A characterization of the simply-laced FC-finite Coxeter groups

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Abstract

We call an element of a Coxeter group fully covering (or a fully covering element) if its length is equal to the number of the elements it covers in the Bruhat ordering. It is easy to see that the notion of fully covering is a generalization of the notion of a 321-avoiding permutation and that a fully covering element is a fully commutative element. Also, we call a Coxeter group bi-full if its fully commutative elements coincide with its fully covering elements. We show that the bi-full Coxeter groups are the ones of type A_n, D_n, E_n with no restriction on n. In other words, Coxeter groups of type $E_9, E_{10}, \ldots$ are also bi-full. According to a result of Fan, a Coxeter group is a simply-laced FC-finite Coxeter group if and only if it is a bi-full Coxeter group.

1 Introduction

There are occasions where certain mathematical objects are associated with Coxeter diagrams (or closely related Dynkin diagrams). Quite often, the objects associated

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with the diagrams of types A, D, E_6, E_7 and E_8 (the diagrams of irreducible simply-laced, finite-type Coxeter systems) form a special class characterized by certain nice properties (sometimes among the ones associated with the irreducible simply-laced diagrams, and sometimes among all irreducible ones). Usually the diagrams E_n with $n \geq 9$ do not join this class. However, in some cases, the diagrams E_n with no restriction on n, along with the diagrams A_n and D_n, form a nice class. As an example, we recall the notion of FC-finite Coxeter groups. A Coxeter group is called *FC-finite* if the number of its fully commutative elements is finite. Here, an element of a Coxeter group is said to be *fully commutative* if any of its reduced expression can be converted into any other by exchanging adjacent commuting generators several times. C. K. Fan gave a result that the irreducible simply-laced FC-finite Coxeter groups are the ones of type A, D, and E ([3, Proposition 2.]). These are also exactly the irreducible simply-laced Coxeter groups with finitely many *minuscule* elements ([7]).

In this paper, we call an element of a Coxeter group *fully covering* if its length is equal to the number of elements it covers in the Bruhat ordering. This notion has appeared in [4, Theorem 1]. Our main goal is to characterize the Coxeter groups whose fully covering elements coincide with its fully commutative elements. We call such a Coxeter group *bi-full*. Fan's result implies that Coxeter groups of type A, D, E_6, E_7, and E_8 are bi-full [4, Theorem 1] and a Coxeter groups of type $\tilde{A}_2$ is not bi-full [4, Conclusion]. However a bi-full Coxeter group was not characterized. Our main result is that the irreducible bi-full Coxeter groups are the ones of type A, D, E. According to a result of Fan, it implies that a Coxeter group is simply-laced and FC-finite if and only if it is bi-full (Theorem 2.14).

An element σ of a symmetric group is called a *321-avoiding permutation* if there is no triple $1 \leq i < j < k \leq n$ such that $\sigma(i) > \sigma(j) > \sigma(k)$. It is easy to see that the notion of being fully covering is a generalization of the notion of a 321-avoiding permutation (see [1]) from the viewpoint of the Bruhat ordering. Also, it is a well known fact that a permutation is 321-avoiding if and only if it is fully commutative [1]. Actually, this fact is a motivation for our present work. There is another interesting generalization of the notion of a 321-avoiding permutation. In [5], Green extended the notion to affine permutation groups (namely the Coxeter groups of type $\tilde{A}_n$) from the viewpoint of a permutation. Our generalization and his generalization are not equivalent. Indeed, in an affine permutation group W, the 321-avoiding permutations in Green's sense are exactly the fully commutative elements. It is known that these are also exactly the minuscule elements in W [6, Theorem 5.1].

Our result can be applied to the theory of Kazhdan-Lusztig polynomials. Let W be a Coxeter group and let x, w be elements of W. Let $p_1(x, w)$ be the coefficient of degree 1 of the Kazhdan-Lusztig polynomial for x, w. M. Dyer showed that $p_1(e, w) = c^-(w) - |\text{supp}(w)|$ and that $p_1(e, w) \geq 0$ (see [2]), where $c^-(w)$ is the number of elements covered by w in the Bruhat ordering. Thus if W is one of type A, D, E and w is a fully commutative element of W then we can rewrite it as $p_1(e, w) = \ell(w) - |\text{supp}(w)|$ by our result.

This paper is organized as follows: In §2, we recall and provide some basic terminology. In §3, we collect some important properties of a fully commutative element. In §4, we show that Coxeter groups of type A, D, and E are bi-full. In §5, we show that a Coxeter group which is neither of type A, D nor E cannot be bi-full.

2 Preliminaries and Notations

In this paper, we assume that (W, S) is a *Coxeter system*.

Notation 2.1 We denote the set of integers by $\mathbb{Z}$ and denote the set of positive integers by $\mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{>0}$, we put $[n] := \{1, 2, \ldots, n\}$. For a set A, we denote its cardinality by $|A|$ or $\sharp A$.

Notation 2.2 Let w be an element of W and let e be the identity element of W. A *length function* ℓ is a mapping from W to $\mathbb{Z}$ defined by $\ell(e)$ equals 0 and $\ell(w)$ equals the smallest m such that there exist elements $s_1, s_2, \ldots, s_m$ of S satisfying $w = s_1 s_2 \ldots s_m$ for $w \neq e$. We call $\ell(w)$ the *length* of w. Let $x_1, x_2, \ldots, x_m$ be elements of W. If we have $w = x_1 x_2 \ldots x_m$ and $\ell(x_1 x_2 \ldots x_m) = \ell(x_1) + \ell(x_2) + \ldots + \ell(x_m)$, then we call $x_1 x_2 \ldots x_m$ an *extended reduced expression* of w. Note that we do not assume that $x_1, x_2, \ldots, x_m$ are elements of S. In particular, we call $x_1 x_2 \ldots x_m$ a *reduced expression* of w if all x_i are elements of S.

Definition 2.3 For $s, t \in S$, we denote the order of st by $m(s, t)$.

(i) If we have $\{m(s,t) | s, t \in S\} \subseteq \{1, 2, 3\}$, then we call (W, S) (resp. W) a *simply-laced* Coxeter system (resp. a simply-laced Coxeter group).

(ii) If a Coxeter diagram of (W, S) is connected then we call (W, S) (resp. W) an *irreducible* Coxeter system (resp. an irreducible Coxeter group).

Definition 2.4 Let (W, S) be a Coxeter system with its relation defined by Figure 1 (resp. Figure 2).

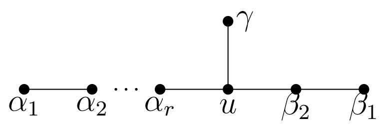

Figure 1: Coxeter diagram of type E_{r+4}

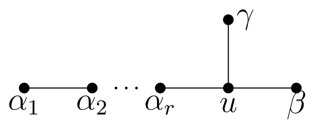

Figure 2: Coxeter diagram of type D_{r+3}

Then we call (W, S) a Coxeter system of type E_{r+4} (resp. type D_{r+3}).

Definition 2.5 Let w be an element of W. We say that w is a *fully commutative element* (or w is *fully commutative*) if any reduced expression of w can be converted into any other reduced expression of w by exchanging adjacent commuting generators several times.

Definition 2.6 For a Coxeter system (W, S), we put

$$W^{FC} := \{w \in W | w \text{ is fully commutative}\}.$$

If the cardinality of W^{FC} is finite then we call (W, S) (resp. W) a *FC-finite* Coxeter system (resp. FC-finite Coxeter group).

From now on, we denote a Coxeter group of type X by $W(X)$.

Theorem 2.7 (C. K. Fan) *The irreducible simply-laced FC-finite Coxeter groups are $W(A_n), W(D_{n+3})$, and $W(E_{n+5})$ for $n \geq 1$ (see [3] for more detailed information).*

We recall the definition of the Bruhat ordering.

Definition 2.8 Put $T := \{wsw^{-1} | s \in S,\ w \in W\}$. For $y, z \in W$, we define its relation and denote it by $y <' z$ if there exists an element t of T such that $\ell(tz) < \ell(z)$ and $y = tz$. Then the *Bruhat ordering* denoted by $\leq$ is defined as follows: For $x, w \in W$, $x \leq w$ if and only if there exist elements $x_0, x_1, \ldots, x_r$ of W such that $x = x_0 <' x_1 <' \cdots <' x_r = w$. For $x, w \in W$, we say that w covers x (or x is covered by w) if $x < w$ and $\ell(x) = \ell(w) - 1$. We denote it by $x \lessdot w$.

The following is well known as the *subword property.* For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced expression of w. For $x \in W$, $x \leq w$ if and only if there exists a sequence of natural numbers $i_1, i_2, \ldots, i_r$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $x = s_{i_1} s_{i_2} \cdots s_{i_r}$. This expression of x is not reduced in general, in other words it may happen that $\ell(x) < r$. However it is known that one can find a sequence of natural numbers $j_1, j_2, \ldots, j_k$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq m$, $x = s_{j_1} s_{j_2} \cdots s_{j_k}$ and $\ell(x) = k$.

In this paper, we assume that an ordering handled with on a Coxeter group is the Bruhat ordering.

Notation 2.9 For $w \in W$, we put

$$\begin{aligned} \operatorname{supp}(w) : &= \{s \in S | s \leq w\}, \\ C^-(w) : &= \{x \in W | x \lessdot w\}, \\ c^-(w) : &= |C^-(w)|. \end{aligned}$$

Definition 2.10 For $w \in W$, we call w *fully covering* (or a *fully covering element*) if $\ell(w) = c^-(w)$.

By the definitions of fully commutative and fully covering, we immediately have the following.

Proposition 2.11 *A fully covering element w of W is fully commutative.*

Proof. Assume that w is not fully commutative. It implies that there exists a reduced expression $s_1 s_2 \dots s_m$ of w and exists an integer $1 \leq i \leq m-2$ such that $s_i = s_{i+2}$. Then $s_1 s_2 \dots s_i \widehat{s_{i+1}} s_{i+2} \dots s_m$ cannot be covered by w, where $x\widehat{y}z$ denotes xz. Thus w is not fully covering. This is a contradiction. ∎

Definition 2.12 Let (W,S) be a Coxeter system. We call (W,S) (resp. W) a *bi-full* Coxeter system or bi-full (resp. a bi-full Coxeter group or bi-full) if it satisfies the following. For any $w \in W$, w is fully commutative if and only if w is fully covering.

Remark 2.13 Let $(W_1,S_1),(W_2,S_2)$ be bi-full Coxeter systems (resp. FC-finite Coxeter systems). If we have $S_1 \cap S_2 = \emptyset$ and $s_1 s_2 = s_2 s_1$ for any $(s_1,s_2) \in S_1 \times S_2$ then $(W_1 W_2, S_1 \cup S_2)$ is also a bi-full Coxeter system (resp. an FC-finite Coxeter system).

Our goal of this paper is to prove the following.

Theorem 2.14 *W is a simply-laced FC-finite Coxeter group if and only if W is a bi-full Coxeter group.*

By Theorem 2.7 and Remark 2.13, we can easily reduce Theorem 2.14 to the following.

Theorem 2.15 *An irreducible bi-full Coxeter group is either of type A, D or E.*

By Proposition 2.11, if the following two claims hold then we can obtain Theorem 2.15.

Claim 1. Any fully commutative element of a Coxeter group of type E is fully covering (Theorem 4.1).

Claim 2. If W is neither of type A, D nor E then there is an element such that it is fully commutative and is not fully covering (Theorem 5.1).

We often use the following fact in this paper (cf [8]).

Fact 2.16 Let J be a subset of S. Put

$$\begin{aligned}
W_J : &= \langle \{s | s \in J\} \rangle, \\
W^J : &= \{x \in W | \ell(xy) = \ell(x) + \ell(y) \text{ for all } y \in W_J\} \\
(&= \{x \in W | \ell(xs) = \ell(x) + 1 \text{ for all } s \in J\}) \text{ and} \\
{}^J W : &= \{x \in W | \ell(yx) = \ell(y) + \ell(x) \text{ for all } y \in W_J\} \\
(&= \{x \in W | \ell(sx) = \ell(x) + 1 \text{ for all } s \in J\}).
\end{aligned}$$

(i) For $w \in W$, there is a unique pair of $(x,y) \in W^J \times W_J$ such that $w = xy$.

(ii) For $w \in W$, there is a unique pair of $(y,z) \in W_J \times {}^J W$ such that $w = yz$.

3 Properties of a fully commutative element

In this section, we collect some basic and important properties of a fully commutative element from a point of view to associate with a fully covering element.

By the definition of fully commutative, we have the following.

Lemma 3.1

(i) *Let w be an element of W. Let $s_1s_2\dots s_m$ and $s'_1s'_2\dots s'_m$ be reduced expressions of w. If w is fully commutative then we have*

$$\{s_1, s_2, \dots, s_m\} = \{s'_1, s'_2, \dots, s'_m\} \text{ as multisets.}$$

(ii) *If $m(s,t)$ is odd or 2 for any $s, t \in S$ then we have the following for any $w \in W$. w is fully commutative if and only if $\{s_1, s_2, \dots, s_r\} = \{s'_1, s'_2, \dots, s'_m\}$ as multisets for any reduced expressions $s_1s_2\dots s_m$, $s'_1s'_2\dots s'_m$ of w.*

(iii) *An element is fully commutative if it has a unique reduced expression.*

(iv) *Let xyz be an extended reduced expression of w. If w is fully commutative then y is also fully commutative.*

(v) *Let W be a simply-laced Coxeter group and let w be an element of W. Then w is not fully commutative if and only if there is a reduced expression $s_1s_2\dots s_m$ of w such that $s_i = s_{i+2}$ for some $1 \le i \le m-2$.*

We omit the proof of the lemma since it is straightforward.

Proposition 3.2 *Let w be a fully commutative element and let $s_1s_2\dots s_r$ be a reduced expression of w ($r \ge 2$). If $w = ss_1s_2\dots s_{r-1}$ for some $s \in S$ then we have the followings.*

(i) $s = s_r$.

(ii) $ss_j = s_js$ *for any* $j \in [r-1]$.

(iii) $s \not\le s_1s_2\dots s_{r-1}$.

We shall state the following lemma before we prove Proposition 3.2.

Lemma 3.3 *Let w be an element of W and let $J = \{a, b\}$ be a subset of S such that $a \ne b$, $wa < w$, $wb < w$, $m(a,b) = m$. Then we have the followings.*

(i) *There exists an element y of W^J such that $w = y(ab)^{\frac{m}{2}} = y(ba)^{\frac{m}{2}}$ and $\ell(w) = \ell(y) + m$ if m is even.*

(ii) *There exists an element y of W^J such that $w = y(ab)^{\frac{m-1}{2}}a = y(ba)^{\frac{m-1}{2}}b$ and $\ell(w) = \ell(y) + m$ if m is odd.*

(iii) *If w is fully commutative then $m = 2$.*

Proof. **(i)** and **(ii)** By Fact 2.16, there exists a pair $(w^J, w_J) \in W^J \times W_J$ such that $wa = w^Jw_J$. It implies that we have $w = w^Jw_Ja$ and $\ell(w) = \ell(w^J) + \ell(w_J) + 1$. By $wb < w$ and $a \ne b$, one of the following properties holds.

(1) There exists $x \in W$ such that x is covered by w^J and that $xw_J ab$ is an extended reduced expression of w.

(2) There exists $z \in W$ such that z is covered by w_J and that $w^J zab$ is an extended reduced expression of w.

Assume (1) holds. By the subword property, we have $w^J \leq w = xw_J ab$. By $x < w^J$, $w_J ab \in W_J$ and the subword property, we have $w^J a \lessdot w^J$ or $w^J b \lessdot w^J$. This is a contradiction. Accordingly (2) holds. Remember that we have $w_J \in W_J$ and $w_J \lessdot w_J a$. It implies $w_J = (ab)^k$ for some $k \geq 1$ or $w_J = b(ab)^h$ for some $h \geq 0$. On the other hand, we have $z \lessdot w_J$ and $z \lessdot za$. It implies that

$$z = \begin{cases} b(ab)^{k-1}, & \text{if } w_J = (ab)^k, \\ (ab)^h, & \text{if } w_J = b(ab)^h. \end{cases}$$

Since $w_J a = zab$, we obtain $w_J a = (ab)^k a = b(ab)^{k-1} ab$ or $w_J a = b(ab)^h a = (ab)^h ab$. Thus $w_J a = (ab)^k a = (ba)^k b$ or $w_J a = (ba)^{h+1} = (ab)^{h+1}$. Hence (i) and (ii) hold.
(iii) By (i),(ii), and the definition of fully commutative, $m \geq 3$ implies that w is not fully commutative. This is a contradiction. Hence (iii) holds. ∎

Proof of Proposition 3.2. By Lemma 3.1(i), we obtain (i). We shall prove (ii) by induction on r.
Case $r = 2$. Now we have $s_1 s_2 = s s_1$. By (i), we obtain $s_1 s_2 = s_2 s_1$. Therefore (ii) holds.
Case $r \geq 3$. Now we have $w = s_1 s_2 \dots s_r = s s_1 s_2 \dots s_{r-1}$. Hence we obtain $ws_r < w$ and $ws_{r-1} < w$. By Lemma 3.3(iii), we have

$$s_{r-1} s_r = s_r s_{r-1}. \tag{1}$$

Thus we have $s_1 s_2 \dots s_{r-2} s_r = s s_1 s_2 \dots s_{r-2}$. Since $s_1 s_2 \dots s_{r-2} s_r s_{r-1}$ is also a reduced expression of w and w is fully commutative, $s_1 s_2 \dots s_{r-2} s_r$ is also fully commutative. By the inductive assumption, we have

$$s s_j = s_j s \quad \text{for any } j \in [r-2]. \tag{2}$$

By (i), (1), and (2), we obtain

$$s s_j = s_j s \text{ for any } j \in [r-1].$$

We can easily show that (iii) holds by (i) and (ii). ∎

The following corollary is useful to find an element which is fully commutative and is not fully covering.

Corollary 3.4 *Let w be an element of W and let $s_1, s_2, \dots, s_m$ be elements of S such that $w = s_1 s_2 \dots s_m$. Note that we do not assume that $s_1 s_2 \dots s_m$ is a reduced expression of w. We define a condition* (FC) *as follows:*

(FC) *If there exists a pair of integers i and j such that $i < j$ and $s_i = s_j$ then there exists a pair of integers a and b such that $i < a < b < j$, $s_a s_i \neq s_i s_a$ and $s_b s_i \neq s_i s_b$.*

Then we have the followings.

(i) If $s_1s_2\dots s_m$ satisfies the condition (FC) *then $s_1s_2\dots s_m$ is a reduced expression of w and w is fully commutative.*

(ii) If W is a simply-laced Coxeter group, $s_1s_2\dots s_m$ is a reduced expression of w and w is fully commutative, then $s_1s_2\dots s_m$ satisfies the condition (FC).

Proof. **(i)** We shall prove the corollary by induction on m.
Case $m \leq 2$**.** It is obvious.
Case $m \geq 3$**.** Assume that $s_1s_2\dots s_m$ is not a reduced expression. By the deletion condition, there exists a pair of integers u and v such that $u < v$ and $w = s_1s_2\dots \widehat{s_u}\dots\widehat{s_v}\dots s_m$. Thus we have

$$s_us_{u+1}\dots s_{v-1} = s_{u+1}\dots s_{v-1}s_v. \tag{3}$$

Note that the condition (FC) holds on $s_us_{u+1}\dots s_{v-1}$. By the inductive assumption, $s_us_{u+1}\dots s_{v-1}$ is a reduced expression and is fully commutative. By (3) and Proposition 3.2, we have $s_u = s_v$, $s_us_k = s_ks_u$ for any $k \in \{u+1, u+2, \dots, v-1\}$. This is a contradiction. Accordingly $s_1s_2\dots s_m$ is a reduced expression of w. If w is not fully commutative then there is a reduced expression $s'_1s'_2\dots s'_m$ of w converted into $s_1s_2\dots s_m$ by exchanging adjacent commuting generators several times such that $s'_i = s'_{i+2}$ for some $i \in [m-2]$. Consequently the condition (FC) does not hold. This is a contradiction. Therefore w is fully commutative.
(ii) Assume that there is a pair of integers i and j such that $i < j$, $s_i = s_j$ and

$$c := \sharp\{k \in \{i+1, i+2, \dots, j-1\} | s_ks_i \neq s_is_k\} \leq 1.$$

Case $c = 0$**.** Then we have $w = s_1\dots\widehat{s_i}\dots\widehat{s_j}\dots s_m$. It implies that $s_1s_2\dots s_m$ cannot be a reduced expression. This is a contradiction.
Case $c = 1$**.** Let k be an integer such that $s_ks_i \neq s_is_k$ and $i+1 \leq k \leq j-1$. By virtue of the case, such k is unique. Then we have

$$w = s_1\dots\widehat{s_i}\dots s_is_ks_j\dots\widehat{s_j}\dots s_m.$$

Since W is a simply-laced Coxeter group, we have $s_is_ks_j = s_ks_is_k$. This is a contradiction. ∎

By Corollary 3.4, we have the following.

Corollary 3.5 *Let W be a simply-laced Coxeter group and let w be an element of W such that $\ell(w^2) = 2\ell(w)$ and w^2 is fully commutative. Then for any $k \in \mathbb{Z}_{>0}$ we have $\ell(w^k) = k\ell(w)$ and w^k is fully commutative. In particular, W is not an FC-finite Coxeter group.*

Proof. Let $s_1s_2\dots s_m$ be a reduced expression of w. Then, $s_1s_2\dots s_ms_1s_2\dots s_m$ is a reduced expression of w^2. By Corollary 3.4(ii) and virtue of the corollary, $s_1s_2\dots s_ms_1s_2\dots s_m$ satisfies the condition (FC). We can easily see that

$$(s_1s_2\dots s_m)(s_1s_2\dots s_m)\cdots(s_1s_2\dots s_m)$$

also satisfies the condition (FC). By Corollary 3.4(i), we have $\ell(w^k) = k\ell(w)$ and w^k is fully commutative. ∎

The following lemma holds on any Coxeter system.

Lemma 3.6 *Let (W, S) be a Coxeter system and let x be an element of W. Let s_1, s_2 be elements of S such that $s_1 s_2 x$ is an extended reduced expression and that $s_2 s_1 s_2$ is a reduced expression. If we have $s_1 \notin$ supp (x) then $s_2 s_1 s_2 x$ is an extended reduced expression.*

Proof. Since $s_1 s_2 x$ is an extended reduced expression, we have $x < s_2 x$. On the other hand, we have $x < s_1 x$ by $s_1 \notin$ supp (x). Thus, we obtain $x \in {}^{\{s_1,s_2\}}W$. Remember that $s_2 s_1 s_2$ is a reduced expression. Hence $s_2 s_1 s_2 x$ is an extended reduced expression. ∎

The following lemma holds on any simply-laced Coxeter system.

Lemma 3.7 *Let (W, S) be a simply-laced Coxeter system and let w be a fully commutative element of W. If $s_1 s_2 \ldots s_m$ is a reduced expression of w then $s_1 \widehat{s_2} s_3 \ldots s_m$ is a reduced expression.*

Proof. Assume that $s_1 \widehat{s_2} s_3 \ldots s_m$ is not a reduced expression. Then there exists an integer j such that $3 \leq j \leq m$ and $s_3 s_4 \ldots s_m = s_1 s_3 \ldots \widehat{s_j} \ldots s_m$. Thus we have $w = s_1 s_2 s_1 \ldots \widehat{s_j} \ldots s_m$. By our assumption, we can see that we have $s_1 s_2 s_1 = s_2 s_1 s_2$. It implies that w is not fully commutative. This is a contradiction. Hence $s_1 \widehat{s_2} s_3 \ldots s_m$ is a reduced expression. ∎

4 $W(E_n)$ is bi-full

Our aim of this section is to prove the following.

Theorem 4.1 *Let W be a Coxeter group of type E and let w be an element of W. If w is fully commutative then w is fully covering.*

The following proposition is well-known. In fact we can easily prove it by the notion of a 321-avoiding permutation. However we prove it without terms of a 321-avoiding permutation.

Proposition 4.2 *Let W be a Weyl group of type A_n. Then a fully commutative element w of W is fully covering.*

Before we prove the proposition above, we show one lemma.

Notation 4.3 Let $s_1 s_2 \ldots s_m$ be a reduced expression of an element of W and let α be an element of S. Put

$$g_\alpha(s_1 s_2 \ldots s_m) := \sharp\{i \in [m] \mid s_i = \alpha\}.$$

By Lemma 3.1(i), if w is fully commutative, then we can define

$$g_\alpha(w) := g_\alpha(s_1 s_2 \ldots s_m),$$

where $s_1 s_2 \ldots s_m$ is a reduced expression of w.

Lemma 4.4 *Let w be an element of W and let $s_1s_2\dots s_m$ be a reduced expression of w. Let $\{\alpha_1,\alpha_2,\dots,\alpha_r\}$ be a subset of supp(w) satisfying the following conditions (1),(2), and (3).*

(1) $\alpha_i s = s\alpha_i$ for any $i \in [r]$ and for any $s \in$ supp(w) $- \{\alpha_1,\alpha_2,\dots,\alpha_r\}$.

(2) $\langle\alpha_1,\alpha_2,\dots,\alpha_r\rangle$ is a Weyl group of type A_r with its relation defined by Figure 3.

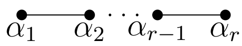

Figure 3: Coxeter diagram of type A_r

(3) $g_{\alpha_1}(s_1s_2\dots s_m) \geq 2$.

Then w is not fully commutative.

Proof. By the condition (3), there exists a pair of integers a and b such that

$$a < b, \quad s_a = s_b = \alpha_1, \quad \alpha_1 \notin \mathrm{supp}(s_{a+1}s_{a+2}\dots s_{b-1}).$$

We shall prove by induction on r.
Case $r = 1$. Since α_1 is commutative to any element of supp(w) $- \{\alpha_1\}$, we have $w = s_1\dots\widehat{s_a}\dots\widehat{s_b}\dots s_m$. It implies that $s_1s_2\dots s_m$ is not a reduced expression of w. This is a contradiction.
Case $r \geq 2$. Note that α_1 is not commutative to α_2 and is commutative to others.

Subcase 1. $g_{\alpha_2}(s_{a+1}s_{a+2}\dots s_{b-1}) = 0$**.** By a similar argument to the case $r = 1$, this is a contradiction.

Subcase 2. $g_{\alpha_2}(s_{a+1}s_{a+2}\dots s_{b-1}) = 1$**.** There exists an integer c such that $a < c < b$, $s_c = \alpha_2$. By virtue of Subcase 2 and the condition (2), we have

$$w = s_1\dots\widehat{s_a}\dots\alpha_1\alpha_2\alpha_1\dots\widehat{s_b}\dots s_m = s_1\dots\widehat{s_a}\dots\alpha_2\alpha_1\alpha_2\dots\widehat{s_b}\dots s_m.$$

Therefore w is not fully commutative.

Subcase 3. $g_{\alpha_2}(s_{a+1}s_{a+2}\dots s_{b-1}) \geq 2$**.** Put $w' := s_{a+1}\dots s_{b-1}$. Then it is easy to see that w' and $\{\alpha_2,\dots,\alpha_r\}$ satisfy the conditions (1), (2), and (3). By the inductive assumption, w' is not fully commutative. It follows from Lemma 3.1(iv) that w is not fully commutative. ∎

Proof of Proposition 4.2. Let m be the length of w, that is, we have $m = \ell(w)$. We shall prove by induction on m.
Case $m \leq 2$. It is obvious.
Case $m \geq 3$. Let $s_1s_2\dots s_m$ be a reduced expression of w.

We check if $s_1s_2\dots\widehat{s_i}\dots s_m$ is a reduced expression or not. It is sufficient to handle with cases $1 < i < m$.

Case 1. $\mathrm{supp}(s_1s_2\dots s_{i-1}) = \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$**.** Since W is a Weyl group of type A_n, there exists an element s_0 of $\mathrm{supp}(s_1s_2\dots s_{i-1})$ such that $\sharp\{s \in \mathrm{supp}(w) | ss_0 \neq$

$s_0s\} \leq 1$. By virtue of Case 1, we have $g_{s_0}(s_1s_2\dots s_m) \geq 2$. By Lemma 4.4, w is not fully commutative. This is a contradiction.

Case 2. $\mathrm{supp}(s_1s_2\dots s_{i-1}) \neq \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$**.**

Subcase 2-1. $\mathrm{supp}(s_1s_2\dots s_{i-1}) - \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m) \neq \emptyset$.
Put $J := \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$. Then there exists a pair of $w^J \in W^J$ and $w_J \in W_J$ such that $w^J w_J s_i s_{i+1}\dots s_m$ is an extended reduced expression of w. By Lemma 3.1(iv), $w_J s_i s_{i+1}\dots s_m$ is also fully commutative. By virtue of Subcase 2-1, we have $w^J \neq e$. It implies that

$$\ell(w_J s_i s_{i+1}\dots s_m) < \ell(w).$$

By the inductive assumption, we have

$$w_J s_{i+1}s_{i+2}\dots s_m \lessdot w_J s_i s_{i+1}\dots s_m, \quad w_J s_{i+1}s_{i+2}\dots s_m \in W_J.$$

By the definition of W^J, we have

$$\ell(w^J w_J s_{i+1}s_{i+2}\dots s_m) = \ell(w) - 1.$$

Thus it follows that $s_1s_2\dots\widehat{s_i}\dots s_m$ is a reduced expression.

Subcase 2-2. $\mathrm{supp}(s_{i+1}s_{i+2}\dots s_m) - \mathrm{supp}(s_1s_2\dots s_{i-1}) \neq \emptyset$. We can prove by a similar discussion above.

Therefore it implies that $s_1s_2\dots\widehat{s_i}\dots s_m$ is a reduced expression. ∎

Furthermore we shall show two lemmas in preparation for proof of Theorem 4.1.

Lemma 4.5 *Let (W, S) be a Coxeter system of type D_{r+3} with its relation defined by Figure 2 ($r \geq 1$). Put $J := S - \{\alpha_1\}$. Let w be a fully commutative element of ${}^JW$ and let $s_1s_2\dots s_m$ be a reduced expression of w. If α_1, β, γ are elements of supp(w) then we have the followings.*

(i) $r + 3 \leq m$, $s_1s_2\dots s_{r+3} = \alpha_1\alpha_2\dots\alpha_r u\beta\gamma$.

(ii) *For any $s \in J$, sw is not fully commutative.*

(iii) $m \leq 2r + 4$.

(iv) *If $m \geq r + 4$ then $s_{r+4}s_{r+5}\dots s_m = u\alpha_r\alpha_{r-1}\dots\alpha_{2r+5-m}$ where $\alpha_{r+1} = u$.*

Proof. In this proof, we sometimes denote u by α_{r+1}.
(i) By $w \in {}^JW$ and $\mathrm{supp}(w) - J = \{\alpha_1\}$, we have $s_1 = \alpha_1$. Assume that $s_2 \neq \alpha_2$. Then we can easily obtain

$$s_2 \in S - \{\alpha_1, \alpha_2\} \subseteq J, \quad w = s_2s_1\widehat{s_2}s_3\dots s_m.$$

This is a contradiction. Thus $s_2 = \alpha_2$. Now we show that if $s_1s_2\dots s_k = \alpha_1\alpha_2\dots\alpha_k$ then $s_{k+1} = \alpha_{k+1}$ for $2 \leq k \leq r$. Note that we have $s_{k+1} \neq \alpha_k$ since $s_1s_2\dots s_m$ is a reduced expression. Assume that $s_{k+1} = \alpha_j$ for some $1 \leq j \leq k - 1$. Then

$$\alpha_j\alpha_{j+1}\dots\alpha_k\alpha_{k+1} = \alpha_j\alpha_{j+1}\alpha_j\alpha_{j+2}\alpha_{j+3}\dots\alpha_k.$$

By $\alpha_j\alpha_{j+1}\alpha_j = \alpha_{j+1}\alpha_j\alpha_{j+1}$, $\alpha_j\alpha_{j+1}\dots\alpha_k\alpha_{k+1}$ is not fully commutative. By Lemma 3.1(iv), w is also not fully commutative. This is a contradiction. If $s_{k+1} \in S -$

$\{\alpha_1, \alpha_2, \ldots, \alpha_{k+1}\} \subseteq J$ then we obtain $s_{k+1}w < w$. This is a contradiction. Hence $s_{k+1} = \alpha_{k+1}$. By the inductive assumption, we obtain $s_1 s_2 \ldots s_r s_{r+1} = \alpha_1\alpha_2 \ldots \alpha_r u$. If $s_{r+2} \in \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ then w is not fully commutative. This is not the case. If $s_{r+2} = u$ then $\ell(w) \neq m$. This is also not the case. Thus we obtain $s_{r+2} = \beta$ or γ. Hence we have

$$s_1 s_2 \ldots s_r s_{r+1} s_{r+2} = \alpha_1\alpha_2 \ldots \alpha_r u\beta \quad \text{or} \quad \alpha_1\alpha_2 \ldots \alpha_r u\gamma.$$

Case $s_1 s_2 \ldots s_r s_{r+1} s_{r+2} = \alpha_1\alpha_2 \ldots \alpha_r u\beta$**.** By a similar argument, we have $s_{r+3} = \gamma$.

Case $s_1 s_2 \ldots s_r s_{r+1} s_{r+2} = \alpha_1\alpha_2 \ldots \alpha_r u\gamma$**.** By a similar argument, we have $s_{r+3} = \beta$.

Since β is commutative to γ, we obtain

$$s_1 s_2 \ldots s_{r+3} = \alpha_1\alpha_2 \ldots \alpha_r u\beta\gamma.$$

Furthermore, by an argument above, we have $r + 3 \leq m$.

(ii) By $w \in {}^J W$, $ss_1 s_2 \ldots s_m$ is a reduced expression of sw. By (i), there is a reduced expression of $ss_1 s_2 \ldots s_{r+3}$ which is

$$\begin{cases} \alpha_1 \ldots \alpha_{k-2}\alpha_k\alpha_{k-1}\alpha_k \ldots \alpha_{r+1}\beta\gamma, & \text{if } s = \alpha_k \ (k = 2, 3, \ldots, r+1), \\ \alpha_1 \ldots \alpha_r \beta u\beta\gamma, & \text{if } s = \beta, \\ \alpha_1 \ldots \alpha_r \gamma u\gamma\beta, & \text{if } s = \gamma. \end{cases}$$

Thus, $s\alpha_1\alpha_2 \ldots \alpha_{r+3}$ is not fully commutative. By Lemma 3.1(iv), sw is also not fully commutative.

(iii) and **(iv)** By Corollary 3.4(ii) and the lemma (i), it is easy to show that we have

$$s_{r+4}s_{r+5} \ldots s_t = u\alpha_r\alpha_{r-1} \ldots \alpha_{2r+5-t}$$

for any t such that $r+4 \leq t \leq 2r+4$ and $t \leq m$. Assume $m > 2r+4$. It implies that

$$s_1 s_2 \ldots s_{2r+5} = \alpha_1\alpha_2 \ldots \alpha_r u\beta\gamma u\alpha_r\alpha_{r-1} \ldots \alpha_1 s_{2r+5}.$$

Since $s_1 s_2 \ldots s_{2r+5}$ is a reduced expression, we have $s_{2r+5} \in J$. By a similar argument of the proof of (ii), it follows that w is not fully commutative. This is a contradiction. Therefore we obtain $m \leq 2r + 4$. ∎

From now on, we assume that (W, S) is a Coxeter system of type E_{r+4} $(r \geq 0)$ with its relation defined by Figure 1. Note that a Coxeter system of type E_4 (resp. E_5) is a Coxeter system of type A_4 (resp. D_5).

Lemma 4.6 *Let (W, S) be a Coxeter system of type E_{r+4} $(r \geq 1)$. Put $J := S - \{\alpha_1\}$. Let w be a fully commutative element of ${}^J W$ and let $s_1 s_2 \ldots s_m$ be a reduced expression of w. Then we have the followings.*

(i) If $\alpha_1, \beta_1, \gamma \in$ supp(w) then sw is not fully commutative for all $s \in J$.

(ii) Assume $\alpha_1, \beta_2, \gamma \in$ supp(w), $\beta_1 \notin$ supp(w) and $s \in J$. If sw is fully commutative then $s = \beta_1$.

(iii) *Assume* $g_{\alpha_1}(w) \geq 2$ *and* $s \in J$ *such that* sw *is fully commutative. Then* $w = \alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r \ldots \alpha_2\alpha_1$ *and* $s = \beta_1$.

(iv) *Assume* $g_{\alpha_1}(w) \geq 3$ *and* $w \in {}^JW \cap W^J$. *Then there exists an element* v *of* $W_{S-\{\alpha_1,\alpha_2\}}$ *such that*

$$(\alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_2)\alpha_1\beta_1 v\beta_1(\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_1)$$

is an extended reduced expression of w *and that* $\beta_1 v\beta_1 \in {}^{S-\{\beta_1\}}W \cap W^{S-\{\beta_1\}}$.

Remark 4.7 Let (W, S) be a Coxeter system of type $\tilde{E}_7$ with its relation defined by Figure 7. Then Lemma 4.6(i) cannot hold on this Coxeter system. For example, put $w := \alpha_1\alpha_2\alpha_3 u\beta_3\beta_2\gamma u\alpha_3\beta_3 u\gamma\alpha_2\alpha_3 u\beta_3\beta_2\beta_1$. Then w is fully commutative and we have $\alpha_1, \beta_1, \gamma \in \mathrm{supp}(w)$ and $w \in {}^{S-\{\alpha_1\}}W$. However $\beta_1 w$ is also fully commutative.

Proof of Lemma 4.6. In this proof, we sometimes denote u by α_{r+1}.
(i) If there exists a pair of not empty subsets S_1 and S_2 of S such that $\mathrm{supp}(w) = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, and that any element of S_1 is commutative to any element of S_2, then w cannot be contained in ${}^JW$. By $\alpha_1, \gamma, \beta_1 \in \mathrm{supp}(w)$, we have $\mathrm{supp}(w) = S$. By Lemma 4.5 and $\{s \in S | \beta_1 s \neq s\beta_1\} = \{\beta_2\}$, we can easily see that there exists an extended reduced expression of w which is $\alpha_1\alpha_2 \ldots \alpha_r u\beta_2\beta_1\gamma y$ for some $y \in W$. By a similar argument of the proof of Lemma 4.5(ii), it follows that sw is not fully commutative.

(ii) Since w is fully commutative and we have $\beta_1 \notin \mathrm{supp}(w)$, $\beta_1 w$ is fully commutative. If we have $s \in J - \{\beta_1\}$ then sw is not fully commutative by Lemma 4.5(ii).

(iii) By our assumption and Corollary 3.4(ii), there exists a pair of elements x_1 and x_2 of W and exists an element z of $\langle \beta_1, \beta_2, \gamma \rangle$ such that we have $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq \mathrm{supp}(x_1) \cap \mathrm{supp}(x_2)$ and that $x_1 uzux_2$ is an extended reduced expression of w. By Corollary 3.4(ii), we can obtain $z \in \{\beta_2\gamma, \beta_1\beta_2\gamma, \beta_2\beta_1\gamma\}$. Thus we have $\{\alpha_1, \alpha_2, \ldots, \alpha_r, u, \gamma, \beta_2\} \subseteq \mathrm{supp}(w)$. Since (i) holds and there exists $s \in J$ such that sw is fully commutative, we have $\beta_1 \notin \mathrm{supp}(w)$. By Lemma 4.5(iii) and $g_{\alpha_1}(w) \geq 2$, we can obtain $w = \alpha_1\alpha_2 \ldots \alpha_r u\beta_2\gamma u\alpha_r\alpha_{r-1} \ldots \alpha_1$. By using (ii), we have $s = \beta_1$.

(iv) By $w \in {}^JW \cap W^J$ and $g_{\alpha_1}(w) \geq 3$, we have $s_1 = s_m = \alpha_1$ and $\alpha_1 \in \mathrm{supp}(s_2 \ldots s_{m-1})$. If we write $s_1 s_2 \ldots s_{m-1} = w_1 w_2$ by some $w_1 \in W^J$, $w_2 \in W_J$ then $w_2 \neq e$ and $g_{\alpha_1}(w_1) \geq 2$. Let s be an element of J and let y be an element of W_J such that $w_2 = sy$ and $\ell(w_2) = 1 + \ell(y)$. Note that $\ell(w_1 s) = \ell(w_1) + 1$. By using (iii), we have

$$w_1 = \alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_1$$

and $s = \beta_1$. Hence $(\alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_1)\beta_1 y\alpha_1$ is an extended reduced expression of w. Rewrite $\alpha_1\beta_1 y\alpha_1 = w_2' w_1'$ for some $w_1' \in {}^JW$ and $w_2' \in W_J$. By $\beta_1\alpha_1 = \alpha_1\beta_1$, we have $w_2' \neq e$ and $g_{\alpha_1}(w_1') \geq 2$. Let s' be an element of J and let z be an element of W_J such that $w_2' = zs'$ and $\ell(w_2') = \ell(z) + 1$. By using (iii), we have

$$w_1' = \alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_1$$

and $s' = \beta_1$. Hence $z\beta_1(\alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_1)$ is an extended reduced expression of $\alpha_1\beta_1 y\alpha_1$. Note that $z\beta_1 w_1'$ is also a fully commutative element and

$\alpha_1 z\beta_1 w_1' < z\beta_1 w_1'$. By Proposition 3.2, we have $\alpha_1 \notin \mathrm{supp}(z\beta_1)$. Thus α_1 is commutative to any element of $\mathrm{supp}(z\beta_1)$. Hence we have $z \in W_{S-\{\alpha_1,\alpha_2\}}$. Therefore we have

$$\begin{aligned} w &= (\alpha_1\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\ldots\alpha_2)z\beta_1\alpha_1(\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\ldots\alpha_1) \\ &= (\alpha_1\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\ldots\alpha_2)\alpha_1 z\beta_1(\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\ldots\alpha_1). \end{aligned}$$

Assume that $z = e$. Then,

$$\alpha_2 z\beta_1\alpha_1\alpha_2 = \alpha_2\beta_1\alpha_1\alpha_2 = \beta_1\alpha_2\alpha_1\alpha_2.$$

This is a contradiction. Thus we have $z \neq e$. Let s'' be an element of $S - \{\alpha_1, \alpha_2\}$ and let v be an element of $W_{S-\{\alpha_1,\alpha_2\}}$ such that $z = s''v$ and $\ell(z) = 1 + \ell(v)$. By using (iii), $s'' = \beta_1$ and

$$(\alpha_1\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\ldots\alpha_2)\alpha_1\beta_1 v\beta_1(\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\ldots\alpha_1)$$

is an extended reduced expression of w. Since $\alpha_2 \notin \mathrm{supp}(\beta_1 v\beta_1)$,

$$(\alpha_1\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\ldots\alpha_2)\beta_1 v\beta_1\alpha_1(\alpha_2\ldots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\ldots\alpha_1)$$

is also an extended reduced expression of w. Moreover, by using (iii), we can easily see the following for a fully commutative element x. If $(\alpha_1\ldots\alpha_r u\gamma\beta_2 u\alpha_r\ldots\alpha_1)x$ (resp. $x(\alpha_1\ldots\alpha_r u\gamma\beta_2 u\alpha_r\ldots\alpha_1)$) is an extended reduced expression then $x \in W^{S-\{\beta_1\}}$ (resp. $x \in {}^{S-\{\beta_1\}}W$). Thus, we can obtain $\beta_1 v\beta_1 \in {}^{S-\{\beta_1\}}W \cap W^{S-\{\beta_1\}}$. ∎

Proof of Theorem 4.1. Let w be a fully commutative element of $W(E_{r+4})$. We shall prove that w is fully covering by induction on r. Note that we sometimes denote u by α_{r+1}.

Case $r = 0$. It has been proven since we regard $W(E_4)$ as $W(A_4)$.

Case $r \geq 1$. If we have $\alpha_1 \notin \mathrm{supp}(w)$ then we can regard $w \in W(E_{r+3})$. By the inductive assumption, w is fully covering. By a similar way, if we have $u \notin \mathrm{supp}(w)$ or $\gamma \notin \mathrm{supp}(w)$ or $\beta_2 \notin \mathrm{supp}(w)$ then w is fully covering. Thus we assume that we have $\alpha_1, u, \gamma, \beta_2 \in \mathrm{supp}(w)$.

Assume that we have $\ell(w) = m$. We shall prove that w is fully covering by induction on m. It is easy to verify in cases $m \leq 2$. Thus we handle with cases $m \geq 3$. Put $J := S - \{\alpha_1\}$ and we check the following three cases.

1. $w \notin W^J$, 2. $w \notin {}^J W$, 3. $w \in {}^J W \cap W^J$.

Case 1 By an assumption of this case, there exists a pair of $w^J \in W^J$ and $w_J \in W_J$ such that $w_J \neq e$ and $w = w^J w_J$. Let $s_1 s_2 \ldots s_m$ be a reduced expression of w such that $s_1 s_2 \ldots s_k = w^J$ and $s_{k+1}\ldots s_m = w_J$. For $1 \leq i \leq m$, we shall prove that $s_1 s_2 \ldots \widehat{s_i} \ldots s_m$ is a reduced expression. Note that by $w^J = s_1 s_2 \ldots s_k \in W^J$, $J = S - \{\alpha_1\}$ and $\alpha_1 \in \mathrm{supp}(w)$, we have $s_k = \alpha_1$ and $\ell(w_J) < \ell(w)$.

Assume that we have $k+1 \leq i \leq m$. Then by Lemma 3.1(iv) w_J is fully commutative. By the inductive assumption on m, $s_{k+1}\ldots\widehat{s_i}\ldots s_m$ is a reduced expression. By the definition of W^J, $s_1\ldots s_k s_{k+1}\ldots\widehat{s_i}\ldots s_m$ is also a reduced expression.

Next assume that we have $1 \leq i \leq k$.

Subcase 1-1 $\mathrm{supp}(s_1s_2\dots s_{i-1}) \neq \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$. By a similar argument in the proof of Case 2 of Proposition 4.2, we can easily see that $s_1s_2\dots\widehat{s_i}\dots s_m$ is a reduced expression.

Subcase 1-2 $\mathrm{supp}(s_1s_2\dots s_{i-1}) = \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$ and $\alpha_1 \notin \mathrm{supp}(s_1s_2\dots s_{i-1})$. By $\alpha_1 \in \mathrm{supp}(w)$, we have $s_i = \alpha_1$. By $s_k = \alpha_1$, we have $i = k$. By $\beta_2, \gamma \in \mathrm{supp}(w)$, we have $\alpha_1, \beta_2, \gamma \in \mathrm{supp}(s_1s_2\dots s_k)$. Assume $\beta_1 \in \mathrm{supp}(s_1s_2\dots s_k)$. By $s_1s_2\dots s_k \in W^J$ and Lemma 4.6(i), $s_1s_2\dots s_ks_{k+1}$ is not fully commutative. This is a contradiction. Assume $\beta_1 \notin \mathrm{supp}(s_1s_2\dots s_k)$. By $s_1s_2\dots s_k \in W^J$ and Lemma 4.6(ii), we have $s_{k+1} = \beta_1$. By $i = k$, we have

$$\beta_1 \in \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m) - \mathrm{supp}(s_1s_2\dots s_{i-1}).$$

This is a contradiction.

Subcase 1-3 $\mathrm{supp}(s_1s_2\dots s_{i-1}) = \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$ and $\alpha_1 \in \mathrm{supp}(s_1s_2\dots s_{i-1})$. By $i \leq k$ and $s_k = \alpha_1$, we have $g_{\alpha_1}(s_1s_2\dots s_k) \geq 2$. Since $s_1s_2\dots s_k \in W^J$ and Lemma 4.6(iii), we obtain

$$s_1s_2\dots s_i\dots s_k = \alpha_1\alpha_2\dots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\dots\alpha_1$$

and $s_{k+1} = \beta_1$. Thus

$$\beta_1 \in \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m) - \mathrm{supp}(s_1s_2\dots s_{i-1}).$$

This is a contradiction.

Case 2 We omit the proof since we can state this case by a similar argument as in the proof of Case 1.

Case 3 Let $s_1s_2\dots s_m$ be a reduced expression of w. For $1 \leq i \leq m$, we prove that $s_1s_2\dots\widehat{s_i}\dots s_m$ is a reduced expression. By Lemma 3.7, it is enough to prove for cases $3 \leq i \leq m-2$. Note that by an assumption of this case we have $s_1 = s_m = \alpha_1$ and $s_2 = s_{m-1} = \alpha_2$.

Subcase 3-1 $\alpha_1 \notin \mathrm{supp}(s_2s_3\dots s_{m-1})$. Assume that $s_1s_2\dots\widehat{s_i}\dots s_m$ is not a reduced expression. By the inductive assumption on m, $s_1\dots\widehat{s_i}\dots s_{m-1}$ is a reduced expression. Thus there exists an integer j such that $1 \leq j \leq i-1$ and $s_1\dots\widehat{s_i}\dots s_{m-1} = s_1\dots\widehat{s_j}\dots\widehat{s_i}\dots s_{m-1}s_m$. It implies $s_j\dots\widehat{s_i}\dots s_{m-1} = s_{j+1}\dots\widehat{s_i}\dots s_m$. Accordingly $s_j\dots\widehat{s_i}\dots s_m$ is not a reduced expression. If $j > 1$ then $s_2\dots\widehat{s_i}\dots s_m$ is not a reduced expression. This is a contradiction. Thus $j = 1$ and $s_1s_2\dots\widehat{s_i}\dots s_{m-1} = s_2\dots\widehat{s_i}\dots s_m$. Put $x := s_3\dots\widehat{s_i}\dots s_{m-1}$. Then $s_2s_1s_2x < s_1s_2x$ and $\alpha_1 \notin \mathrm{supp}(x)$. By Lemma 3.6, we have $s_1 = \alpha_1$, $s_2 = \alpha_2$ and $\ell(\alpha_2\alpha_1\alpha_2) = 3$. This is a contradiction. Hence $s_1s_2\dots\widehat{s_i}\dots s_m$ is a reduced expression.

Subcase 3-2 $\alpha_1 \in \mathrm{supp}(s_2s_3\dots s_{m-1})$. By Lemma 4.6(iv), there exists an element v of W such that we have $\alpha_1, \alpha_2 \notin \mathrm{supp}(v)$ and that

$$(\alpha_1\alpha_2\dots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\dots\alpha_2)\alpha_1\beta_1 v\beta_1(\alpha_2\dots\alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1}\dots\alpha_1)$$

is an extended reduced expression of w. Note $v = s_{2r+6}s_{2r+7}\dots s_{m-2r-4}$. If $1 \leq i \leq 2r+5$ or $m-2r-3 \leq i \leq m$ then $\mathrm{supp}(s_1s_2\dots s_{i-1}) \neq \mathrm{supp}(s_{i+1}s_{i+2}\dots s_m)$. By a similar discussion in the proof of Case 2 of Proposition 4.2, $s_1s_2\dots\widehat{s_i}\dots s_m$

is a reduced expression. The rest case is $2r+6 \leq i \leq m-2r-4$. Assume that $s_1 s_2 \ldots \widehat{s_i} \ldots s_m$ is not a reduced expression. By a similar discussion in Subcase 3-1, we have $s_1 s_2 \ldots \widehat{s_i} \ldots s_{m-1} = s_2 \ldots \widehat{s_i} \ldots s_m$. Put $v' := s_{2r+6} s_{2r+7} \ldots \widehat{s_i} \ldots s_{m-2r-4}$. Then

$$\begin{aligned} &\alpha_1(\alpha_2 \ldots \alpha_r u \gamma \beta_2 u \alpha_r \alpha_{r-1} \ldots \alpha_2)\alpha_1 \beta_1 v' \beta_1 (\alpha_2 \ldots \alpha_r u \gamma \beta_2 u \alpha_r \alpha_{r-1} \ldots \alpha_2) \\ &= (\alpha_2 \ldots \alpha_r u \gamma \beta_2 u \alpha_r \alpha_{r-1} \ldots \alpha_2)\alpha_1 \beta_1 v' \beta_1 (\alpha_2 \ldots \alpha_r u \gamma \beta_2 u \alpha_r \alpha_{r-1} \ldots \alpha_2)\alpha_1 \end{aligned}$$

and both sides are extended reduced expressions.

By the subword property, we have

$$\begin{aligned} &\alpha_1 \alpha_2 \ldots \alpha_r u \beta_2 \beta_1 \\ &\leq (\alpha_2 \ldots \alpha_r u \gamma \beta_2 u \alpha_r \alpha_{r-1} \ldots \alpha_2)\alpha_1 \beta_1 v' \beta_1 (\alpha_2 \ldots \alpha_r u \gamma \beta_2 u \alpha_r \alpha_{r-1} \ldots \alpha_2)\alpha_1. \end{aligned}$$

On the other hand, a reduced expression of $\alpha_1 \alpha_2 \ldots \alpha_r u \beta_2 \beta_1$ is unique. This is a contradiction. Therefore $s_1 s_2 \ldots \widehat{s_i} \ldots s_m$ is a reduced expression. ∎

Remark 4.8 Let w be an element of a Coxeter group. In [4], w is said to be *short-braid avoiding* if and only if any reduced expression $s_1 s_2 \ldots s_m$ for w satisfies $s_i \neq s_{i+2}$ for all $i \in [m-2]$. It is easy to see that a fully covering element is short-braid avoiding, and that a short-braid avoiding element is fully commutative. By the same method as the one adopted in the proof of [4, Theorem 1] and Theorem 4.1, we can easily obtain the following which includes Fan's result [4, Theorem 1]. Let (W, S) be a Coxeter system and let (W_0, S_0) be a Coxeter system defined by $S_0 := S$ as a set and $m(s,t) := 3$ if $m(s,t) > 3$ in W or $m(s,t)$ in W_0 is defined as $m(s,t)$ in W if $m(s,t) \leq 3$ for $s, t \in S_0$. If W_0 is a Coxeter group of type A, D or E then for $w \in W$, w is a short-braid avoiding element if and only if w is a fully covering element.

Although it is already shown by Fan that a Coxeter group of type E is an FC-finite Coxeter group, we give another proof.

Proposition 4.9 *For $n \geq 3$, we have*

$$\max\{\ell(w) | w \in W(E_n)^{FC}\} \leq 2^{n-1} - 1,$$

where we put $W(E_3) := \langle \beta_1, \beta_2, \gamma \rangle$. In particular, we have $|W(E_n)^{FC}| < \infty$.

Note that the above inequality is not best possible (see the proof of this proposition).

Remark 4.10 *In [10], H. Tagawa showed*

$$\max\{c^-(x) | x \in W(A_n)\} = \lfloor (n+1)^2/4 \rfloor,$$

where $\lfloor a \rfloor$ is the largest integer equal or less than a. By the formula, it is easy to show

$$\max\{\ell(x) | x \in W(A_n)^{FC}\} = \lfloor (n+1)^2/4 \rfloor.$$

Note that it does not hold on case of type D. In fact, we have

$$\max\{c^-(x) | x \in W(D_4)\} = 8 > 6 = \max\{\ell(x) | x \in W(D_4)^{FC}\}.$$

Proof of Proposition 4.9. For $n \geq 3$, we put

$$a_n := \max\{\ell(w) | w \in W(E_n)^{FC}\}$$

and we shall prove $a_n \leq 2^{n-1} - 1$ by induction on n.
Case $n = 3, 4$. By Remark 4.10, we have

$$a_3 = 3 = 2^2 - 1, \quad a_4 = 6 < 2^3 - 1.$$

Case $n \geq 5$. We claim $\ell(w) \leq 2^{n-1} - 1$ for any $w \in W(E_n)^{FC}$. If $g_{\alpha_1}(w) = 0$ then we can regard $w \in W(E_{n-1})^{FC}$. Thus

$$\ell(w) \leq a_{n-1} \leq 2^{n-2} - 1 < 2^{n-1} - 1.$$

Now we assume $g_{\alpha_1}(w) \geq 1$. Put $W := W(E_n)$ and $J := S - \{\alpha_1\}$.

Subcase $w \notin W^J$. Then there exists a pair of $w^J \in W^J$ and $w_J \in W_J$ such that $w_J \neq e$ and $w = w^J w_J$. Assume $g_{\alpha_1}(w) = 1$. Then since $w^J \in W^J$ and $J = S - \{\alpha_1\}$, there exists an element z of W such that $\alpha_1 \notin \mathrm{supp}(z)$ and $w^J = z\alpha_1$. Then $z\alpha_1 w_J$ is an extended reduced expression of w and we can regard $z, w_J \in W(E_{n-1})^{FC}$. Thus

$$\ell(w) \leq 2a_{n-1} + 1 \leq 2(2^{n-2} - 1) + 1 = 2^{n-1} - 1.$$

Assume $g_{\alpha_1}(w) \geq 2$. Then by Lemma 4.6, we have $w^J = \alpha_1\alpha_2 \dots \alpha_{n-4} u\gamma\beta_2 u\alpha_{n-4}\alpha_{n-5} \dots \alpha_1$. On the other hand, we can regard $w_J \in W(E_{n-1})^{FC}$. Hence we have

$$\ell(w) \leq 2n - 4 + a_{n-1} \leq 2(n-2) + 2^{n-2} - 1 \leq 2^{n-1} - 1.$$

Subcase $w \notin {}^J W$. We can prove this case by a similar discussion above.

Subcase $w \in {}^J W \cap W^J$. Then there exists an element z of $W(E_n)$ such that $\alpha_1 z \alpha_1$ is an extended reduced expression of w. If we have $g_{\alpha_1}(z) = 0$ then we have $z \in W(E_{n-1})^{FC}$. Thus we have

$$\ell(w) \leq a_{n-1} + 2 \leq 2^{n-2} + 1 \leq 2^{n-1} - 1.$$

Assume that we have $g_{\alpha_1}(z) \geq 1$. Then there exists an element v of $W(E_n)^{FC}$ such that $v \in W_{S - \{\alpha_1, \alpha_2\}}$ and that

$$(\alpha_1\alpha_2 \dots \alpha_{n-4} u\gamma\beta_2 u\alpha_{n-4}\alpha_{n-5} \dots \alpha_2)\alpha_1\beta_1 v\beta_1(\alpha_2 \dots \alpha_{n-4} u\gamma\beta_2 u\alpha_{n-4}\alpha_{n-5} \dots \alpha_1)$$

is an extended reduced expression of w by Lemma 4.6(iv). Hence we have

$$\ell(w) \leq 4n - 9 + a_{n-2} \leq 4n - 9 + 2^{n-3} - 1 \leq 2^{n-1} - 1.$$

This completes the proof of the proposition. ∎

5 Not bi-full Coxeter groups

Our aim of this section is to prove the following.

Theorem 5.1 *Let W be an irreducible Coxeter group which is neither of type A, D nor E. Then W is not a bi-full Coxeter group. In other words, there is an element of W which is fully commutative and which is not fully covering. In particular, if W is a simply-laced Coxeter group then we have $|W^{FC}| = \infty$.*

First we prove the following.

Proposition 5.2 *Let (W_1, S_1) (resp. (W_2, S_2), (W_3, S_3), (W_4, S_4), (W_5, S_5)) be a Coxeter system of type $\tilde{A}_n$ ($n \geq 2$) (resp. $\tilde{D}_{r+3}$ ($r \geq 1$), $\tilde{E}_6$, $\tilde{E}_7$, $I_2(m)$ ($m \geq 4$)) with its relation defined by Figure 4 (resp Figure 5, Figure 6, Figure 7, Figure 8). Then for each $1 \leq i \leq 5$ there exists an element w_i of W_i such that w_i is fully commutative and w_i is not fully covering. Furthermore we have $|W_i^{FC}| = \infty$ for any $1 \leq i \leq 4$.*

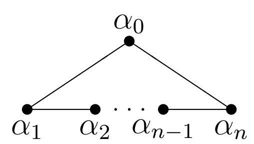

Figure 4: Coxeter diagram of type $\tilde{A}_n$

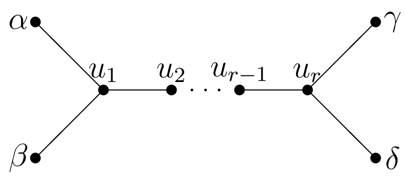

Figure 5: Coxeter diagram of type $\tilde{D}_{r+3}$

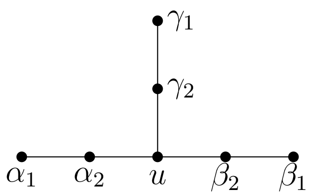

Figure 6: Coxeter diagram of type $\tilde{E}_6$.

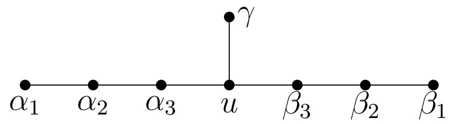

Figure 7: Coxeter diagram of type $\tilde{E}_7$.

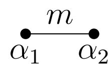

Figure 8: Coxeter diagram of type $I_2(m)$.

Proof. **Case (1),(2),(3),(4).** For $1 \leq i \leq 4$, let w_i and y_i be elements of W_i defined by putting as follows:

$$
\begin{aligned}
w_1 &:= \alpha_1\alpha_2\ldots\alpha_n\alpha_0\alpha_1\alpha_2\ldots\alpha_n,\\
y_1 &:= \alpha_1\alpha_2\ldots\alpha_n\widehat{\alpha_0}\alpha_1\alpha_2\ldots\alpha_n,\\
w_2 &:= u_ru_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta u_ru_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta u_r,\\
y_2 &:= u_ru_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta\widehat{u_r}u_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta u_r,\\
w_3 &:= u\beta_2\beta_1\alpha_2u\beta_2\gamma_2u\alpha_2\alpha_1\gamma_1\gamma_2u\alpha_2\beta_2u\gamma_2\gamma_1\beta_1\beta_2u\gamma_2\alpha_2u\beta_2\beta_1\alpha_1\alpha_2u\beta_2\gamma_2u,\\
y_3 &:= u\beta_2\beta_1\alpha_2u\beta_2\gamma_2u\alpha_2\alpha_1\gamma_1\gamma_2u\alpha_2\beta_2\widehat{u}\gamma_2\gamma_1\beta_1\beta_2u\gamma_2\alpha_2u\beta_2\beta_1\alpha_1\alpha_2u\beta_2\gamma_2u,\\
w_4 &:= \beta_1\beta_2\beta_3u\alpha_3\gamma u\beta_3\beta_2\beta_1\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\beta_1\\
&\quad\times\alpha_1\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\beta_1,\\
y_4 &:= \beta_1\beta_2\beta_3u\alpha_3\gamma u\beta_3\beta_2\beta_1\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\beta_1\\
&\quad\times\widehat{\alpha_1}\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\beta_1.
\end{aligned}
$$

By Corollary 3.4, we can easily see that all w_i are fully commutative. By direct calculation, we can obtain

$$
\begin{aligned}
y_1 &= \widehat{\alpha_1}\alpha_2\ldots\alpha_n\widehat{\alpha_0}\alpha_1\alpha_2\ldots\widehat{\alpha_n},\\
y_2 &= \widehat{u_r}u_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta\widehat{u_r}u_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta\widehat{u_r},\\
y_3 &= \widehat{u}\beta_2\beta_1\alpha_2u\beta_2\gamma_2u\alpha_2\alpha_1\gamma_1\gamma_2u\alpha_2\beta_2\widehat{u}\gamma_2\gamma_1\beta_1\beta_2u\gamma_2\alpha_2u\beta_2\beta_1\alpha_1\alpha_2u\beta_2\gamma_2\widehat{u},\\
y_4 &= \widehat{\beta_1}\beta_2\beta_3u\alpha_3\gamma u\beta_3\beta_2\beta_1\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\beta_1\\
&\quad\times\widehat{\alpha_1}\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\widehat{\beta_1}.
\end{aligned}
$$

Thus, y_i is not covered by w_i, that is, w_i is not fully covering for $1 \leq i \leq 4$. For $1 \leq i \leq 4$, let x_i be an element of W_i defined by putting as follows:

$$
\begin{aligned}
x_1 &:= \alpha_0\alpha_1\ldots\alpha_n,\\
x_2 &:= u_ru_{r-1}\ldots u_1\alpha\beta u_1u_2\ldots u_r\gamma\delta,\\
x_3 &:= u\beta_2\gamma_2u\alpha_2\alpha_1\gamma_1\gamma_2u\alpha_2\beta_2u\gamma_2\gamma_1\beta_1\beta_2u\gamma_2\alpha_2u\beta_2\beta_1\alpha_1\alpha_2,\\
x_4 &:= \alpha_1\alpha_2\alpha_3u\beta_3\beta_2\gamma u\alpha_3\beta_3u\gamma\alpha_2\alpha_3u\beta_3\beta_2\beta_1.
\end{aligned}
$$

Then we can easily see that x_i^2 is also fully commutative and that we have $\ell(x_i^2) = 2\ell(x_i)$ for $1 \leq i \leq 4$. Therefore we have $|W_i^{FC}| = \infty$ for any $1 \leq i \leq 4$ by Corollary 3.5.

Case (5). Put $w_5 := \alpha_1\alpha_2\alpha_1$. By Lemma 3.1(iii), w_5 is fully commutative. Since $\alpha_1\widehat{\alpha_2}\alpha_1(= e)$ is not covered by w_5, w_5 is not fully covering. ∎

Proof of Theorem 5.1. Recall that W is neither of type A, D, nor E. It is easy to show that a Coxeter diagram associated to W contains at least one of the Coxeter diagrams in Figure 4,5,6,7 and 8. Therefore W is not a bi-full Coxeter group, by Proposition 5.2. Furthermore if W is a simply-laced Coxeter group then we can easily see that there are infinite its fully commutative elements by Proposition 5.2. ∎

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