as the Littlewood formulas and the identity cited above from [Ma], our formulas may describe an irreducible decomposition of certain $\mathrm{GL}_n(\mathbb{C})$-modules which is not multiplicity free under the action of $\mathrm{GL}_n(\mathbb{C})$. For example, if we take $a=b=1$ then the identity (1.1) (resp. (1.2)) asserts that a finite-dimensional polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to λ appears $N(\lambda)$-times (resp., $N(\lambda')$-times) as an irreducible component in the decomposition of $\mathscr{P}(\mathbb{C}^n)^{\otimes 2}\otimes\mathscr{P}(\mathrm{Skew}_{n\times n})$ (resp., $\Lambda^1(\mathbb{C})^{\otimes n}\otimes\mathscr{P}(\mathbb{C}^n)\otimes\mathscr{P}(\mathrm{Skew}_{n\times n})$) where $N(\lambda)=\prod_{i=1}^{\infty}(\lambda_i-\lambda_{i+1}+1)$. Here $\mathscr{P}(\mathrm{Skew}_{n\times n})$ represents the polynomial ring of the skew-symmetric matrices, and the action is obviously coming from the usual actions of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Skew}_{n\times n}$ by $X\to gX{}^tg$ for $g\in\mathrm{GL}_n(\mathbb{C})$, $X\in\mathrm{Skew}_{n\times n}$.

(3) If we put $a=\omega$ and $b=\omega^2$ in (1,1), where ω is a cubic root of unity, then we obtain the formula (5.10) in [LP].

(4) During the evaluation of our special choice of Pfaffians, we have encountered various formulas which resemble certain special values of hypergeometric series, e.g., on a finite field in appearance, but we have not yet clarified this matter.

REFERENCES

[IOW] M. Ishikawa, S. Okada, and M. Wakayama, Minor summation of Pfaffians and generalized Littlewood type formulas, *J. Algebra* **183** (1996), 193–216.

[IW1] M. Ishikawa and M. Wakayama, Minor summation of Pfaffians, *Linear and Multilinear Algebra* **39** (1995), 258–305.

[IW2] M. Ishikawa and M. Wakayama, Minor summation of Pfaffians and Schur functions identities, *Proc. Japan Acad. Ser. A* **71** (1995), 54–57.

[IW3] M. Ishikawa and M. Wakayama, New Schur function series, *J. Algebra* **208** (1998), 480–525.

[Ku] D. Knuth, Overlapping Pfaffians, *Electron J. Combin* **3** (1995), 151–163.

[LP] A. Lascoux and P. Pragacz, *S*-function series, *J. Phys. A: Math. Gen.* **21** (1988), 4105–4114.

[MA] D. E. Littlewood, "Symmetric Functions and Hall Polynomials, Second Edition," Oxford Univ. Press, Oxford, 1950.

[Ok] I. G. Macdonald, Applications of minor-summation formulas to rectangular-shaped representations of classical groups, preprint.

[Ste] J. Stembridge, Nonintersecting paths, Pfaffians and plane partitions, *Adv. Math.* **83** (1990), 96–131.

[Su] T. Sundquist, "Pfaffians, Involutions, and Schur Functions," University of Minnesota, Ph.D. thesis.

[Wy] H. Weyl, "The Classical Groups, Their Invariants and Representations," 2nd ed., Princeton Univ. Press, Princeton, NJ, 1946.

[YW] M. Yang and B. G. Wybourne, New *S* function series and non-compact Lie groups, *J. Phys. A: Math. Gen.* **19** (1986), 3513–3525.