Generalizations of Cauchy's Determinant and Schur's Pfaffian

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Abstract

We present several generalizations of Cauchy's determinant $\det(1/(x_i+y_j))$ and Schur's Pfaffian $\operatorname{Pf}((x_j-x_i)/(x_j+x_i))$ by considering matrices whose entries involve some generalized Vandermonde determinants. Special cases of our formulae include previous formulae due to S. Okada and T. Sundquist. As an application, we give a relation for the Littlewood–Richardson coefficients involving a rectangular partition.

Résumé

On présente plusieurs généralisations du déterminant de Cauchy $\det(1/(x_i+y_j))$ et du Pfaffian de Schur $\operatorname{Pf}((x_j-x_i)/(x_j+x_i))$ en considérant des matrices dont les coefficients impliquent des déterminants de Vandermonde généralisés. Des cas particuliers de nos formules contiennent celles obtenues précédemment par S. Okada et T. Sundquist. Comme une application, on donne une relation pour les coefficients de Littlewood–Richardson associés aux trois partitions dont une est de forme rectangle.

1 Introduction

Identities for determinants and Pfaffians are of great interest in many branches of mathematics. Some people need relations among minors or subPfaffians of a general matrix, others have to evaluate special determinants or Pfaffians. In combinatorics and representation theory, an important role is played by Cauchy's determinant identity [3]

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{n}(x_i+y_j)}, \tag{1.1}$$

and Schur's Pfaffian identity [20]

$$\operatorname{Pf}\left(\frac{x_j-x_i}{x_j+x_i}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n}\frac{x_j-x_i}{x_j+x_i}. \tag{1.2}$$

Also their variations and generalizations have many applications. See, for example, [5], [8], [12], [16], [17], [22], [23]. Also see [10] and [11] for a survey of determinant evaluations.

In this article, we establish several identities of Cauchy-type determinants and Schur-type Pfaffians involving generalized Vandermonde determinants. Let $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{a}=(a_1,\cdots,a_n)$ be two vectors of variables of length n. For nonnegative integers p and q with $p+q=n$, we define a generalized Vandermonde matrix $V^{p,q}(\boldsymbol{x};\boldsymbol{a})$ to be the $n\times n$ matrix with ith row

$$(1,x_i,\cdots,x_i^{p-1},a_i,a_ix_i,\cdots,a_ix_i^{q-1}).$$

We introduce another generalized Vandermonde matrix $W^n(\boldsymbol{x};\boldsymbol{a})$ as the $n\times n$ matrix with ith row

$$(1+a_ix_i^{n-1},x_i+a_ix_i^{n-2},\cdots,x_i^{n-1}+a_i).$$

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If $q = 0$, then $V^{n,0}(\boldsymbol{x};\boldsymbol{a}) = \left(x_i^{j-1}\right)_{1 \le i,j \le n}$ and the determinant $\det V^{n,0}(\boldsymbol{x};\boldsymbol{a}) = \prod_{1 \le i < j \le n}(x_j - x_i)$ is the usual Vandermonde determinant.

The main purpose of this paper is to prove the following identities for the determinants and Pfaffians involving these generalized Vandermonde determinants. In the later sections, we also give several variants of these determinants and Pfaffians.

Theorem 1.1. (a) Let n be a positive integer and let p and q be nonnegative integers. For six vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_n),\ \boldsymbol{y} = (y_1, \cdots, y_n),\ \boldsymbol{a} = (a_1, \cdots, a_n),\ \boldsymbol{b} = (b_1, \cdots, b_n),$$
$$\boldsymbol{z} = (z_1, \cdots, z_{p+q}),\ \boldsymbol{c} = (c_1, \cdots, c_{p+q}),$$

we have

$$\det \left(\frac{\det V^{p+1,q+1}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})}{y_j - x_i} \right)_{1 \le i,j \le n}$$
$$= \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n}(y_j - x_i)} \det V^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1} \det V^{n+p,n+q}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \quad (1.3)$$

(b) Let n be a positive integer and let p, q, r, s be nonnegative integers. For seven vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_{2n}),\ \boldsymbol{a} = (a_1, \cdots, a_{2n}),\ \boldsymbol{b} = (b_1, \cdots, b_{2n}),$$
$$\boldsymbol{z} = (z_1, \cdots, z_{p+q}),\ \boldsymbol{c} = (c_1, \cdots, c_{p+q}),$$
$$\boldsymbol{w} = (w_1, \cdots, w_{r+s}),\ \boldsymbol{d} = (d_1, \cdots, d_{r+s}),$$

we have

$$\operatorname{Pf} \left(\frac{\det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i} \right)_{1 \le i,j \le 2n}$$
$$= \frac{1}{\prod_{1 \le i < j \le 2n}(x_j - x_i)} \det V^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1} \det V^{r,s}(\boldsymbol{w};\boldsymbol{d})^{n-1}$$
$$\times \det V^{n+p,n+q}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c}) \det V^{n+r,n+s}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}). \quad (1.4)$$

(c) Let n be a positive integer and let p be a nonnegative integer. For six vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_n),\ \boldsymbol{y} = (y_1, \cdots, y_n),\ \boldsymbol{a} = (a_1, \cdots, a_n),\ \boldsymbol{b} = (b_1, \cdots, b_n),$$
$$\boldsymbol{z} = (z_1, \cdots, z_p),\ \boldsymbol{c} = (c_1, \cdots, c_p),$$

we have

$$\det \left(\frac{\det W^{p+2}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})}{(y_j - x_i)(1 - x_i y_j)} \right)_{1 \le i,j \le n}$$
$$= \frac{1}{\prod_{i,j=1}^{n}(y_j - x_i)(1 - x_i y_j)} \det W^{p}(\boldsymbol{z};\boldsymbol{c})^{n-1} \det W^{2n+p}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \quad (1.5)$$

(d) Let n be a positive integer and let p and q be nonnegative integers. For seven vectors of variables

$$\boldsymbol{x} = (x_1, \cdots, x_{2n}),\ \boldsymbol{a} = (a_1, \cdots, a_{2n}),\ \boldsymbol{b} = (b_1, \cdots, b_{2n}),$$
$$\boldsymbol{z} = (z_1, \cdots, z_p),\ \boldsymbol{c} = (c_1, \cdots, c_p),$$
$$\boldsymbol{w} = (w_1, \cdots, w_q),\ \boldsymbol{d} = (d_1, \cdots, d_q),$$

we have

$$\operatorname{Pf} \left(\frac{\det W^{p+2}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det W^{q+2}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{(x_j - x_i)(1 - x_i x_j)} \right)_{1 \le i,j \le 2n}$$
$$= \frac{1}{\prod_{1 \le i < j \le 2n}(x_j - x_i)(1 - x_i x_j)} \det W^{p}(\boldsymbol{z};\boldsymbol{c})^{n-1} \det W^{q}(\boldsymbol{w};\boldsymbol{d})^{n-1}$$
$$\times \det W^{2n+p}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c}) \det W^{2n+q}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}). \quad (1.6)$$

These identities were conjectured by S. Okada in [18]. If we put $p = q = 0$ in (1.3) or $p = q = r = s = 0$ in (1.4), then the identities read

$$\det \left(\frac{b_j - a_i}{y_j - x_i} \right)_{1 \le i,j \le n} = \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n} (y_j - x_i)} \det V^{n,n}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}), \tag{1.7}$$

$$\operatorname{Pf} \left(\frac{(a_j - a_i)(b_j - b_i)}{x_j - x_i} \right)_{1 \le i,j \le 2n} = \frac{1}{\prod_{1 \le i < j \le 2n} (x_j - x_i)} \det V^{n,n}(\boldsymbol{x}; \boldsymbol{a}) \det V^{n,n}(\boldsymbol{x}; \boldsymbol{b}). \tag{1.8}$$

These special cases, as well as the identities (1.5) with $p = 0$ and (1.6) with $p = q = 0$, are first given by S. Okada [16, Theorems 4.2, 4.7, 4.3, 4.4] in his study of rectangular-shaped representations of classical groups. Another special case of the identity (1.5) with $p = 1$ is given in [17] and applied to the enumeration of vertically and horizontally symmetric alternating sign matrices. These special cases are the starting point of our study.

Under the specialization

$$x_i \leftarrow x_i^2, \quad y_i \leftarrow y_i^2, \quad z_i \leftarrow z_i^2, \quad w_i \leftarrow w_i^2, \quad a_i \leftarrow x_i, \quad b_i \leftarrow y_i, \quad c_i \leftarrow z_i, \quad d_i \leftarrow w_i,$$

one can deduce from (1.3) and (1.4) the following identities:

$$\det \left(\frac{s_{\delta(k)}(x_i, y_j, \boldsymbol{z})}{x_i + y_j} \right)_{1 \le i,j \le n} = \frac{\prod_{1 \le i < j \le n} (x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n} (x_i + y_j)} s_{\delta(k)}(\boldsymbol{z})^{n-1} s_{\delta(k)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \tag{1.9}$$

$$\operatorname{Pf} \left(\frac{x_j - x_i}{x_j + x_i} s_{\delta(k)}(x_i, x_j \boldsymbol{z}) s_{\delta(l)}(x_i, x_j, \boldsymbol{w}) \right)_{1 \le i,j \le 2n}$$
$$= \prod_{1 \le i < j \le 2n} \frac{x_j - x_i}{x_j + x_i} s_{\delta(k)}(\boldsymbol{z})^{n-1} s_{\delta(l)}(\boldsymbol{w})^{n-1} s_{\delta(k)}(\boldsymbol{x}, \boldsymbol{z}) s_{\delta(l)}(\boldsymbol{x}, \boldsymbol{w}), \tag{1.10}$$

where s_λ denotes the Schur function corresponding to a partition λ and $\delta(k) = (k, k-1, \cdots, 1)$ denotes the staircase partition. If we take $k = 0$ in (1.9) and $k = l = 0$ in (1.10), we obtain Cauchy's determinant identity (1.1) and Schur's Pfaffian identity (1.2). Another special case of (1.9) with $k = l = 1$ is the rational case of Frobenius' identity [4]. Also, if we take $k = l = 1$ in (1.10), we obtain the rational case of an elliptic generalization of (1.2) given in [19].

2 A Sketch of the Proof of Our Main Theorem

In this section, we give an outline of the proof of Theorem 1.1. First S. Okada presented the identities in Theorem 1.1 at the workshop on "Aspects of Combinatorial Representation Theory" and "2nd East Asian Conference on Algebra and Combinatorics". At the point they were conjectures, and we tried a couple of methods to prove some of these identities, for example, the inductions, the Desnanot–Jacobi formula, and the complex analysis. Among such methods, the best and simplest way we found is to use the Desnanot–Jacobi formula (Lemma 2.1) and a homogeneous version $U^{p,q}$ of the generalized Vandermonde matrix $V^{p,q}$.

The proof of Theorem 1.1 consists of two parts. In the first part, we prove (1.4) by applying the Desnanot–Jacobi formula for Pfaffians to reduce the general case to the case $n = 2$, and then by using the induction on $p + q + r + s$ to show the case $n = 2$. In the second part, we translate (1.4) into the homogeneous version (2.5) and derive (1.3), (1.5), (1.6) from this 'master' identity. Only a sketch of the proof is given below, and the details can be found in our paper [6].

2.1 Proof of (1.4)

For the first part, we recall the Desnanot–Jacobi formulae for determinants and Pfaffians. Given a square matrix A and indices $i_1, \cdots, i_r, j_1, \cdots, j_r$, we denote by $A^{i_1, \cdots, i_r}_{j_1, \cdots, j_r}$ the matrix obtained by removing the rows $i_1, \cdots, i_r$ and the columns $j_1, \cdots, j_r$ of A.

Lemma 2.1. (1) If A is a square matrix, then we have

$$\det A^1_1 \cdot \det A^2_2 - \det A^1_2 \cdot \det A^2_1 = \det A \cdot \det A^{1,2}_{1,2}. \tag{2.1}$$

(2) If A is a skew-symmetric matrix, then we have

$$\operatorname{Pf} A_{1,2}^{1,2} \cdot \operatorname{Pf} A_{3,4}^{3,4} - \operatorname{Pf} A_{1,3}^{1,3} \cdot \operatorname{Pf} A_{2,4}^{2,4} + \operatorname{Pf} A_{1,4}^{1,4} \cdot \operatorname{Pf} A_{2,3}^{2,3} = \operatorname{Pf} A \cdot \operatorname{Pf} A_{1,2,3,4}^{1,2,3,4}. \tag{2.2}$$

This Pfaffian analogue of Desnanot–Jacobi formula is given in [9], [8], and is called the Plücker relation in [8].

By applying Desnanot–Jacobi formula for Pfaffians to the skew-symmetric matrix on the left hand side of (1.4) and using the induction on n, we can see that the proof of (1.4) is reduced to the case $n = 2$ with $\boldsymbol{z}$, $\boldsymbol{c}$, $\boldsymbol{w}$, $\boldsymbol{d}$ replaced by

$$\boldsymbol{z} \leftarrow (\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{z}), \quad \boldsymbol{c} \leftarrow (\boldsymbol{a}^{(1,2,3,4)}, \boldsymbol{c}), \quad \boldsymbol{w} \leftarrow (\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{w}), \quad \boldsymbol{d} \leftarrow (\boldsymbol{b}^{(1,2,3,4)}, \boldsymbol{d}),$$

respectively, where $\boldsymbol{x}^{(1,2,3,4)}$ denotes the vector obtained by removing x_1, x_2, x_3, x_4 from $\boldsymbol{x}$. Then the identity (1.4) in the case $n = 2$ can be proven by the induction on $p + q + r + s$ with the help of the following relations between $\det V^{p,q}$ and $\det V^{p-1,q}$ (or $\det V^{q,p}$).

Lemma 2.2. (1) If $p \geq q$ and $p \geq 1$, then we have

$$\det V^{p,q}(\boldsymbol{x}; \boldsymbol{a}) = \prod_{i=1}^{p+q-1} (x_{p+q} - x_i) \cdot \det V^{p-1,q}(x_1, \cdots, x_{p+q-1}; a'_1, \cdots, a'_{p+q-1}), \tag{2.3}$$

where we put

$$a'_i = \frac{a_i - a_{p+q}}{x_i - x_{p+q}} \quad (1 \leq i \leq p + q - 1).$$

(2) For nonnegative integers p and q, we have

$$\det V^{p,q}(\boldsymbol{x}; \boldsymbol{a}) = (-1)^{pq} \prod_{i=1}^{p+q} a_i \cdot \det V^{q,p}(\boldsymbol{x}; \boldsymbol{a}^{-1}), \tag{2.4}$$

where $\boldsymbol{a}^{-1} = (a_1^{-1}, \cdots, a_{p+q}^{-1})$.

Remark 2.3. We can also reduce the proof of the other identities (1.3), (1.5) and (1.6) in Theorem 1.1 to the case of $n = 2$ by using Desnanot-Jacobi formulae. It is easy to show the case of $n = 2$ of (1.3) by using the relations in Lemma 2.2 and the induction on $p + q$. We can prove (1.5) (resp. (1.6)) in the case of $n = 2$, by regarding the both sides as polynomials in z_{p+q} (resp. z_p) and showing that the values coincide at appropriate points by a brute force. Also the special cases of these identities can be obtained by regarding the both sides as meromorphic functions and computing the principal parts at their poles.

2.2 Homogeneous version and proof of (1.3), (1.5) and (1.6)

For the second part of the proof, we introduce a homogeneous version of the matrix $V^{p,q}(\boldsymbol{x}; \boldsymbol{a})$. For vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, $\boldsymbol{b}$ of length n and nonnegative integers p, q with $p+q = n$, we set $U^{p,q}\begin{pmatrix} \boldsymbol{x} & \boldsymbol{a} \\ \boldsymbol{y} & \boldsymbol{b} \end{pmatrix}$ to be the $n \times n$ matrix with ith row

$$(a_i x_i^{p-1}, a_i x_i^{p-2} y_i, \cdots, a_i y_i^{p-1}, b_i x_i^{q-1}, b_i x_i^{q-2} y_i, \cdots, b_i y_i^{q-1}).$$

Then we have the following relation among $\det U^{p,q}$, $\det V^{p,q}$ and $\det W^p$. Here use the following notation for vectors $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$:

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n), \quad \boldsymbol{x}\boldsymbol{y} = (x_1 y_1, \ldots, x_n y_n),$$

and, for integers k and l,

$$\boldsymbol{x}^k = (x_1^k, \ldots, x_n^k), \quad \boldsymbol{x}^k \boldsymbol{y}^l = (x_1^k y_1^l, \ldots, x_n^k y_n^l).$$

Lemma 2.4.

$$U^{p,q}\left(\begin{matrix}\boldsymbol{x}\\ \boldsymbol{y}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{a}\\ \boldsymbol{b}\end{matrix}\right)=\prod_{k=1}^{p+q}a_kx_k^{p-1}\cdot V^{p,q}\left(\boldsymbol{x}^{-1}\boldsymbol{y};\boldsymbol{a}^{-1}\boldsymbol{b}\boldsymbol{x}^{q-p}\right), \tag{2.5}$$

$$V^{p,q}(\boldsymbol{x};\boldsymbol{a})=U^{p,q}\left(\begin{matrix}\boldsymbol{1}\\ \boldsymbol{x}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1}\\ \boldsymbol{a}\end{matrix}\right), \tag{2.6}$$

$$\det U^{n,n}\left(\begin{matrix}\boldsymbol{x}\\ \boldsymbol{1}+\boldsymbol{x}^2\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1}+\boldsymbol{a}\boldsymbol{x}\\ \boldsymbol{x}+\boldsymbol{a}\end{matrix}\right)=(-1)^{n(n-1)/2}\det W^{2n}(\boldsymbol{x};\boldsymbol{a}), \tag{2.7}$$

$$\det U^{n,n+1}\left(\begin{matrix}\boldsymbol{x}\\ \boldsymbol{1}+\boldsymbol{x}^2\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1}+\boldsymbol{a}\boldsymbol{x}^2\\ \boldsymbol{1}+\boldsymbol{a}\end{matrix}\right)=(-1)^{n(n-1)/2}\det W^{2n+1}(\boldsymbol{x};\boldsymbol{a}), \tag{2.8}$$

where $\boldsymbol{1}=(1,\cdots,1)$.

We can "homogenize" the identity (1.4). It follows from (2.5) and (2.6) that the following theorem is equivalent to (1.4).

Theorem 2.5. Let n be a positive integer and let p, q, r and s be nonnegative integers. Suppose that the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ have length $2n$, the vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ have length $p+q$, and the vectors $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ have length $r+s$. Then we have

$$\begin{aligned}&\operatorname{Pf}\left(\frac{\det U^{p+1,q+1}\left(\begin{matrix}x_i,x_j,\boldsymbol{\xi}\\ y_i,y_j,\boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix}a_i,a_j,\boldsymbol{\alpha}\\ b_i,b_j,\boldsymbol{\beta}\end{matrix}\right)\det U^{r+1,s+1}\left(\begin{matrix}x_i,x_j,\boldsymbol{\zeta}\\ y_i,y_j,\boldsymbol{\omega}\end{matrix}\,\middle|\,\begin{matrix}c_i,c_j,\boldsymbol{\gamma}\\ d_i,d_j,\boldsymbol{\delta}\end{matrix}\right)}{\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}}\right)_{1\le i<j\le 2n}\\ &=\frac{1}{\prod_{1\le i<j\le 2n}\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}}\det U^{p,q}\left(\begin{matrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right)^{n-1}\det U^{r,s}\left(\begin{matrix}\boldsymbol{\zeta}\\ \boldsymbol{\omega}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{\gamma}\\ \boldsymbol{\delta}\end{matrix}\right)^{n-1}\\ &\qquad\times\det U^{n+p,n+q}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{\xi}\\ \boldsymbol{y},\boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{a},\boldsymbol{\alpha}\\ \boldsymbol{b},\boldsymbol{\beta}\end{matrix}\right)\det U^{n+r,n+s}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{\zeta}\\ \boldsymbol{y},\boldsymbol{\omega}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{c},\boldsymbol{\gamma}\\ \boldsymbol{d},\boldsymbol{\delta}\end{matrix}\right).\end{aligned} \tag{2.9}$$

The special case of $p=q=r=s=0$ of this identity (2.9) is given by M. Ishikawa [5, Theorem 3.1], and is one of the key ingredients of his proof of Stanley's conjecture on a certain weighted summation of Schur functions. (See [21].)

In this setting, a homogeneous version of (1.3) is a direct consequence of (2.9). A key is the following relation between determinant and Pfaffian. If A is any $m\times(2n-m)$ matrix, then we have

$$\operatorname{Pf}\begin{pmatrix}O & A\\ -{}^tA & O\end{pmatrix}=\begin{cases}(-1)^{n(n-1)/2}\det A & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases} \tag{2.10}$$

Corollary 2.6. Let n be a positive integer and let p and q be fixed nonnegative integers. For vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{w}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ of length n, and vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of length $p+q$, we have

$$\begin{aligned}&\det\left(\frac{\det U^{p+1,q+1}\left(\begin{matrix}x_i,z_j,\boldsymbol{\xi}\\ y_i,w_j,\boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix}a_i,c_j,\boldsymbol{\alpha}\\ b_i,d_j,\boldsymbol{\beta}\end{matrix}\right)}{\det\begin{pmatrix}x_i & z_j\\ y_i & w_j\end{pmatrix}}\right)_{1\le i,j\le n}\\ &=\frac{(-1)^{n(n-1)/2}}{\prod_{1\le i,j\le n}\det\begin{pmatrix}x_i & z_j\\ y_i & w_j\end{pmatrix}}\det U^{p,q}\left(\begin{matrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right)^{n-1}\det U^{n+p,n+q}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi}\\ \boldsymbol{y},\boldsymbol{w},\boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{a},\boldsymbol{c},\boldsymbol{\alpha}\\ \boldsymbol{b},\boldsymbol{d},\boldsymbol{\beta}\end{matrix}\right).\end{aligned} \tag{2.11}$$

Proof. In (2.9), we take $r=s=0$ and put

$$\begin{aligned}c_1&=\cdots=c_n=1, & c_{n+1}&=\cdots=c_{2n}=0,\\ d_1&=\cdots=d_n=0, & d_{n+1}&=\cdots=d_{2n}=1.\end{aligned} \tag{2.12}$$

Then we can apply (2.10) to obtain (2.11). □

Now the identity (1.3) follows from this corollary (2.11), (2.6) and an appropriate replacement of variables. Also the remaining identities (1.5) and (1.6) are immediate from (2.11) and (2.9) by using the relations (2.7) and (2.8). This completes the proof of Theorem 1.1.

3 A variation of the determinant and Pfaffian identities

In this section, we give a variation of the identities in Theorem 1.1, which can be regarded as a generalization of an identity of T. Sundquist [23].

Let n be a positive integer and let p and q be nonnegative integers with $p+q=n$. Let $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{a}=(a_1,\cdots,a_n)$ be vectors of variables. For partitions λ and μ with $l(\lambda)\le p$ and $l(\mu)\le q$, we define a matrix $V^{p,q}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a})$ to be the $n\times n$ matrix with ith row

$$(x_i^{\lambda_p},x_i^{\lambda_{p-1}+1},x_i^{\lambda_{p-2}+2},\cdots,x_i^{\lambda_1+p-1},a_ix_i^{\mu_q},a_ix_i^{\mu_{q-1}+1},a_ix_i^{\mu_{q-2}+2},\cdots,a_ix_i^{\mu_1+q-1}).$$

For example, if $\lambda=\mu=\emptyset$, then we have $V^{p,q}_{\emptyset,\emptyset}(\boldsymbol{x};\boldsymbol{a})=V^{p,q}(\boldsymbol{x};\boldsymbol{a})$. Let $\mathcal{P}_n$ denote the set of partitions λ with length $\le n$ which are of the form $\lambda=(\alpha_1,\cdots,\alpha_r|\alpha_1+1,\cdots,\alpha_r+1)$ in the Frobenius notation. We define

$$F^{p,q}(\boldsymbol{x};\boldsymbol{a})=\sum_{\lambda\in\mathcal{P}_p,\,\mu\in\mathcal{P}_q}(-1)^{(|\lambda|+|\mu|)/2}\det V^{p,q}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a}).$$

The main result of this section is the following theorem.

Theorem 3.1. (a) Let n be a positive integer and let p and q be nonnegative integers. For six vectors of variables

$$\boldsymbol{x}=(x_1,\cdots,x_n),\quad \boldsymbol{y}=(y_1,\cdots,y_n),\quad \boldsymbol{z}=(z_1,\cdots,z_{p+q}),$$
$$\boldsymbol{a}=(a_1,\cdots,a_n),\quad \boldsymbol{b}=(b_1,\cdots,b_n),\quad \boldsymbol{c}=(c_1,\cdots,c_{p+q}),$$

we have

$$\det\left(\frac{F^{p+1,q+1}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}{(y_j-x_i)(1-x_iy_j)}\right)_{1\le i,j\le n}
=\frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n}(y_j-x_i)(1-x_iy_j)}F^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1}F^{n+p,n+q}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \quad (3.1)$$

(b) Let n be a positive integer and let p, q, r, s be nonnegative integers. For seven vectors of variables

$$\boldsymbol{x}=(x_1,\cdots,x_{2n}),\quad \boldsymbol{a}=(a_1,\cdots,a_{2n}),\quad \boldsymbol{b}=(b_1,\cdots,b_{2n}),$$
$$\boldsymbol{z}=(z_1,\cdots,z_{p+q}),\quad \boldsymbol{c}=(c_1,\cdots,c_{p+q}),$$
$$\boldsymbol{w}=(w_1,\cdots,w_{r+s}),\quad \boldsymbol{d}=(d_1,\cdots,d_{r+s}),$$

we have

$$\operatorname{Pf}\left(\frac{F^{p+1,q+1}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})F^{r+1,s+1}(x_i,x_j,\boldsymbol{w};b_i,b_j,\boldsymbol{d})}{(x_j-x_i)(1-x_ix_j)}\right)_{1\le i,j\le 2n}
=\frac{1}{\prod_{1\le i<j\le 2n}(x_j-x_i)(1-x_ix_j)}F^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1}F^{r,s}(\boldsymbol{w};\boldsymbol{d})^{n-1}
\times F^{n+p,n+q}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c})F^{n+r,n+s}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}). \quad (3.2)$$

In particular, by putting $p=q=r=s=0$ and $b_i=x_i$ for $1\le i\le 2n$ in (3.2), we obtain Sundquist's identity [23, Theorem 2.1].

Corollary 3.2.

$$\operatorname{Pf}\left(\frac{a_j-a_i}{1-x_ix_j}\right)_{1\le i<j\le 2n}=\frac{(-1)^{n(n-1)/2}}{\prod_{1\le i<j\le 2n}(1-x_ix_j)}\sum_{\lambda,\mu\in\mathcal{P}_n}(-1)^{(|\lambda|+|\mu|)/2}\det V^{n,n}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a}). \quad (3.3)$$

The key ingredient to prove Theorem 3.1 and Corollary 3.2 is the following relation between $F^{p,q}(\boldsymbol{x};\boldsymbol{a})$ and $\det V^{p,q}(\boldsymbol{y};\boldsymbol{b})$.

Proposition 3.3. We have

$$\begin{aligned} F^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= (-1)^{\binom{p}{2}+\binom{q}{2}} \prod_{i=1}^{p+q} x_i^{p-1} \cdot \det V^{p,q}(\boldsymbol{x}+\boldsymbol{x}^{-1};\boldsymbol{a}\boldsymbol{x}^{q-p}), \\ &= (-1)^{\binom{p}{2}+\binom{q}{2}} \det U^{p,q} \left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{1} \\ \boldsymbol{1}+\boldsymbol{x}^2 & \boldsymbol{a} \end{array} \right). \end{aligned} \tag{3.4}$$

This proposition can be proven by the Cauchy-Binet formula and the computation of minors in the following lemma.

Lemma 3.4. Let D_r be the following $r \times (2r-1)$ matrix with columns indexed by $0, 1, \cdots, 2r-2$:

$$\begin{array}{cc} & \begin{matrix} 0 & & r-2 & r-1 & r & & 2r-2 \end{matrix} \\ D_r = & \begin{pmatrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ 1 & & & & & & 1 \end{pmatrix}. \end{array}$$

Then the minor of D_r corresponding to a partition λ is given by

$$\det \Delta_{I(\lambda)}^{[r]}(D_r) = \begin{cases} (-1)^{r(r-1)/2+|\lambda|/2} & \text{if } \lambda \in \mathcal{P}_r, \\ 0 & \text{otherwise}, \end{cases}$$

where $\Delta_{I(\lambda)}^{[r]}(D_r)$ is the $r \times r$ submatrix of D_r consisting of columns $\lambda_r, \lambda_{r-1}+1, \cdots, \lambda_1+(r-1)$.

Concluding this section, we should remark that, in Theorem 3.1, we can replace $F^{p,q}(\boldsymbol{x};\boldsymbol{a})$ by the following alternatives:

$$\begin{aligned} G^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= \sum_{\lambda \in \mathcal{Q}_p,\, \mu \in \mathcal{Q}_q} (-1)^{(|\lambda|+|\mu|)/2} \det V_{\lambda,\mu}^{p,q}(\boldsymbol{x};\boldsymbol{a}), \\ H^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= \sum_{\lambda \in \mathcal{R}_p,\, \mu \in \mathcal{R}_q} (-1)^{(|\lambda|+p(\lambda)+|\mu|+p(\mu))/2} \det V_{\lambda,\mu}^{p,q}(\boldsymbol{x};\boldsymbol{a}), \end{aligned}$$

where $\mathcal{Q}_n$ (resp. $\mathcal{R}_n$) is the set of partitions λ with length $\leq n$ which are of the form $\lambda = (\alpha+1|\alpha)$ (resp. $\lambda = (\alpha|\alpha)$) in the Frobenius notation. Then we obtain similar relations between $G^{p,q}(\boldsymbol{x};\boldsymbol{a})$, $H^{p,q}(\boldsymbol{x};\boldsymbol{a})$ and $\det V^{p,q}(\boldsymbol{y};\boldsymbol{b})$.

4 Another generalization of Cauchy's determinant identity

In this section, we give another type of generalized Cauchy's determinant identities involving $\det V^{p,q}$ and $\det W^p$. The following is the main result of this section:

Theorem 4.1.

$$\begin{aligned} &\det \left(\frac{1}{\det V^{p+1,q+1}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})} \right)_{1 \leq i,j \leq n} \\ &= \frac{(-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} \det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det V^{p+1,q+1}(y_i, y_j, \boldsymbol{z}; b_i, b_j, \boldsymbol{c})}{\prod_{i,j=1}^{n} \det V^{p+1,q+1}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\det \left(\frac{1}{\det W^{p+2}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})} \right)_{1 \leq i,j \leq n} \\ &= \frac{(-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} \det W^{p+2}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det W^{p+2}(y_i, y_j, \boldsymbol{z}; b_i, b_j, \boldsymbol{c})}{\prod_{i,j=1}^{n} \det W^{p+2}(x_i, y_j, \boldsymbol{z}; a_i, b_j, \boldsymbol{c})}. \end{aligned} \tag{4.2}$$

If $p = q = 0$, then the identity (4.1) becomes

$$\det\left(\frac{1}{b_j - a_i}\right)_{1\le i,j\le n} = \frac{(-1)^{n(n-1)/2}\prod_{1\le i<j\le n}(a_j-a_i)(b_j-b_i)}{\prod_{i,j=1}^{n}(b_j-a_i)},$$

which is equivalent to Cauchy's determinant identity (1.1).

Proof. It follows from Desnanot–Jacobi formula (2.1) that it suffices to show the identities in the case $n = 2$. If we put

$$f(x,y;a,b) = \det V^{p+1,q+1}(x,y,\boldsymbol{z};a,b,\boldsymbol{c}), \quad \text{or} \quad \det W^{p+2}(x,y,\boldsymbol{z};a,b,\boldsymbol{c}),$$

then the case $n = 2$ is equivalent to the following quadratic relation:

$$f(x_1,x_2;a_1,a_2)f(y_1,y_2;b_1,b_2) - f(x_1,y_1;a_1,b_1)f(x_2,y_2;a_2,b_2) + f(x_1,y_2;a_1,b_2)f(x_2,y_1;a_2,b_1) = 0. \quad (4.3)$$

This relation can be obtained by the Plücker relation for determinants. □

5 A hyperpfaffian expression

The purpose of this section is to obtain a hyperpfaffian expression of $\det V^{p,q}(\boldsymbol{x};\boldsymbol{a})$ when $p = q$ is even. First we recall the definition of hyperpfaffians ([1], see also [14]). Let n and r be positive integers. Define a subset $\mathcal{E}_{rn,n}$ of the symmetric groups $\mathcal{S}_{rn}$ by

$$\mathcal{E}_{rn,n} = \{\sigma\in\mathcal{S}_{rn} : \sigma(n(i-1)+1) < \sigma(n(i-1)+2) < \cdots < \sigma(ni) \text{ for } 1\le i\le n\}.$$

For example, if $n = r = 2$, then $\mathcal{E}_{4,2}$ is composed of the following 6 elements:

$$\mathcal{E}_{4,2} = \{(1,2,3,4),(1,3,2,4),(1,4,2,3),(3,4,1,2),(2,4,1,3),(2,3,1,4)\}.$$

Let $a = (a_{i_1\dots i_n})_{1\le i_1<\cdots<i_n\le nr}$ be an alternating tensor, i.e. $a_{i_{\sigma(1)}\cdots i_{\sigma(n)}} = \operatorname{sgn}(\sigma)a_{i_1\dots i_n}$ for any permutations $\sigma\in\mathcal{S}_{nr}$. The hyperpfaffian of a is, by definition,

$$\operatorname{Pf}^{[n]}(a) = \frac{1}{r!}\sum_{\sigma\in\mathcal{E}_{nr,n}}\operatorname{sgn}(\sigma)\prod_{i=1}^{r}a_{\sigma(n(i-1)+1),\dots,\sigma(ni)}.$$

An alternating 2-tensor a is a skew-symmetric matrix and the hyperpfaffian $\operatorname{Pf}^{[2]}(a)$ is the usual Pfaffian of the skew-symmetric matrix.

The main result of this section is the following Theorem. (Similar expressions are obtained when n is odd.)

Theorem 5.1. If n is even, then

$$\det V^{n,n}(\boldsymbol{x};\boldsymbol{a}) = \operatorname{Pf}^{[n]}\left[\left(1+\prod_{s=1}^{n}a_{i_s}\right)\prod_{1\le s<t\le n}(x_{i_t}-x_{i_s})\right]_{1\le i_1<\cdots<i_n\le 2n}, \quad (5.1)$$

$$\det U^{n,n}\left(\begin{array}{c|c}\boldsymbol{x} & \boldsymbol{a}\\ \boldsymbol{y} & \boldsymbol{b}\end{array}\right) = \operatorname{Pf}^{[n]}\left[\left(\prod_{s=1}^{n}a_{i_s}+\prod_{s=1}^{n}b_{i_s}\right)\prod_{1\le s<t\le n}\det\begin{pmatrix}y_{i_s} & x_{i_s}\\ y_{i_t} & x_{i_t}\end{pmatrix}\right]_{1\le i_1<\cdots<i_n\le 2n}. \quad (5.2)$$

The essential part of the proof of the theorem is to compute the following special Pfaffian and hyperpfaffian.

Lemma 5.2. Let n and r be positive integers and assume $n = 2m$ is even. Then we have

$$\operatorname{Pf}\left(\frac{(x_j^m - x_i^m)^2}{x_j - x_i}\right)_{1\le i,j\le nr} = \begin{cases}\prod_{1\le i<j\le n}(x_j-x_i) & \text{if } r=1,\\ 0 & \text{if } r\ge 2,\end{cases} \quad (5.3)$$

$$\operatorname{Pf}^{[n]}\left[\prod_{1\le s<t\le n}(x_{i_t}-x_{i_s})\right]_{1\le i_1<\cdots<i_n\le nr} = \begin{cases}\prod_{1\le i<j\le n}(x_j-x_i) & \text{if } r=1,\\ 0 & \text{if } r\ge 2.\end{cases} \quad (5.4)$$

Proof. The first identity (5.3) is obtained from (1.4) by putting $p = q = r = s = 0$ and $a_i = b_i = x_i^m$. The second identity (5.4) follows from the first one (5.3) and the composition of hyperpfaffians given in [14, Eq. (82)]. $\square$

6 Application to Littlewood–Richardson coefficients

In this section, we use the Pfaffian identity (1.5) in Theorem 1.1 and the minor-summation formula [7] to derive a relation between Littlewood–Richardson coefficients.

For three partitions λ, μ and ν, we denote by $\mathrm{LR}^{\lambda}_{\mu,\nu}$ the Littlewood-Richardson coefficient. These numbers $\mathrm{LR}^{\lambda}_{\mu,\nu}$ appear in the following expansions (see [15]) :

$$s_\mu(X)s_\nu(X) = \sum_{\lambda} \mathrm{LR}^{\lambda}_{\mu,\nu}\, s_\lambda(X),$$

$$s_{\lambda/\mu}(X) = \sum_{\nu} \mathrm{LR}^{\lambda}_{\mu,\nu}\, s_\nu(X),$$

$$s_\lambda(X,Y) = \sum_{\mu,\nu} \mathrm{LR}^{\lambda}_{\mu,\nu}\, s_\mu(X)s_\nu(Y).$$

We are concerned with the Littlewood–Richardson coefficients involving rectangular partitions. Let $\square(a,b)$ denote the partition whose Young diagram is the rectangle $a \times b$, i.e.

$$\square(a,b) = (b^a) = (\underbrace{b,\ldots,b}_{a}).$$

For a partition $\lambda \subset \square(a,b)$, we define $\lambda^\dagger = \lambda^\dagger(a,b)$ by

$$\lambda^\dagger_i = b - \lambda_{a+1-i} \quad (1 \le i \le a).$$

Okada [16] used the special case of the identities (1.3) and (1.4) (i.e., the case of $p = q = 0$ and $p = q = r = s = 0$) to prove the following proposition. (This proposition is also derived by the combinatorial algorithm called Littlewood–Richardson rule.)

Proposition 6.1. Let n be a positive integer and let e and f be nonnegative integers.

(1) For partitions μ, ν, we have

$$\mathrm{LR}^{\square(n,e)}_{\mu,\nu} = \begin{cases} 1 & \text{if } \nu = \mu^\dagger(n,e), \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

(2) For a partition λ of length $\le 2n$, we have

$$\mathrm{LR}^{\lambda}_{\square(n,e),\square(n,f)} = \begin{cases} 1 & \text{if } \lambda_{n+1} \le \min(e,f) \text{ and } \lambda_i + \lambda_{2n+1-i} = e+f \ (1 \le i \le n), \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

The main result of this section is the following theorem, which generalizes (6.2). It would be interesting to find a bijective proof of the equality (6.5).

Theorem 6.2. Let n be a positive integer and let e and f be nonnegative integers. Let λ and μ be partitions such that $\lambda \subset \square(2n, e+f)$ and $\mu \subset \square(n,e)$. Then we have

(1) $\mathrm{LR}^{\lambda}_{\mu,\square(n,f)} = 0$ unless

$$\lambda_n \ge f \quad \text{and} \quad \lambda_{n+1} \le \min(e,f). \tag{6.3}$$

(2) If λ satisfies the above condition (6.3) and we define two partitions α and β by

$$\alpha_i = \lambda_i - f, \quad \beta_i = e - \lambda_{2n+1-i}, \quad (1 \le i \le n), \tag{6.4}$$

then we have

$$\mathrm{LR}^{\lambda}_{\mu,\square(n,f)} = \mathrm{LR}^{\beta}_{\alpha,\mu^\dagger(n,e)}. \tag{6.5}$$

In particular, $\mathrm{LR}^{\lambda}_{\mu,\square(n,f)} = 0$ unless $\alpha \subset \beta$.

In particular, if $\mu = \square(n,e)$ is a rectangle, then this theorem reduces to (6.2). If μ is a near-rectangle, then we have the following corollary by using Pieri's rule [15, (5.16), (5.17)].

Corollary 6.3. Suppose that a partitions $\lambda \subset \square(2n, e+f)$ satisfies the condition (6.3) in Theorem 6.2. Define two partitions α and β by (6.4). Then we have

$$\mathrm{LR}^{\lambda}_{(e^{n-1},e-k),(f^n)} = \begin{cases} 1 & \text{if } \beta/\alpha \text{ is a horizontal strip of length } k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{LR}^{\lambda}_{(e^{n-k},(e-1)^k),(f^n)} = \begin{cases} 1 & \text{if } \beta/\alpha \text{ is a vertical strip of length } k, \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove Theorem 6.2, we substitute

$$a_i = x_i^{e+p+n}, \quad b_i = x_i^{f+r+n}, \quad c_i = z_i^{e+p+n}, \quad d_i = w_i^{f+r+n} \tag{6.6}$$

in the Pfaffian identity (1.4). By the bi-determinant definition of Schur functions, we have

$$\det V^{p,q}(\boldsymbol{x}; \boldsymbol{x}^k) = \begin{cases} s_{\square(q,k-p)}(\boldsymbol{x})\Delta(\boldsymbol{x}) & \text{if } k \geq p, \\ 0 & \text{if } k < p, \end{cases}$$

where $\Delta(\boldsymbol{x}) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$. Hence, under the substitution (6.6), the identity (1.4) gives us the following Pfaffian identity.

Proposition 6.4. We have

$$\frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf}\left((x_j - x_i) s_{\square(q+1,e+n-1)}(x_i, x_j, \boldsymbol{z}) s_{\square(s+1,f+n-1)}(x_i, x_j, \boldsymbol{w})\right)_{1 \leq i,j \leq 2n}$$
$$= s_{\square(q,e+n)}(\boldsymbol{z})^{n-1} s_{\square(s,f+n)}(\boldsymbol{w})^{n-1} s_{\square(n+q,e)}(\boldsymbol{x}, \boldsymbol{z}) s_{\square(n+s,f)}(\boldsymbol{x}, \boldsymbol{w}). \tag{6.7}$$

Remark 6.5. If we substitute

$$a_i = x_i^{e+p+n}, \quad b_i = y_i^{e+p+n} \quad (1 \leq i \leq n)$$

in the determinant identity (1.3), then we have

$$\frac{1}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} \det\left(s_{\square(q+1,e+n-1)}(x_i, y_j, \boldsymbol{z})\right)_{1 \leq i,j \leq n}$$
$$= (-1)^{n(n-1)/2} s_{\square(q,e+n)}(\boldsymbol{z})^{n-1} s_{\square(q+n,e)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}). \tag{6.8}$$

The special case ($q = e+n-1$) of this identity is given in [13, Proposition 8.4.3], and the proof there works in the general case of (6.8).

If we take $q = s = 0$ in (6.7), we have

$$\frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf}\left((x_j - x_i) h_{e+n-1}(x_i, x_j, \boldsymbol{z}) h_{f+n-1}(x_i, x_j, \boldsymbol{w})\right)_{1 \leq i,j \leq 2n}$$
$$= s_{\square(n,e)}(\boldsymbol{x}, \boldsymbol{z}) s_{\square(n,f)}(\boldsymbol{x}, \boldsymbol{w}), \tag{6.9}$$

where h_r denotes the rth complete symmetric function. We use the minor-summation formula [7] to expand the left hand side in the Schur function bases $\{s_\lambda(\boldsymbol{x})\}$.

Lemma 6.6. Let $b_{k,l}$ be the coefficient of $x^k y^l$ in

$$(y - x) h_{e+n-1}(x, y, \boldsymbol{z}) h_{f+n-1}(x, y, \boldsymbol{w}).$$

Then we have $b_{kl} = -b_{lk}$, and b_{kl}, $k < l$, is given by

$$b_{kl} = \sum_{i,j} h_i(\boldsymbol{z}) h_j(\boldsymbol{w}),$$

where the sum is taken over all pairs of integers (i,j) satisfying

$$i + j = (e+n-1) + (f+n-1) + 1 - k - l, \quad 0 \leq i \leq (e+n-1) - k, \quad 0 \leq j \leq (f+n-1) - k.$$

Here we recall the minor summation formula [7].

Lemma 6.7. Let X be a $2n \times N$ matrix and A be an $N \times N$ skew-symmetric matrix. Then we have

$$\sum_I \operatorname{Pf} \Delta_I^I(A) \det \Delta_I^{[2n]}(X) = \operatorname{Pf}(XA^tX),$$

where I runs over all $2n$-element subsets of $[N]$, and $\Delta_J^I(M)$ denotes the submatrix of a matrix M obtained by picking up the rows indexed by I and the columns indexed by J.

By applying this minor-summation formula, we obtain

Proposition 6.8. Let $B = (b_{ij})_{0 \le i,j \le e+f+2n-1}$ be the skew-symmetric matrix, whose entries b_{ij} are given in Lemma 6.6. Then, for a partition $\lambda \subset \square(2n, e+f)$, we have

$$\sum_{\substack{\mu \subset \square(n,e) \\ \nu \subset \square(n,f)}} \mathrm{LR}_{\mu,\nu}^{\lambda} \, s_{\mu^\dagger(n,e)}(\boldsymbol{z}) s_{\nu^\dagger(n,f)}(\boldsymbol{w}) = \operatorname{Pf} \Delta_{I(\lambda)}^{I(\lambda)}(B), \tag{6.10}$$

where $I(\lambda) = \{\lambda_{2n}, \lambda_{2n-1}+1, \dots, \lambda_2 + 2n-2, \lambda_1 + 2n-1\}$.

Now we can finish the proof of Theorem 6.2.

Proof of Theorem 6.2. In the above argument, we take $p \ge n$ and $r = 0$. In this case, the variables $\boldsymbol{w}$ disappear and we see that

$$b_{kl} = \begin{cases} h_{(e+n-1)+(f+n-1)+1-k-l}(\boldsymbol{z}) & \text{if } 0 \le k \le \min(e+n-1, f+n-1) \text{ and } l \ge f+n-1, \\ 0 & \text{otherwise} \end{cases}$$

and the equation (6.10) becomes

$$\sum_{\mu \subset \square(n,e)} \mathrm{LR}_{\mu,\square(n,f)}^{\lambda} \, s_{\mu^\dagger(n,e)}(\boldsymbol{z}) = \operatorname{Pf} \Delta_{I(\lambda)}^{I(\lambda)}(B).$$

The skew-symmetric matrix B has the form $B = \begin{pmatrix} O & C \\ -{}^tC & O \end{pmatrix}$ with

$$C = \left(h_{e+n-1-i-j}(\boldsymbol{z})\right)_{0 \le i \le f+n-1, 0 \le j \le e+n-1}.$$

¿From the relation (2.10), we see that the subPfaffian Pf $B_{I(\lambda)}$ vanishes unless

$$\lambda_{n+1} \le \min(e, f), \quad \lambda_n \ge f.$$

If these conditions are satisfied, then we have

$$\begin{aligned} \operatorname{Pf} B_{I(\lambda)} &= (-1)^{n(n-1)/2} \det \left(h_{\beta_i - \alpha_{n+1-j} - i + (n+1-j)}(\boldsymbol{z})\right)_{1 \le i,j \le n} \\ &= s_{\beta/\alpha}(\boldsymbol{z}). \end{aligned}$$

Hence we have

$$\sum_{\mu \subset \square(n,e)} \mathrm{LR}_{\mu,\square(n,f)}^{\lambda} \, s_{\mu^\dagger(n,e)}(\boldsymbol{z}) = s_{\beta/\alpha}(\boldsymbol{z}).$$

Comparing the coefficients of $s_{\mu^\dagger(n,e)}(\boldsymbol{z})$ completes the proof. □

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