EULER-MAHONIAN STATISTICS ON ORDERED PARTITIONS

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Abstract. An ordered partition with k blocks of $[n] := \{1, 2, \ldots, n\}$ is a sequence of k disjoint and nonempty subsets, called blocks, whose union is $[n]$. Clearly the number of such ordered partitions is $k!S(n, k)$, where $S(n, k)$ is the Stirling number of the second kind. A statistic on ordered partitions of $[n]$ with k blocks is called *Euler-Mahonian statistics* if its generating polynomial is $[k]_q!S_q(n, k)$, which is a natural q-analogue of $k!S(n, k)$.

Motivated by Steingrímsson's conjectures, we consider two different methods to produce Euler-Mahonian statistics on ordered partitions: (a) we give a bijection between ordered partitions and weighted Motzkin paths by using a variant of Françon-Viennot's bijection to derive many Euler-Mahonian statistics by expanding the generating function of $[k]_q!S_q(n, k)$ as an explicit continued fraction; (b) we encode ordered partitions by walks in some digraphs and then derive new Euler-Mahonian statistics by computing their generating functions using the transfer-matrix method. In particular, we prove several conjectures of Steingrímsson.

Contents

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1. Introduction

The systematic study of statistics on permutations and words has its origins in the work of MacMahon, at the turn of the last century. MacMahon [11] considered four different statistics for a permutation π: The number of descents ($\operatorname{des}\pi$), the number of excedances ($\operatorname{cinv}\pi$), the number of inversions ($\operatorname{inv}\pi$), and the major index ($\operatorname{maj}\pi$). These are defined as follows: A descent in a permutation $\pi = a_1a_2\cdots a_n$ is an i such that $a_i > a_{i+1}$, an excedance is an i such that $a_i > i$, an inversion is a pair (i,j) such that $i<j$ and $a_i > a_j$, and the major index of π is the sum of the descents in π.

In fact, MacMahon studied these statistics in greater generality, namely over the rearrangement class of an arbitrary word w with repeated letters. The rearrangement class $R(w)$ of a word $w = a_1a_2\cdots a_n$ is the set of all words obtained by permutating the letters of w. All of the statistics mentioned above generalize to words, and in each case, except for that of cinv, the generalization is trivial.

MacMahon showed, algebraically, that cinv is *equidistributed* with des, and that inv is equidistributed with maj, over $R(w)$ for any word w. That is to say,

$$\sum_{z\in R(w)} t^{\operatorname{cinv} z} = \sum_{z\in R(w)} t^{\operatorname{des} z} \quad \text{and} \quad \sum_{z\in R(w)} t^{\operatorname{inv} z} = \sum_{z\in R(w)} t^{\operatorname{maj} z}.$$

Any statistic that is equidistributed with des is said to be *Eulerian*, while any statistic equidistributed with inv is said to be *Mahonian*. Foata [5] coined the name *Euler-Mahonian* statistic for a bivariate statistic $(\operatorname{eul}\pi, \operatorname{mah}\pi)$, where eul is Eulerian and mah is Mahonian and carried out the first study of such pairs.

In 1997 Steingrímsson [18] introduced the notion of *Euler-Mahonian* statistic on ordered partitions. In order to motivate this definition we recall a well-known basic identity relating Eulerian and Stirling numbers along with some basic definitions.

For a permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ of $[n] = \{1,2,\ldots,n\}$, the integer $i\in[n-1]$ is called a *descent* (resp. *ascent*) of σ if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) < \sigma(i+1)$). Let $\operatorname{des}\sigma$ be the number of descents of σ and S_n be the set of permutations of $[n]$, then the *Eulerian polynomial* $A_n(x)$ is the generating polynomial of permutations in S_n with respect to the number of descents, i.e.,

$$A_n(x) = \sum_{\sigma\in S_n} x^{1+\operatorname{des}\sigma} = \sum_{k=0}^{n-1} A(n,k)x^{1+k}.$$

where the coefficients $A(n,k)$ are called the *Eulerian numbers*. The Eulerian polynomials can also be defined by the following exponential generating function:

$$1 + \sum_{n\geq 1} A_n(x)\frac{z^n}{n!} = \frac{1-x}{1-xe^{(1-x)z}}. \tag{1.1}$$

A partition $\pi = B_1/B_2/\cdots/B_k$ of $[n]$ is a collection of disjoint and nonempty subsets $B_i's$, called *blocks*, whose union is $[n]$, where we write π in the standard way, i.e., the blocks B_i are arranged in increasing order of their minimal elements and separated by /. Let $\mathcal{P}_n^k$ be the set of partitions of $[n]$ with k blocks. Now, if σ is a permutation in S_k, the sequence $\pi_\sigma = B_{\sigma(1)}/B_{\sigma(2)}/\cdots/B_{\sigma(k)}$ is called an *ordered partition* of $[n]$ with k blocks. We set $\sigma = \operatorname{perm}(\pi)$. Let $\mathcal{OP}_n^k$ be the set of ordered partitions of $[n]$ into k blocks.

A *preferential arrangement* of $[n]$ is a permutation of $[n]$, of which each ascent is *underlined or not.* A *run* in a permutation σ is an increasing sequence of consecutive terms $\sigma(i) < \sigma(i+1) < \ldots < \sigma(j)$ such that $\sigma(i-1) > \sigma(i)$ and $\sigma(j) > \sigma(j+1)$, where $\sigma(0) = n+1$ and $\sigma(n+1) = 0$. Clearly each preferential arrangement can be decomposed uniquely, from left to right, into *runs* or *increasing sequences ending with an underlined ascent.* Hence a preferential arrangement of $[n]$ can be identified with an *ordered set partition* of $[n]$. For example, we have the following correspondence between a preferential arrangement σ and an ordered partition π of $[9]$:

$$\sigma = \underline{2}\,9\,3\,1\,4\,8\,5\,\underline{6}\,7 \longleftrightarrow \pi = 2/9/3/1\,4\,8/5\,6/7.$$

Clearly, if k is the number of blocks in the ordered partition π then $k-1$ equals the number of descents and underlined ascents of σ. Since the number of ordered partitions of $[n]$ into k blocks is $k!S(n,k)$, where $S(n,k)$ denotes the *Stirling numbers of the second kind*, this correspondence implies immediately the identity:

$$k!S(n,k) = \sum_{m=1}^{k} \binom{n-m}{n-k} A(n, m-1). \tag{1.2}$$

Indeed, we can construct the corresponding preferential arrangements of $[n]$ by first choosing a permutation of $[n]$ with $m-1$ descents and then underline $k-1-(m-1) = k-m$ ascents.

Let $\mathcal{OP}_n$ be the set of all ordered partitions of $[n]$, and $\underline{S_n}$ be the set of all preferential arrangements of $[n]$. Clearly the cardinality of $\mathcal{OP}_n$ is given by $|\mathcal{OP}_n| = \sum_{k=0}^{n} k!S(n,k)$. On the other hand, it is easy to derive from (1.1) the following exponential generating function :

$$1 + \sum_{n\geq 1} |\mathcal{OP}_n| \frac{z^n}{n!} = \frac{1}{2-e^z} = 1 + z + 3\,\frac{z^2}{3!} + 13\,\frac{z^3}{3!} + 75\,\frac{z^4}{4!} + \cdots .$$

Carlitz [1, 2] was the first to study q-analogues of *Eulerian numbers* and *Stirling numbers of the second kind*, that we recall in the following.

Define the p,q-integer $[n]_{p,q} = \frac{p^n - q^n}{p-q}$, the p,q-factorial $[n]_{p,q}! = [1]_{p,q}[2]_{p,q}\cdots[n]_{p,q}$ and the *p,q-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} \qquad n \geq k \geq 0.$$

If $p = 1$, we shall write $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for $[n]_{1,q}$, $[n]_{1,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{1,q}$, and usually called *$q$-integer*, *$q$-factorial* and *$q$-binomial coefficient*, respectively. In this paper we need a variation of the q-binomial coefficient which seems fairly non-standard. Put

$$]n,k[_{q,r} = [n]_{qr} - q^{n-k}[k]_{qr},$$

and let

$$\left]\begin{matrix} n \\ k \end{matrix}\right[_{q,r} = \begin{cases} \frac{\prod_{i=0}^{k-1}]n,i[_{q,r}}{[k]_{qr}!} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, $]3,1[_{q,r} = 1 + qr + q^2r^2 - q^2$ and $\left]\begin{matrix} 3 \\ 2 \end{matrix}\right[_{q,r} = \frac{(1+qr+q^2r^2)(1+qr+q^2r^2-q^2)}{(1+qr)}$. Note that $]n,k[_{q,1} = [n-k]_q$ and $\left]\begin{matrix} n \\ k \end{matrix}\right[_{q,1} = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

The q-Eulerian numbers $A_q(n,k)$ $(n \geq k \geq 0)$ are defined by

$$A_q(n,k) = q^k[n-k]_q A_q(n-1,k-1) + [k+1]_q A_q(n-1,k).$$

The first values of the q-Eulerian numbers $\left\langle {n \atop k} \right\rangle_q$ $(n \geq k \geq 0)$ read

$n \setminus k$	0	1	2	3
1	1			
2	1	q		
3	1	$2q+2q^2$	q^3	
4	1	$3q+5q^2+3q^3$	$3q^3+5q^4+3q^5$	q^6.

The *major index* of σ, noted $\operatorname{maj}\sigma$, is the sum of its descents, i.e., $\operatorname{maj}\sigma = \sum_i i$, where the summation is over all descents i of σ. It is well-known that maj is a Mahonian statistic on S_n, i.e.,

$$\sum_{\sigma \in S_n} q^{\operatorname{maj}\sigma} = [n]_q!.$$

Carlitz [2] gave the following combinatorial interpretation of q-Eulerian numbers:

$$A_q(n,k) = \sum_{\sigma} q^{\operatorname{maj}\sigma},$$

where the summation is over all permutations of $[n]$ with k descents.

The two natural q-Stirling numbers $S_q(n,k)$ and $\widetilde{S}_q(n,k)$ of the second kind are defined by:

$$S_q(n,\, k) = S_q(n-1,\, k-1) + [k]_q S_q(n-1,\, k) \qquad (n \geq k \geq 0), \tag{1.3}$$

and

$$\widetilde{S}_q(n,\, k) = q^{k-1}\widetilde{S}_q(n-1,\, k-1) + [k]_q \widetilde{S}_q(n-1,\, k) \qquad (n \geq k \geq 0), \tag{1.4}$$

where $S_q(n,k) = \widetilde{S}_q(n,k) = \delta_{n\,k}$ if $n = 0$ or $k = 0$. It is easy to see that $\widetilde{S}_q(n,k) = q^{k(k-1)/2} S_q(n,k)$. The first values of the q-Stirling numbers $\widetilde{S}_q(n,\, k)$ read

$n \setminus k$	1	2	3	4
1	1			
2	1	q		
3	1	$2q+q^2$	q^3	
4	1	$3q+3q^2+q^3$	$3q^3+2q^4+q^5$	q^6.

There has been a considerable amount of recent interest in properties and combinatorial interpretations of the q-Eulerian numbers and q-Stirling numbers and related numbers (see e.g. [1, 2, 3, 10, 12, 13, 14, 15, 16, 18, 21, 20, 22]).

Garsia [8] found a q-analogue of Frobenius formula relating q-Eulerian numbers and q-Stirling numbers of the second kind:

$$\sum_{k=1}^{n} \frac{[k]!\widetilde{S}_q(n,k)x^k}{(x;q)_{k+1}} = \sum_{k=1}^{\infty} [k]^n x^k = \frac{\sum_{\sigma \in S_n} x^{1+\operatorname{des}\sigma} q^{\operatorname{maj}\sigma}}{(x;q)_{n+1}}, \tag{1.5}$$

where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$. It is then easy (see [23]) to derive the following q-analogue of (1.2):

$$[k]_q!\, \widetilde{S}_q(n,k) = \sum_{m=1}^{k} q^{k(k-m)} \begin{bmatrix} n-m \\ n-k \end{bmatrix}_q A_q(n,m-1). \tag{1.6}$$

In his attempt to give a combinatorial proof of (1.6), Steingrímsson [18] introduced the following definition.

Definition 1.1. *A statistic* STAT *on* $\mathcal{OP}_n^k$ *is called Euler-Mahonian if its generating function is equal to* $[k]_q!\widetilde{S}_q(n,k)$, *i.e.,*

$$\sum_{\pi\in\mathcal{OP}_n^k} q^{\mathrm{STAT}\,\pi} = [k]_q!\, \widetilde{S}_q(n,k).$$

An Euler-Mahonian statistic on ordered partitions can be derived from a result of Wachs [12, Theorem 2.1] (see also [18, Theorem 4]). Steingrímsson [18] gave several Euler-Mahonian statistics and conjectured more such statistics.

The aim of this paper is to provide much more such statistics on ordered partitions by taking two different approaches. In the first approach our starting point is an explicit continued fraction expansion of the generating function $\sum_{n,k}[k]_q!S_q(n,k)u^kz^n$. By embedding the ordered partitions into preferential arrangements we can encode the later by weighted Motzkin paths, whose generating function is known to have an explicit continued fraction expansion. By identifying these two continued fractions we find many Euler-Mahonian statistics on ordered partitions. In the second approach we shall construct a bijection ψ between ordered partitions and walks in some digraphs (see section 3). This bijection keeps track of several statistics on ordered partitions. Then, by transfer-matrix method, we evaluate the generating functions of these statistics on ordered partitions. Finally, by identifying the generating functions we derive several Euler-Mahonian statistics. We conclude this paper with some open problems.

2. Definitions and main results

Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \underline{S_n}$. By convention, we assume that $\sigma_0 = n+1$ and $\sigma_{n+1} = 0$. For $1 \le i \le n$, we say that σ_i is a

- *valley* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$;
- *underlined valley* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$ and σ_i is underlined;
- *peak* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- *double ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- *double underlined ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ and σ_i is underlined;
- *double descent* if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

We will denote by $\mathrm{DA}(\sigma)$, $\underline{\mathrm{DA}}(\sigma)$, $\mathrm{VA}(\sigma)$, $\underline{\mathrm{VA}}(\sigma)$, $\mathrm{DD}(\sigma)$, $\mathrm{PE}(\sigma)$ the set of double ascents, underlined double ascents, valleys, underlined valleys, double descents and peaks of σ respectively. For instance, if $\sigma = \underline{2}\,9\,3\,1\,4\,8\,5\,\underline{6}\,7$, then

$$\mathrm{DA}(\sigma) = \{4\},\quad \underline{\mathrm{DA}}(\sigma) = \{6\},\quad \mathrm{VA}(\sigma) = \{1,5\},$$
$$\underline{\mathrm{VA}}(\sigma) = \{2\},\quad \mathrm{DD}(\sigma) = \{3\},\quad \mathrm{PE}(\sigma) = \{7,8,9\}.$$

For $i \in [n]$, define $\mathrm{lsg}_i(\sigma)$ (resp. $\mathrm{rsg}_i(\sigma)$) as the number of runs of σ strictly to the left (resp. right) of i which contain an element smaller than i and an element greater than i. Then we define

$$\mathrm{lsg}(\sigma) = \sum_{i=1}^{n} \mathrm{lsg}_i(\sigma), \quad \mathrm{rsg}(\sigma) = \sum_{i=1}^{n} \mathrm{rsg}_i(\sigma).$$

More generally, for any set of positive integers A and a statistics STAT_i on ordered partitions, we define the statistic $\mathrm{STAT}(A)$ by

$$\mathrm{STAT}(A) = \sum_{i \in A} \mathrm{STAT}_i.$$

Consider the generating polynomial of preferential arrangements:

$$\begin{aligned}\mu_n = \sum_{\pi \in \underline{S_n}} & \alpha^{\#\mathrm{DD}(\sigma)} \beta^{\#\mathrm{DA}(\sigma)} \gamma^{\#\underline{\mathrm{DA}}(\sigma)} \delta^{\#\mathrm{VA}(\sigma)} \xi^{\#\underline{\mathrm{VA}}(\sigma)} \eta^{\#\mathrm{PE}(\sigma)} \\ & \times a^{\mathrm{lsg}(\mathrm{DD})(\sigma)} b^{\mathrm{rsg}(\mathrm{DD})(\sigma)} c^{\mathrm{lsg}(\mathrm{DA})(\sigma)} d^{\mathrm{rsg}(\mathrm{DA})(\sigma)} f^{\mathrm{lsg}(\underline{\mathrm{DA}})(\sigma)} g^{\mathrm{rsg}(\underline{\mathrm{DA}})(\sigma)} \\ & \times s^{\mathrm{lsg}(\mathrm{VA})(\sigma)} t^{\mathrm{rsg}(\mathrm{VA})(\sigma)} x^{\mathrm{lsg}(\underline{\mathrm{VA}})(\sigma)} y^{\mathrm{rsg}(\underline{\mathrm{VA}})(\sigma)} v^{\mathrm{lsg}(\mathrm{PE})(\sigma)} w^{\mathrm{rsg}(\mathrm{PE})(\sigma)}.\end{aligned}$$

Our first result is an explicit continued fraction expansion for the ordinary generating function of μ_n.

Theorem 2.1. *We have the following continued fraction expansion*

$$1 + \sum_{n \geq 1} \mu_n z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{\ddots}}}, \tag{2.1}$$

where $b_n = \alpha[n+1]_{a,b} + \beta[n]_{c,d} + \gamma[n]_{f,g}$, $\quad \lambda_n = (\delta[n]_{s,t} + \xi[n]_{x,y})\, \eta[n]_{v,w}$.

In order to derive Euler-Mahonian statistics on ordered partitions we need the following lemma.

Lemma 2.2. *We have*

$$\sum_{n,k} [k]_q! S_q(n,k)\, u^k\, z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{\ddots}}}, \tag{2.2}$$

where $b_n = uq^n[n+1]_q + [n]_q(1 + uq^n)$, $\quad \lambda_n = uq^{n-1}[n]_q^2(1 + uq^n)$.

There are several possibilities to specialize the parameters in (2.1) to obtain (2.2). Clearly such a specialization will induce an Euler-Mahonian statistic on ordered partitions. For example, setting $\alpha = \eta = u$, $\gamma = \xi = u\,q$, $\beta = \delta = s = 1$, $a = f = x = v = q^2$ and $b = d = g = t = y = w = q$ in Theorem 2.1, the right-hand side of (2.1) reduces to that of (2.2). On the other hand, for $\sigma \in \underline{S_n}$, let

$$s(\sigma) = \#\underline{\mathrm{VA}}(\sigma) + \#\underline{\mathrm{DA}}(\sigma) + (2\,\mathrm{lsg} + \mathrm{rsg} - \mathrm{lsg}(\mathrm{DA}) - \mathrm{lsg}(\mathrm{VA}) - \mathrm{rsg}(\mathrm{DA}) - \mathrm{rsg}(\mathrm{VA}))(\sigma).$$

Then

$$\sum_{\pi \in \underline{S_n}} u^{n-\#\mathrm{DA}(\sigma)-\#\mathrm{VA}(\sigma)} q^{s(\sigma)} = \sum_{k=1}^{n} [k]_q! S_q(n,k) u^k. \tag{2.3}$$

As $n - \#\mathrm{DA}(\sigma) - \#\mathrm{VA}(\sigma)$ is the number of blocks of σ identified as an ordered partition, extracting the coefficients of u^k in (2.3) yields the following result.

Theorem 2.3. *We have*

$$\sum_{\pi \in \mathcal{OP}_n^k} q^{s(\sigma)} = [k]_q! S_q(n,k). \tag{2.4}$$

Remark 2.4. *By varying the coefficients* lsg *and* rsg *in the specialization we have* $2^6 = 64$ *statistics satisfying* (2.4)*, i.e.,* $\{a,b\} = \{f,g\} = \{x,y\} = \{v,w\} = \{q,q^2\}$ *and* $\{c,d\} = \{s,t\} = \{1,q\}$*.*

Given an ordered partition π in $\mathcal{OP}_n^k$, the elements of $[n]$ are divided into four classes:

- *singletons*: elements of the singleton blocks;
- *openers*: smallest elements of the non singleton blocks;
- *closers*: largest elements of the non singleton blocks;
- *transients*: all other elements, i.e., non extremal elements of non singleton blocks.

The sets of openers, closers, singletons and transients of π will be denoted by $\mathcal{O}(\pi)$, $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$ and $\mathcal{T}(\pi)$, respectively. The 4-tuple $(\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$ is called the *type* of π. For instance, if $\pi = 3\,5/2\,4\,6/1/7\,8$, then

$$\mathcal{O}(\pi) = \{2,3,7\}, \quad \mathcal{C}(\pi) = \{5,6,8\}, \quad \mathcal{S}(\pi) = \{1\} \quad \text{and} \quad \mathcal{T}(\pi) = \{4\}.$$

Let $\pi = B_1/B_2/\cdots/B_k$ be an ordered partition in $\mathcal{OP}_n^k$. We define a *partial order* on blocks B_i's as follows : $B_i > B_j$ if all the letters of B_i are greater than those of B_j; in other words, if the opener of B_i is greater than the closer of B_j.

Definition 2.5. *For a permutation σ of $[n]$, the pair (i,j) is an* inversion *if* $1 \le i < j \le n$ *and* $\sigma(i) > \sigma(j)$*. Let* $\operatorname{inv}\sigma$ *be the number of inversions in σ and*

$$\operatorname{cinv}\sigma = \binom{n}{2} - \operatorname{inv}\sigma.$$

By convention, for any ordered partition π, we put $\operatorname{inv}\pi = \operatorname{inv}(\operatorname{perm}(\pi))$ *and* $\operatorname{cinv}\pi = \operatorname{cinv}(\operatorname{perm}(\pi))$*.*

Definition 2.6. *A* block inversion *in π is a pair (i,j) such that $i < j$ and $B_i > B_j$. We denote by* $\operatorname{bInv}\pi$ *the number of block inversions in π. We also set* $\operatorname{cbInv} = \binom{k}{2} - \operatorname{bInv}$*.*

Let w_i be the index of the block (counting from left to right) containing i, namely the integer j such that $i \in B_j$. Following Steingrímsson [18], for $1 \le i \le n$ we define ten coordinate statistics on $\pi \in \mathcal{OP}_n^k$:

$$\begin{aligned}
\mathrm{ros}_i(\pi) &= \#\{j \in (\mathcal{O}\cup\mathcal{S})(\pi) \,|\, i > j,\, w_j > w_i\},\\
\mathrm{rob}_i(\pi) &= \#\{j \in (\mathcal{O}\cup\mathcal{S})(\pi) \,|\, i < j,\, w_j > w_i\},\\
\mathrm{rcs}_i(\pi) &= \#\{j \in (\mathcal{C}\cup\mathcal{S})(\pi) \,|\, i > j,\, w_j > w_i\},\\
\mathrm{rcb}_i(\pi) &= \#\{j \in (\mathcal{C}\cup\mathcal{S})(\pi) \,|\, i < j,\, w_j > w_i\},\\
\mathrm{los}_i(\pi) &= \#\{j \in (\mathcal{O}\cup\mathcal{S})(\pi) \,|\, i > j,\, w_j < w_i\},\\
\mathrm{lob}_i(\pi) &= \#\{j \in (\mathcal{O}\cup\mathcal{S})(\pi) \,|\, i < j,\, w_j < w_i\},\\
\mathrm{lcs}_i(\pi) &= \#\{j \in (\mathcal{C}\cup\mathcal{S})(\pi) \,|\, i > j,\, w_j < w_i\},\\
\mathrm{lcb}_i(\pi) &= \#\{j \in (\mathcal{C}\cup\mathcal{S})(\pi) \,|\, i < j,\, w_j < w_i\},
\end{aligned}$$

where $(\mathcal{O}\cup\mathcal{S})(\pi) = \mathcal{O}(\pi)\cup\mathcal{S}(\pi)$, and let $\mathrm{rsb}_i(\pi)$ (resp. $\mathrm{lsb}_i(\pi)$) be the number of blocks B in π to the right (resp. left) of the block containing i such that the opener of B is smaller than i and the closer of B is greater than i. Then define ros, rob, rcs, rcb, lob, los, lcs, lcb, lsb and rsb as the sum of their coordinate statistics, e.g.

$$\mathrm{ros} = \sum_i \mathrm{ros}_i\,.$$

Remark 2.7. *Note that* ros *is the abbreviation of "right, opener, smaller", while* lcb *is the abbreviation of "left, closer, bigger", etc.*

Given an ordered partition π, let π^r be the ordered partition obtained from π by reversing the order of the blocks. This turns a left (resp. right) opener into a right (resp. left) opener, and likewise for the closers. Moreover, let π^c be the ordered partition obtained by complementing each of the letters in π, that is, by replacing the letter i by $n+1-i$. Then, it is easy to see that $\mathrm{rob}(\pi^c) = \mathrm{rcs}(\pi)$ and $\mathrm{ros}(\pi^c) = \mathrm{rcb}(\pi)$, and likewise for the left and closer statistics. Thus the eight statistics obtained by independently varying left/right, opener/closer and smaller/bigger fall into only two categories when it comes to their distribution on ordered partitions. One of these categories consists of $\mathrm{rob}, \mathrm{lob}, \mathrm{rcs}$ and lcs, and the other contains $\mathrm{ros}, \mathrm{los}, \mathrm{rcb}$ and lcb. Note that these results are completely false on the unordered set partitions.

For instance, we give the values of the coordinate statistics computed on the ordered partition $\pi = 6\,8/5/1\,4\,7/3\,9/2$:

$\pi =$	6 8	/	5	/	1 4 7	/	3 9	/	2
los_i :	0 0	/	0	/	0 0 2	/	1 3	/	1
ros_i :	4 4	/	3	/	0 2 2	/	1 1	/	0
lob_i :	0 0	/	1	/	2 2 0	/	2 0	/	3
rob_i :	0 0	/	0	/	2 0 0	/	0 0	/	0
lcs_i :	0 0	/	0	/	0 0 1	/	0 3	/	0
rcs_i :	2 3	/	1	/	0 1 1	/	1 1	/	0
lcb_i :	0 0	/	1	/	2 2 1	/	3 0	/	4
rcb_i :	2 1	/	2	/	2 1 1	/	0 0	/	0
lsb_i :	0 0	/	0	/	0 0 1	/	1 0	/	1
rsb_i :	2 1	/	2	/	0 1 1	/	0 0	/	0

Note that there are four block inversions: $\{6,8\} > \{5\}$, $\{6,8\} > \{2\}$, $\{5\} > \{2\}$ and $\{3,9\} > \{2\}$, and two block descents at $i = 1$ and 4; thus $\mathrm{bInv}\ \pi = 4$ and $\mathrm{bmaj}\ \pi = 1 + 4 = 5$. Note also that $\mathrm{cbInv}\ \pi = \binom{4}{2} - 4 = 4$. Moreover, $perm(\pi) = 54132$ and thus $\mathrm{inv}(\pi) = 8$ and $\mathrm{cinv}(\pi) = \binom{5}{2} - 8 = 2$.

Inspired by a statistic mak due to Foata & Zeilberger [6] on the permutations, Steingrímsson introduced its analogous on $\mathcal{OP}_n^k$ as follows:

$$\mathrm{mak} = \mathrm{ros} + \mathrm{lcs}, \tag{2.5}$$
$$\mathrm{lmak} = n(k-1) - [\mathrm{los} + \mathrm{rcs}], \tag{2.6}$$
$$\mathrm{mak}' = \mathrm{lob} + \mathrm{rcb}, \tag{2.7}$$
$$\mathrm{lmak}' = n(k-1) - [\mathrm{lcb} + \mathrm{rob}]. \tag{2.8}$$

The following result due to Ksavrelof and Zeng [10] permits to reduce the original conjectures in [18] almost by half. For completeness, we include a more straightforward proof.

Proposition 2.8. *On $\mathcal{OP}_n^k$ the following functional identities hold:*

$$\mathrm{mak} = \mathrm{lmak}' \quad \textit{and} \quad \mathrm{mak}' = \mathrm{lmak}.$$

Proof. Let $i \in [n]$. For any $\pi \in \mathcal{OP}_n^k$ the value $(\mathrm{los}_i + \mathrm{lob}_i)\pi$ (resp. $(\mathrm{ros}_i + \mathrm{rob}_i)\pi$) is equal to the number of blocks on the left (resp. right) of the block containing i, so $(\mathrm{los}_i + \mathrm{lob}_i + \mathrm{ros}_i + \mathrm{rob}_i)\pi = k - 1$. Similarly, we have $(\mathrm{lcs}_i + \mathrm{lcb}_i + \mathrm{rcs}_i + \mathrm{rcb}_i)\pi = k - 1$. Summing over all $i \in [n]$ yields that $\mathrm{los} + \mathrm{lob} + \mathrm{ros} + \mathrm{rob} = \mathrm{lcs} + \mathrm{lcb} + \mathrm{rcs} + \mathrm{rcb} = n(k-1)$. The proposition is then equivalent to $\mathrm{lob} + \mathrm{los} = \mathrm{lcb} + \mathrm{lcs}$, which is obvious. □

Let $\mathcal{OP}^k$ be the set of all ordered partitions with k blocks. Define also the statistic cinvLSB on $\mathcal{OP}^k$ by

$$\mathrm{cinvLSB} := \mathrm{lsb} + \mathrm{cbInv} + \binom{k}{2}. \tag{2.9}$$

Consider the following two generating functions of ordered partitions with $k \geq 0$ blocks:

$$\phi_k(a; x, y, t, u) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{mak} + \mathrm{bInv})\pi} y^{\mathrm{cinvLSB}\ \pi} t^{\mathrm{inv}\ \pi} u^{\mathrm{cinv}\ \pi} a^{|\pi|}, \tag{2.10}$$
$$\varphi_k(a; x, y, t, u) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{lmak} + \mathrm{bInv})\pi} y^{\mathrm{cinvLSB}\ \pi} t^{\mathrm{inv}\ \pi} u^{\mathrm{cinv}\ \pi} a^{|\pi|}, \tag{2.11}$$

where $|\pi| = n$ if π is an ordered partition of $[n]$.

One of the main results of this paper is the following theorem, whose proof will be given in Sections 4 and 5.

Theorem 2.9. *We have*

$$\phi_k(a; x, y, t, u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{2.12}$$
$$\varphi_k(a; x, y, t, u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}. \tag{2.13}$$

We first show how to derive Euler-Mahonian statistics on ordered partitions from the above theorem.

By (2.10) and (2.11) we have

$$\sum_{\pi\in\mathcal{OP}^k} q^{(\mathrm{mak}\,+\,\mathrm{bInv})\pi}\, a^{|\pi|} = \phi_k(a;q,1,1,1),$$
$$\sum_{\pi\in\mathcal{OP}^k} q^{(\mathrm{lmak}\,+\,\mathrm{bInv})\pi}\, a^{|\pi|} = \varphi_k(a;q,1,1,1),$$
$$\sum_{\pi\in\mathcal{OP}^k} q^{\mathrm{cinvLSB}\;\pi}\, a^{|\pi|} = \phi_k(a;1,q,1,1) = \varphi_k(a;1,q,1,1).$$

Theorem 2.9 infers that the right-hand sides of the above three identities are all equal to

$$\frac{a^k\, q^{\binom{k}{2}}\,[k]_q!}{\prod_{i=1}^k(1-a[i]_q)} = \sum_{n\geq k}[k]_q!\widetilde{S}_q(n,k)\,a^n, \tag{2.14}$$

where the last equality follows directly from (1.4). Thus we have proved the following result, which was conjectured by Steingrímsson [18].

Theorem 2.10. *The following inversion-like statistics are Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak}+\mathrm{bInv}, \quad \mathrm{lmak}+\mathrm{bInv}, \quad \mathrm{cinvLSB}\,.$$

In other words, the generating functions of the above statistics over $\mathcal{OP}_n^k$ are all equal to $[k]_q!\widetilde{S}_q(n,k)$.

Similarly, we have

$$\sum_{\pi\in\mathcal{OP}^k} q^{(\mathrm{mak}\,+\,\mathrm{bInv}\,-\,\mathrm{inv}\,+\,\mathrm{cinv})\pi}\, a^{|\pi|} = \phi_k(a;q,1,1/q,q),$$
$$\sum_{\pi\in\mathcal{OP}^k} q^{(\mathrm{lmak}\,+\,\mathrm{bInv}\,-\,\mathrm{inv}\,+\,\mathrm{cinv})\pi}\, a^{|\pi|} = \varphi_k(a;q,1,1/q,q),$$
$$\sum_{\pi\in\mathcal{OP}^k} q^{(\mathrm{cinvLSB}\,+\,\mathrm{inv}\,-\,\mathrm{cinv})\,\pi}\, a^{|\pi|} = \phi_k(a;1,q,q,1/q) = \varphi_k(a;1,q,q,1/q),$$

and the three right-hand sides are all equal to (2.14). In other word, we have the following result.

Theorem 2.11. *The following statistics are Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak}+\mathrm{bInv}-(\mathrm{inv}-\mathrm{cinv}), \quad \mathrm{lmak}+\mathrm{bInv}-(\mathrm{inv}-\mathrm{cinv}), \quad \mathrm{cinvLSB}+(\mathrm{inv}-\mathrm{cinv}).$$

As a further consequence of Theorem 2.9, we can give an alternative proof of the following "hard" combinatorial interpretations for q-Stirling numbers of the second kind, where the first two interpretations were proved by Ksavrelof and Zeng [10] and the third interpretation was first proved by Stanton (see [20]).

Corollary 2.12. *We have*

$$\widetilde{S}_q(n,k) = \sum_{\pi\in\mathcal{P}_n^k} q^{\operatorname{mak}\pi} = \sum_{\pi\in\mathcal{P}_n^k} q^{\operatorname{lmak}\pi} = \sum_{\pi\in\mathcal{P}_n^k} q^{\operatorname{lsb}\pi+\binom{k}{2}}.$$

Proof. An unordered partition can be identified with an ordered partition without inversion, i.e., $\mathcal{P}^k = \{\pi \in \mathcal{OP}^k \mid \operatorname{inv}\pi = 0\}$. Since the statistic bInv vanishes on $\mathcal{P}^k$, we derive from (2.10) and (2.11) that

$$\sum_{\pi\in\mathcal{P}^k} q^{\operatorname{mak}\pi} a^{|\pi|} = \phi_k(a;q,1,0,1),$$
$$\sum_{\pi\in\mathcal{P}^k} q^{\operatorname{lmak}\pi} a^{|\pi|} = \varphi_k(a;q,1,0,1).$$

Now, by definition (2.9), the identity cinvLSB $=$ lsb $+2\binom{k}{2}$ holds on $\mathcal{P}^k$. It then follows from (2.10) that:

$$\sum_{\pi\in\mathcal{P}^k} q^{\binom{k}{2}+\operatorname{lsb}\pi} a^{|\pi|} = q^{-\binom{k}{2}}\phi_k(a;1,q,0,1).$$

Now, applying Theorem 2.9 we see that the right-hand sides of the above three identities are equal to $\sum_{n\geq k}\widetilde{S}_q(n,k)\,a^n$ in view of (2.14). ☐

3. Ordered partitions and Motzkin paths

3.1. **Encoding ordered partitions by weighted Motzkin paths.** Recall that a Motzkin path of length n is a path $\gamma = (\gamma_0,\gamma_1,\ldots,\gamma_n)$ from $\gamma_0 = (0,0)$ to $\gamma_n = (n,0)$ in the first quadrant with the unit step $\gamma_i - \gamma_{i-1} = (1,0)$, $(1,1)$ or $(1,-1)$, which is called East, North-east and South-east respectively. For the convenience, we will distinguish three kinds of East steps: $\mathbf{E}$, $\overline{\mathbf{E}}$ and $\widetilde{\mathbf{E}}$, and two kinds of North-east steps: $\mathbf{NE}$ and $\overline{\mathbf{NE}}$. The weight of a step at height h is a_h (resp. $\bar{a}_h$, c_h, b'_h, b''_h and $b_h{}'''$) if it is $\mathbf{NE}$ (resp. $\overline{\mathbf{NE}}$, $\mathbf{SE}$, $\mathbf{E}$, $\overline{\mathbf{E}}$ and $\widetilde{\mathbf{E}}$).

There is a classical bijection due to Françon and Viennot [7] between the symmetric group S_n and the set Γ_n of weighted Motzkin paths of length n, that we can adapt to give a bijection between $\underline{S_n}$ and Γ_n (see also [19, II-38]).

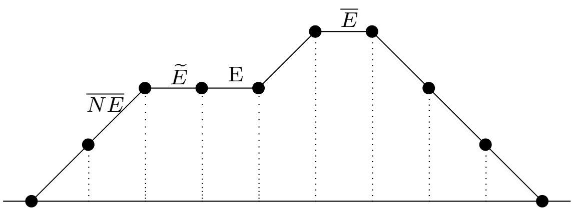
Consider the weighted Motzkin paths with six kinds of steps and weights defined by the following array:

step	NE	$\overline{NE}$	E	$\overline{E}$	$\widetilde{E}$	SE
weight	$a_h = h+1$	$\bar{a}_h = h+1$	$b_h = h$	$\bar{b}_h = h$	$\widetilde{b}_h = h+1$	$c_h = h$

A *Meixner diagram* of length n is a pair $h = (\omega,\xi)$, where ω is a Motzkin path of length n and $\xi = (\xi_i)_{1\leq i\leq n}$ is a sequence of integers such that if the i-th step of ω is of height h then :

$$\begin{aligned} 1 \leq \xi_i \leq h+1, &\qquad \text{if the step is NE, } \overline{NE} \text{ or } \widetilde{E},\\ 2 \leq \xi_i \leq h+1, &\qquad \text{if the step is E, } \overline{E} \text{ or SE.}\end{aligned}$$

We denote by Γ_n the set of Meixner diagrams of length n.



i	1	2	3	4	5	6	7	8	9
$h_i + 1$	1	2	3	3	3	4	4	3	2
ξ_i	1	1	2	3	3	4	4	3	2

FIGURE 1. A Meixner diagram of length 9

Let $h = (\omega, \xi) \in \Gamma_n$. We construct a sequence of words w_i on the alphabet $\{-\} \cup [i]$, $i = 0 \ldots n$, satisfying the following conditions:

(a) for $i \in [n]$, the number of occurrences of $-$ in w_{i-1} is equal to $h_i + 1$, where h_i is the height of the i-th step of ω,

(b) for each i, $0 \le i \le n$, the first letter of the word w_i is the symbol $-$,

and defined by induction as follows:

(1) $w_0 = -$,

(2) for $i \in [n]$, the word w_i is obtained from w_{i-1} by replacing the ξ_i-th occurrences of $-$ (from left to right) in w_{i-1} by i (resp. $-i-$, $-\underline{i}-$, $i-$, $\underline{i}-$, $-i$) if the i-th step of ω is SE (resp. NE, $\overline{NE}$, E, $\overline{E}$, $\widetilde{E}$).

It is clear that w_0 satisfy (a) and (b). Now, let $i \in [n]$ and suppose that $w_0, \ldots, w_{i-1}$ are defined and satisfied the conditions (a) and (b). The definition of a path diagram and the condition (a) allows us to construct w_i thanks to the condition (2). The step (2) shows also that w_i satisfy the conditions (a) and (b). Then, the word w_n has only one occurrence of $-$ in the beginning. We then replace it by $n+1$. We can identify such permutations with underlined permutations on $[n]$ by deleting the first letter.

For instance, if h is the Meixner diagram in Figure 1, the step-by-step construction of $\phi(h)$ goes as follows:

$$w_0 = -, \quad w_1 = -1-, \quad w_2 = -\underline{2} - 1-, \quad w_3 = -\underline{2} - 3\,1-, \quad w_4 = -\underline{2} - 3\,1\,4-,$$
$$w_5 = -\underline{2} - 3\,1\,4 - 5-, \quad w_6 = -\underline{2} - 3\,1\,4 - 5\,\underline{6}-, \quad w_7 = -\underline{2} - 3\,1\,4 - 5\,\underline{6}\,7,$$
$$w_8 = -\underline{2} - 3\,1\,4\,8\,5\,\underline{6}\,7, \quad w_9 = -\underline{2}\,9\,3\,1\,4\,8\,5\,\underline{6}\,7.$$

Hence $\phi(h) = \underline{2}\,9\,3\,1\,4\,8\,5\,\underline{6}\,7$.

As in [15, 19, 3], we have the following result. The proof is left to the reader.

Theorem 3.1. *The application ϕ is a bijection between Γ_n and $\underline{S_n}$. Moreover if $h = (\omega, \xi) \in \Gamma_n$ and $\sigma = \phi(h) \in \underline{S_n}$, for $1 \le i \le n$:*

(a) *$i \in \mathrm{VA}(\sigma)$ (resp. $\underline{\mathrm{VA}}(\sigma)$, $\mathrm{DA}(\sigma)$, $\underline{\mathrm{DA}}(\sigma)$, $\mathrm{DD}(\sigma)$, $\mathrm{PE}(\sigma)$) if and only if the i-th step of ω is NE (resp. $\overline{NE}$, E, $\overline{E}$, $\widetilde{E}$, SE).*

(b) $\mathrm{rsg}_i(\sigma) = h_i + 1 - \xi_i$ *and*

$$\begin{cases} \mathrm{lsg}_i(\sigma) = \xi_i - 1, & \text{if the } i\text{-th step is NE, } \overline{NE} \text{ or } \widetilde{E}; \\ \mathrm{lsg}_i(\sigma) = \xi_i - 2, & \text{if the } i\text{-th step is E, } \overline{E} \text{ or SE.} \end{cases}$$

Theorem 2.1 follows then immediately from Theorem 3.1 by applying Flajolet's combinatorial theorem of continued fractions [4].

3.2. **A continued fraction expansion.** The recurrence (1.3) is equivalent to

$$\sum_{n\geq k} S_q(n,k)z^n = \frac{z^k}{(1-z)(1-[2]_q z)\cdots(1-[k]_q z)}.$$

Let $[b;n]_q! = b(b+q)\cdots(b+q+\ldots+q^{n-1})$. Then

$$\sum_{n,k}[b;k]_q! S_q(n,k)u^k z^n = \sum_{k\geq 0}\frac{[b;k]_q! u^k z^k}{(1-z)(1-[2]_q z)\cdots(1-[k]_q z)}. \tag{3.1}$$

Using the method in [24], we can derive an explicit continued fraction expansion of the last expression.

Theorem 3.2. *We have*

$$\sum_{n,k}[b;k]_q! S_q(n,k)\, u^k\, z^n = \cfrac{1}{1-b_0\, z-\cfrac{\lambda_1\, z^2}{1-b_1\, z-\cfrac{\lambda_2\, z^2}{\ddots}}},$$

where

$$\begin{cases} b_n &= uq^n(b+q+\cdots q^n)+[n]_q(1+uq^n), \quad n\geq 0, \\ \lambda_n &= uq^{n-1}(b+q+\cdots q^{n-1})[n]_q(1+uq^n), \quad n\geq 1. \end{cases} \tag{3.2}$$

Proof. Let $f(b,z) := \sum_{n,k}[b;k]_q! S_q(n,k)\, u^k\, z^n$. Then it is easy to see that f satisfies the following functional equation:

$$f(b,z) = 1+\frac{ubz}{1-z} f\left(\frac{b}{q}+1,\ \frac{qz}{1-z}\right). \tag{3.3}$$

Suppose that

$$f(b,z) = \cfrac{1}{1-\cfrac{c_1(b)\, z}{1-\cfrac{c_2(b)\, z}{\ddots}}}.$$

By contraction from the 1st row we have

$$f(b,z) = 1+\cfrac{c_1(b)\, z}{1-(c_1(b)+c_2(b))z-\cfrac{c_2(b)c_3(b)\, z^2}{\ddots}}.$$

By contraction from the second row we have

$$f(b,z) = \cfrac{1}{1-c_1(b)\, z-\cfrac{c_1(b)c_2(b)\, z^2}{1-(c_2(b)+c_3(b))\, z-\cfrac{c_3(b)c_4(b)\, z^2}{\ddots}}}. \tag{3.4}$$

It follows that

$$\frac{ubz}{1-z} f\left(\frac{b}{q}+1,\, \frac{qz}{1-z}\right) = \cfrac{ubz}{1-(qc_1(b/q+1)+1)\,z - \cfrac{c_1(b/q+1)c_2(b/q+1)q^2\,z^2}{\ddots}}.$$

Hence

$$\begin{aligned}
c_1(b) &= u\,b,\\
c_1(b)+c_2(b) &= qc_1(b/q+1)+1 \Longrightarrow c_2(b) = u\,q+1,\\
c_2(b)c_3(b) &= c_1(b/q+1)c_2(b/q+1)q^2 \Longrightarrow c_3(b) = u(b+q)q,\\
c_3(b)+c_4(b) &= 1+qc_2(b/q+1)+qc_3(b/q+1) \Longrightarrow c_4(b) = (1+q)(1+uq^2),\\
c_4(b)c_5(b) &= q^2c_3(b/q+1)c_4(b/q+1) \Longrightarrow c_5(b) = uq^2(b+q+q^2).
\end{aligned}$$

This infers immediately by induction that

$$\begin{aligned}
c_{2i+1}(b) &= uq^i(b+q+\cdots q^i), \qquad i \geq 0,\\
c_{2i}(b) &= [i]_q(1+uq^i), \qquad i \geq 1.
\end{aligned}$$

Substituting these in (3.4) yields the desired result.

□

Note that when $b=1$ Theorem 3.2 reduces to Lemma 1.

4. Ordered partitions and walks in digraphs

4.1. **Encoding ordered partitions by walk diagrams.** Let $\pi = B_1/B_2/\cdots/B_k$ be an ordered partition of $[n]$ and i an integer in $[n]$. The restriction $B_j \cap [i]$ of a block B_j on $[i]$ is said to be *active* if $B_j \neq [i]$ and $B_j \cap [i] \neq \emptyset$, *complete* if $B_j \subseteq [i]$. We define the *trace* of the ordered partition π on $[i]$ as an ordered partition $T_i(\pi)$ on $[i]$ with two kinds of blocks, active or complete, i.e.,

$$T_i(\pi) = B_1(\leq i)/B_2(\leq i)/\cdots/B_k(\leq i),$$

where $B_j(\leq i)$ is the complete or active block $B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the *trace* of the ordered partition π.

Definition 4.1. *Let $D=(V,E)$ be the digraph on $V=\mathbb{N}^2$ with edge set E defined by*

$$E = \{(u,v) \in V^2 \mid u = v = (x,y) \text{ with } y>0 \quad \text{or} \quad u-v = (0,1),\ (1,0),\ (1,-1)\}.$$

For any integer $k \geq 0$, let $V_k = \{(i,j) \in V | i+j \leq k\}$ and D_k be the restriction of the digraph D on V_k.

An illustration of D_k is given in Figure 2.

Clearly there are four types of edges in D. For convenience, an edge (u,v) is called:

- *North* if $v = u+(0,1)$;
- *East* if $v = u+(1,0)$;
- *South-East* if $v = u+(1,-1)$;
- *Null* if $v = u$.

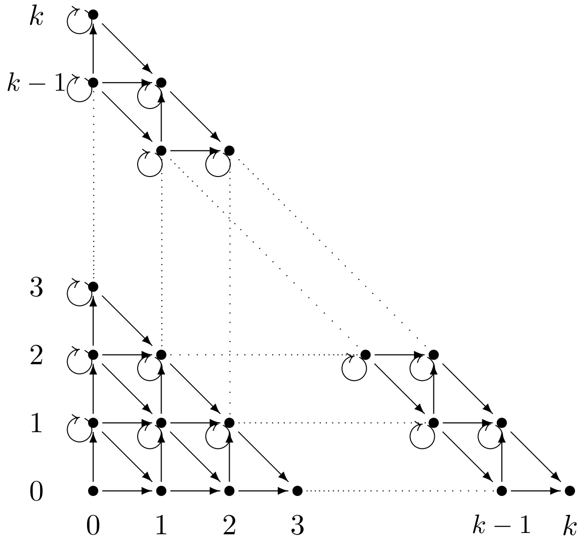


FIGURE 2. The digraph D_k

Definition 4.2. *A walk of depth k and length n in D is a sequence $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ of vertices in D such that $\omega_0 = (0,0)$, $\omega_n = (k,0)$ and (ω_i, ω_{i+1}) is an edge of D for $i = 0, \ldots, n-1$. Moreover, the* abscissa *and* ordinate *of ω_i are called the* abscissa *and* height *of the step (ω_i, ω_{i+1}), respectively. Let Ω_n^k be the set of walks of depth k and length n and Ω^k be the set of walks of depth k.*

We can visualize a walk ω by drawing a segment from ω_{i-1} to ω_i in the xy plane. For instance, if

$$\omega = ((0,0),\ (0,1),\ (0,2),\ (0,3),\ (0,3),\ (0,3),\ (1,3),\ (2,2),\ (3,1),\ (4,1),\ (5,0)),$$

then the illustration is given in in Figure 3.

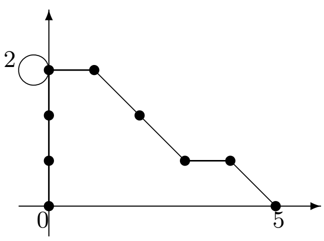


FIGURE 3. A walk in Ω_{10}^5 with two successive Null steps from $(0,3)$ to $(0,3)$.

Definition 4.3. *A walk diagram of depth k and length n is a pair (ω, ξ), where $\omega = ((p_i, q_i))_{0 \le i \le n}$ is a walk in Ω_n^k and $\xi = (\xi_i)_{1 \le i \le n}$ is a sequence of integers such that*

- $1 \le \xi_i \le q_{i-1}$ *if the i-th step of ω is Null or South-East,*
- $1 \le \xi_i \le p_{i-1} + q_{i-1} + 1$ *if the i-th step of ω is North or East.*

Denote by Δ_n^k the set of walk diagrams of depth k and length n and by $\Delta^k = \bigcup_{n \ge 0} \Delta_n^k$ the set of walk diagrams of depth k. The following is the main result of this section.

Theorem 4.4. *For each $n \geq k \geq 1$, there is a bijection $\psi : \Delta_n^k \to \mathcal{OP}_n^k$ such that if $\psi((\omega, \xi)) = \pi$ for $(\omega, \xi) \in \Delta_n^k$, then*

(a) *if the i-th step of ω is North (resp. East), then $i \in \mathcal{O}(\pi)$ (resp. $i \in \mathcal{S}(\pi)$) and*

$$(\mathrm{lcs}+\mathrm{rcs})_i(\pi) = p_{i-1}, \quad \mathrm{los}_i(\pi) = \xi_i - 1,$$
$$(\mathrm{lsb}+\mathrm{rsb})_i(\pi) = q_{i-1}, \quad \mathrm{ros}_i(\pi) = p_{i-1} + q_{i-1} + 1 - \xi_i;$$

(b) *if the i-th step of ω is South-East (resp. Null), then $i \in \mathcal{C}(\pi)$ (resp. $i \in \mathcal{T}(\pi)$) and*

$$(\mathrm{lcs}+\mathrm{rcs})_i(\pi) = p_{i-1}, \quad \mathrm{lsb}_i(\pi) = \xi_i - 1,$$
$$(\mathrm{lsb}+\mathrm{rsb})_i(\pi) = q_{i-1} - 1, \quad \mathrm{rsb}_i(\pi) = q_{i-1} - \xi_i.$$

Proof. Given a walk diagram $(\omega, \xi) \in \Delta_n^k$, we define the ordered partition $\pi = \psi((\omega, \xi)) \in \mathcal{OP}_n^k$ by constructing successively its traces $T_0 := \emptyset, T_1, \ldots, T_n := \pi$. For $1 \leq i \leq n$, assume that $T_{i-1} = B_{i1}/B_{i2}/\cdots/B_{il}$ with $l = p_{i-1} + q_{i-1}$. Then there are $l+1$ places to insert an active block with i as an opener or a singleton block $\{i\}$: before B_{i1}, between B_{ij} and $B_{i(j+1)}$, where $1 \leq j \leq l-1$, or after B_{il}. There are q_{i-1} places to insert i as a transient or closer: in any of the q_{i-1} active blocks. We label these places from left to right. We extend T_{i-1} to T_i as follows:

- if the i-th step of ω is North (resp. East), then we create an active block (resp. singleton) with i in T_{i-1} at the ξ_i-th place and close it if the step is East;
- if the i-th step of ω is Null (resp. South-East), then we insert i as a transient (resp. closer) in the ξ_i-th active block of T_{i-1}.

To show that ψ is bijective, we construct its inverse ϕ. Given an ordered partition π in $\mathcal{OP}_n^k$ we denote by $\mathrm{act}_i\, \pi$ and $\mathrm{com}_i\, \pi$ the numbers of active and complete blocks in $T_i(\pi)$, respectively. Let $\omega_0(\pi) = (0,0)$ and $\omega_i(\pi) = (\mathrm{com}_i\pi, \mathrm{act}_i\pi)$ for $1 \leq i \leq n$. If i is a singleton or opener, let ξ_i be the index (from left to right) of the block containing i in $T_i(\pi)$. If i is a transient or closer, let ξ_i be the index (from left to right) of the active block which will contain i in $T_i(\pi)$ among the active blocks in the trace $T_{i-1}(\pi)$. Let $\omega = (\omega_i(\pi))_{0 \leq i \leq n}$ and $\xi = (\xi_i)_{1 \leq i \leq n}$. It is readily seen that (ω, ξ) is a walk diagram in Ω_n^k and ϕ is the inverse of ψ.

It remains to check the other properties.

(a) If the i-th step of ω is North (resp. East), by construction, the element i will be an opener (resp. singleton) of the partitions T_j for $j \geq i$. Since $(\mathrm{lcs}+\mathrm{rcs})_i(\pi)$ (resp. $(\mathrm{lsb}+\mathrm{rsb})_i(\pi)$) is equal to the number of complete (resp. active) blocks in $T_{i-1}(\pi)$, we have

$$(\mathrm{lcs}+\mathrm{rcs})_i(\pi) = p_{i-1} \quad \text{and} \quad (\mathrm{lsb}+\mathrm{rsb})_i(\pi) = q_{i-1}.$$

Since $\mathrm{los}_i\, \pi$ (resp. $\mathrm{ros}_i\, \pi$) is the number of blocks in $T_i(\pi)$ on the left (resp. right) of the block containing i, we get $\mathrm{los}_i(\pi) = \xi_i - 1$ and $\mathrm{ros}_i(\pi) = p_{i-1} + q_{i-1} + 1 - \xi_i$.

(b) If the i-th step of ω is South-East (resp. Null), by construction, the element i is a closer (resp. transient) of the ordered partition T_j for $j \geq i$. As in (a), we have $(\mathrm{lcs}+\mathrm{rcs})_i(\pi) = p_{i-1}$. Note that $\mathrm{lsb}_i(\pi)$ (resp. $\mathrm{rsb}_i(\pi)$) is equal to the number of active blocks in $T_i(\pi)$ on the left (resp. on the right) of the block containing i, as we insert i in the ξ_i-th active block in $T_{i-1}(\pi)$ among q_{i-1} active blocks, we have $\mathrm{lsb}_i(\pi) = \xi_i - 1$ and $\mathrm{rsb}_i(\pi) = q_{i-1} - \xi_i$. Finally we have $(\mathrm{lsb} + \mathrm{rsb})_i\,(\pi) = (\xi_i - 1) + (q_{i-1} - \xi_i) = q_{i-1} - 1$.

□

Consider the walk diagram $h = (\omega, \xi) \in \Omega^5_{10}$, where ω is the walk in Figure 2 and $\xi = (1, 2, 1, 2, 1, 1, 1, 2, 4, 1)$, then the step by step construction of $\psi(h)$ goes as follows:

i	ω_i	step_i	ξ_i	T_i
0	$(0,0)$			$\emptyset$
1	$(0,1)$	*North*	1	$\{1,\cdots\}$
2	$(0,2)$	*North*	2	$\{1,\cdots\}-\{2,\cdots\}$
3	$(0,3)$	*North*	1	$\{3,\cdots\}-\{1,\cdots\}-\{2,\cdots\}$
4	$(0,3)$	*Null*	2	$\{3,\cdots\}-\{1,4,\cdots\}-\{2,\cdots\}$
5	$(0,3)$	*Null*	1	$\{3,5,\cdots\}-\{1,4,\cdots\}-\{2,\cdots\}$
6	$(1,3)$	*East*	1	$\{6\}-\{3,5,\cdots\}-\{1,4,\cdots\}-\{2,\cdots\}$
7	$(2,2)$	*South-East*	1	$\{6\}-\{3,5,7\}-\{1,4,\cdots\}-\{2,\cdots\}$
8	$(3,1)$	*South-East*	2	$\{6\}-\{3,5,7\}-\{1,4,\cdots\}-\{2,8\}$
9	$(4,1)$	*East*	4	$\{6\}-\{3,5,7\}-\{1,4,\cdots\}-\{9\}-\{2,8\}$
10	$(5,0)$	*South-East*	1	$\{6\}-\{3,5,7\}-\{1,4,10\}-\{9\}-\{2,8\}.$

Thus $\psi(h) = \{6\}/\{3,5,7\}/\{1,4,10\}/\{9\}/\{2,8\}$.

4.2. **Generating functions of walks.** Given a walk ω of finite length in D_k, define the valuation of a step (ω_i, ω_{i+1}) of abscissa p and height q by

$$\mathrm{v}(\omega_i, \omega_{i+1}) = \begin{cases} t_1^p\, t_7^q\, [p+q+1]_{t_5,t_6} & \text{if the step is North or East;} \\ t_2^p\, [q]_{t_3,t_4} & \text{if the step is Null or South-East.} \end{cases} \tag{4.1}$$

The valuation $\mathrm{v}(\omega)$ of ω is the product of the weights of all its steps.

Let $\boldsymbol{t} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7)$. For $k \geq 0$, define

$$Q_k(a; \boldsymbol{t}) := \sum_{w \in \Omega^k} \mathrm{v}(w)\, a^{|\omega|}, \tag{4.2}$$

where $|\omega|$ denotes the length of the walk ω.

By Theorem 4.4 one can rewrite the generating function $Q_k(a; \boldsymbol{t})$ of ordered partitions as

$$Q_k(a; \boldsymbol{t}) = \sum_{\pi \in \mathcal{OP}^k} t_1^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{O}\cup\mathcal{S})\pi}\, t_2^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{T}\cup\mathcal{C})\pi}\, t_3^{\mathrm{rsb}(\mathcal{T}\cup\mathcal{C})\pi} \times t_4^{\mathrm{lsb}(\mathcal{T}\cup\mathcal{C})\pi}\, t_5^{\mathrm{ros}(\mathcal{O}\cup\mathcal{S})\pi}\, t_6^{\mathrm{los}(\mathcal{O}\cup\mathcal{S})\pi}\, t_7^{(\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})\pi}\, a^{|\pi|}. \tag{4.3}$$

It is obvious that the number of vertices of D_k is equal to

$$\widehat{k} := 1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.$$

We label the vertices of D_k by their ranks in the following total ordering: $(i,j) \le (i',j')$ if and only if $i+j < i'+j'$ or ($i+j = i'+j'$ and $j \ge j'$). Thus $v_1 = (0,0)$, $v_2 = (0,1)$, $v_3 = (1,0)$, $v_4 = (0,2)$, $v_5 = (1,1)$, $v_6 = (2,0), \cdots, v_{\widehat{k}} = (k,0)$.

The *adjacency matrix* of D_k relative to the valuation v is the $\widehat{k} \times \widehat{k}$ matrix A_k defined by

$$A_k(i,j) = \begin{cases} \mathrm{v}(v_i, v_j) & \text{if } (v_i, v_j) \text{ is an edge of } D_k; \\ 0 & \text{otherwise.} \end{cases}$$

Applying the transfer-matrix method (see e.g. [17, Theorem 4.7.2]), we derive

$$Q_k(a;\boldsymbol{t}) = \frac{(-1)^{1+\widehat{k}} \det(I_k - aA_k; \widehat{k}, 1)}{\det(I_k - aA_k)}, \tag{4.4}$$

where $(B; i, j)$ denotes the matrix obtained by removing the i-th row and j-th column of B and I_k is the $\widehat{k} \times \widehat{k}$ identity matrix.

For instance, when $k = 2$, we have

$$A_2 = \left(\begin{array}{c|cc|ccc} 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & t_7\,[2]_{t_5,t_6} & t_7\,[2]_{t_5,t_6} & 0 \\ 0 & 0 & 0 & 0 & t_1\,[2]_{t_5,t_6} & t_1\,[2]_{t_5,t_6} \\ \hline 0 & 0 & 0 & [2]_{t_3,t_4} & [2]_{t_3,t_4} & 0 \\ 0 & 0 & 0 & 0 & t_2 & t_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and

$$Q_2(a;\boldsymbol{t}) = -\frac{\det(I_2 - aA_2; 6, 1)}{\det(I_2 - aA_2)} = \frac{a^2[2]_{t_5,t_6}(at_2t_7 + t_1(1 - a[2]_{t_3,t_4}))}{(1-a)(1-a[2]_{t_3,t_4})(1-at_2)}.$$

In order to prove Theorem 2.9 it is sufficient to evaluate the following special cases of $Q_k(a;\boldsymbol{t})$:

$$f_k(a;x,y,t,u) = Q_k(a;x,x,x,y,t,u,y), \tag{4.5}$$

$$g_k(a;x,y,t,u) = Q_k(a;1,x,1,xy,t,u,y). \tag{4.6}$$

Invoking (4.3), we have the combinatorial interpretations:

$$f_k(a;x,y,t,u) = \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{lcs} + \mathrm{rcs} + \mathrm{rsb})(\mathcal{T} \cup \mathcal{C})\,\pi} y^{((\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}))\,\pi} t^{\mathrm{inv}\,\pi} u^{\mathrm{cinv}\,\pi} a^{|\pi|},$$

$$g_k(a;x,y,t,u) = \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{lcs} + \mathrm{rcs} + \mathrm{lsb})(\mathcal{T} \cup \mathcal{C})\pi} y^{((\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}))\,\pi} t^{\mathrm{inv}\,\pi} u^{\mathrm{cinv}\,\pi} a^{|\pi|}.$$

In the next section, we shall prove the following result.

Theorem 4.5. *For $k \geq 1$, we have*

$$f_k(a; x, y, t, u) = \frac{a^k x^{\binom{k}{2}} [k]_{t,u}!}{\prod_{i=1}^{k}(1 - a[i]_{x,y})}, \tag{4.7}$$

$$g_k(a; x, y, t, u) = \frac{a^k \, [k]_{t,u}!}{\prod_{i=1}^{k}(1 - ax^{k-i}[i]_{xy})}. \tag{4.8}$$

The f_k's and g_k's are closely related to the ϕ_k's and φ_k's, respectively. To explain a relation between these generating functions, we need the following lemma.

Lemma 4.6. *The following functional identities hold on $\mathcal{OP}_n^k$:*

$$\begin{aligned}
\mathrm{mak} + \mathrm{bInv} &= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{rsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv},\\
\mathrm{lmak} + \mathrm{bInv} &= n(k-1) - (\mathrm{lcs} + \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) - \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) - \mathrm{cinv},\\
\mathrm{cinvLSB} &= (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv} + 2\ \mathrm{cinv}.
\end{aligned}$$

Proof. Since $\mathrm{bInv} = \mathrm{rcs}(\mathcal{O} \cup \mathcal{S})$ and $\mathrm{inv} = \mathrm{ros}(\mathcal{O} \cup \mathcal{S})$ we can rewrite

$$\begin{aligned}
\mathrm{mak} + \mathrm{bInv} &= \mathrm{lcs} + \mathrm{ros} + \mathrm{rcs}(\mathcal{O} \cup \mathcal{S})\\
&= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{ros} + (\mathrm{rcs}(\mathcal{O} \cup \mathcal{S}) - \mathrm{rcs})\\
&= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{ros} - \mathrm{rcs}(\mathcal{T} \cup \mathcal{C})\\
&= (\mathrm{lcs} + \mathrm{rcs}) + (\mathrm{ros} - \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) + \mathrm{ros}(\mathcal{O} \cup \mathcal{S})\\
&= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{rsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{cinvLSB} &= k(k-1) + \mathrm{lsb} - \mathrm{bInv}\\
&= 2(\mathrm{inv} + \mathrm{cinv}) + \mathrm{lsb} - \mathrm{rcs}(\mathcal{O} \cup \mathcal{S})\\
&= 2(\mathrm{inv} + \mathrm{cinv}) + (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S})\\
&\qquad - (\mathrm{rcs} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C})\\
&= 2(\mathrm{inv} + \mathrm{cinv}) + (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) - \mathrm{ros}(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C})\\
&= (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv} + 2\ \mathrm{cinv}.
\end{aligned}$$

Invoking (2.6) we have

$$\begin{aligned}
n(k-1) - (\mathrm{lmak} + \mathrm{bInv}) &= (\mathrm{los} + \mathrm{rcs}) - \mathrm{rcs}(\mathcal{O} \cup \mathcal{S})\\
&= \mathrm{los} + \mathrm{rcs}(\mathcal{T} \cup \mathcal{C})\\
&= \mathrm{los}(\mathcal{O} \cup \mathcal{S}) + \mathrm{rcs}(\mathcal{T} \cup \mathcal{C}) + \mathrm{los}(\mathcal{T} \cup \mathcal{C})\\
&= \mathrm{cinv} + (\mathrm{lcs} + \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) + (\mathrm{los} - \mathrm{lcs})(\mathcal{T} \cup \mathcal{C})\\
&= \mathrm{cinv} + (\mathrm{lcs} + \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}),
\end{aligned}$$

This completes the proof of Lemma 4.6. $\square$

The following lemma is an immediate consequence of Lemma 4.6.

Lemma 4.7. *For $k \geq 1$, we have*

$$\phi_k(a; x, y, t, u) = f_k(a; x, y, txy, uy^2), \tag{4.9}$$
$$\varphi_k(a; x, y, t, u) = g_k(ax^{k-1}; 1/x, y, ty, uy^2/x). \tag{4.10}$$

Theorem 2.9 follows immediately from Theorem 4.5 and Lemma 4.7.

5. Determinantal computations

Let A'_k and A''_k be the matrices obtained from A_k by making the substitutions corresponding to (4.5) and (4.6), respectively. Now, for each $k \geq 0$, let

$$M_k = I_k - aA'_k \quad \text{and} \quad N_k = I_k - aA''_k.$$

Then we derive from (4.4), (4.5) and (4.6) that for each $k \geq 1$,

$$f_k(a; x, y, t, u) = \frac{(-1)^{1+\widehat{k}} \det(M_k; \widehat{k}, 1)}{\det M_k}, \tag{5.1}$$
$$g_k(a; x, y, t, u) = \frac{(-1)^{1+\widehat{k}} \det(N_k; \widehat{k}, 1)}{\det N_k}. \tag{5.2}$$

It is not difficult to see that M_n and N_n are upper triangular matrices and to obtain then that for each $n \geq 1$

$$\det M_n = \prod_{m=1}^{n} \prod_{i=0}^{m} (1 - ax^i[m-i]_{x,y}), \tag{5.3}$$
$$\det N_n = \prod_{m=1}^{n} \prod_{k=0}^{n-m} (1 - ax^k[m]_{xy}). \tag{5.4}$$

The evaluation of $\det(M_n; \widehat{n}, 1)$ and $\det(N_n; \widehat{n}, 1)$ is highly non-trivial.

Theorem 5.1. *Let $n \geq 1$ be a positive integer. Then*

$$\det(M_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n x^{\binom{n}{2}} [n]_{t,u}! \prod_{m=1}^{n-1} \prod_{i=1}^{m} (1 - ax^i[m-i+1]_{x,y}), \tag{5.5}$$

Theorem 5.2. *Let $n \geq 1$ be a positive integer. Then*

$$\det(N_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n \, [n]_{t,u}! \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} (1 - ax^{k-1}[m]_{xy}). \tag{5.6}$$

Note that in view of (5.1) and (5.2), the above results complete the proof of Theorem 4.5.

5.1. **Proof of Theorem 5.1.** By definition of the matrix M_n we have

$$M_1 = \left(\begin{array}{c|cc} 1 & -a & -a \\ \hline 0 & 1-a & -a \\ 0 & 0 & 1 \end{array}\right)$$

and

$$M_2 = \left(\begin{array}{ccc|ccc} 1 & -a & -a & 0 & 0 & 0 \\ 0 & 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ 0 & 0 & 1 & 0 & -ax(t+u) & -ax(t+u) \\ \hline 0 & 0 & 0 & 1-a(x+y) & -a(x+y) & 0 \\ 0 & 0 & 0 & 0 & 1-ax & -ax \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Clearly the matrix M_n is of order $\hat{n}$ and can be defined recursively by

$$M_0 = (1)\,, \quad M_n = \left(\begin{array}{c|c} M_{n-1} & \overline{M}_{n-1} \\ \hline O_{n+1,\widehat{n-1}} & \widehat{M}_{n-1} \end{array}\right), \tag{5.7}$$

where $n \geq 1$,

$$\widehat{M}_{n-1} = \left(\delta_{ij} - ax^{i-1}[n+1-i]_{x,y}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i,j \leq n+1} \tag{5.8}$$

and $\overline{M}_{n-1}$ is the $\widehat{n-1} \times (n+1)$ matrix

$$\overline{M}_{n-1} = \left(\begin{array}{c} O_{\widehat{n-2},n+1} \\ \hline \check{M}_{n-1} \end{array}\right)$$

with the $n \times (n+1)$ matrix

$$\check{M}_{n-1} = \left(-ax^{i-1}y^{n-i}[n]_{t,u}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i \leq n,\, 1 \leq j \leq n+1}. \tag{5.9}$$

Here δ_{ij} stands for the Kronecker delta and $O_{m,n}$ denotes the $m \times n$ zero matrix.

Let

$$\mathrm{K}_n = \widehat{n} - 1 = \frac{n(n+3)}{2},$$

and let $P_n = (M_n; \widehat{n}, 1)$, i.e. the $\mathrm{K}_n \times \mathrm{K}_n$ matrix obtained from M_n by deleting the $\widehat{n}$th row and the first column. For instance, we have

$$P_1 = \begin{pmatrix} -a & -a \\ 1-a & -a \end{pmatrix}$$

and

$$P_2 = \left(\begin{array}{cc|ccc} -a & -a & 0 & 0 & 0 \\ 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ \hline 0 & 1 & 0 & -ax(t+u) & -ax(t+u) \\ 0 & 0 & 1-a(x+y) & -a(x+y) & 0 \\ 0 & 0 & 0 & 1-ax & -ax \end{array}\right).$$

In general we can define P_n as follows:

$$P_n = \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1} \end{array} \right),$$

where $\overline{P}_{n-1}$ is a $\mathrm{K}_{n-1} \times (n+1)$ matrix, X_{n-1} is a $(n+1) \times \mathrm{K}_{n-1}$ matrix, and $\widehat{P}_{n-1}$ is a $(n+1) \times (n+1)$ matrix. We shall compute $\det P_n$ by the following well-known formula for any block matrix with an invertible square matrix A,

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det A \cdot \det \left(D - CA^{-1}B \right).$$

Since the entries of $CA^{-1}B$ are also written by minors, we guess these entries and prove it by induction (see Theorem 5.3). Thus, looking at P_2 as the block matrix composed of P_1, X_1, $\overline{P}_1$ and $\widehat{P}_1$, we have

$$\overline{P}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -ay(t+u) & -ay(t+u) & 0 \end{pmatrix}$$

and

$$\widehat{P}_1 = \begin{pmatrix} 0 & -ax(t+u) & -ax(t+u) \\ 1-a(x+y) & -a(x+y) & 0 \\ 0 & 1-ax & -ax \end{pmatrix}.$$

Since $\overline{P}_n$ is an $\mathrm{K}_n \times (n+2)$ matrix, we can write

$$\overline{P}_n = \left(\frac{O_{\mathrm{K}_{n-1},n+2}}{U_n} \right),$$

where U_n is the $(n+1) \times (n+2)$ matrix composed of the last $(n+1)$ rows of $\overline{P}_n$. For $1 \leq k \leq n+2$, let

$$P_n^k = \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1}^k \end{array} \right)$$

denote the $\mathrm{K}_n \times \mathrm{K}_n$ matrix obtained from P_n by replacing the right-most column with the kth column of $\overline{P}_n$. Here $\widehat{P}_{n-1}^k$ is the $(n+1) \times (n+1)$ matrix obtained from $\widehat{P}_{n-1}$ by replacing the right-most column with the kth column of U_n. For example,

$$P_2^2 = \left(\begin{array}{cc|ccc} -a & -a & 0 & 0 & 0 \\ 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ \hline 0 & 1 & 0 & -ax(t+u) & 0 \\ 0 & 0 & 1-a(x+y) & -a(x+y) & -ay^2(t^2+tu+t^2) \\ 0 & 0 & 0 & 1-ax & -axy(t^2+tu+t^2) \end{array} \right).$$

Here is our key result.

Theorem 5.3. *Let $n \geq 1$ be a positive integer. Then we have*

$$\frac{\det P_n}{\det P_{n-1}} = (-1)^{n-1} a x^{n-1} [n]_{t,u} \prod_{i=1}^{n-1} (1 - a x^i [n-i]_{x,y}), \tag{5.10}$$

and

$$\frac{\det P_n^k}{\det P_n} = a\, x^{\frac{(k-1)(k-2)}{2} - \frac{n(n-1)}{2}}\, y^{\frac{(n+1-k)(n+2-k)}{2}}\, [n+1]_{t,u} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y} \tag{5.11}$$

for $1 \leq k \leq n$,

$$\frac{\det P_n^{n+1}}{\det P_n} = a\, y\, [n+1]_{t,u}\, [n]_{x,y} \tag{5.12}$$

and $\det P_n^{n+2} = 0$.

We first prove an elementary identity, which is obvious if it is written in a suitable manner.

Lemma 5.4. *For $0 \leq m \leq n$,*

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^{m-k} x^{\binom{k}{2}} y^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{x,y} \prod_{i=0}^{k-1} \left\{1 - a x^i [n-i]_{x,y}\right\} \prod_{i=k}^{m-1} \left\{-a x^i [n-i]_{x,y}\right\} \\ &\qquad = x^{\binom{m}{2}} y^{\binom{n-m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_{x,y} \prod_{i=1}^{m} \left\{1 - a x^i [n-i]_{x,y}\right\}. \end{aligned} \tag{5.13}$$

Proof. Note that

$$[n]_{x,y} = y^{n-1} [n]_{x/y}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{x,y} = y^{kn-k^2} \begin{bmatrix} n \\ k \end{bmatrix}_{x/y}.$$

Since $\binom{n-k}{2} + \binom{k}{2} + kn - k^2 = \binom{n}{2}$, setting $c = a y^{n-1}$ and $q = x/y$, we can rewrite (5.13) as follows:

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^{m-k} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} \left\{1 - c q^i [n-i]_q\right\} \prod_{i=k}^{m-1} \left\{-c q^i [n-i]_q\right\} \\ &\qquad = q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \prod_{i=1}^{m} \left\{1 - c q^i [n-i]_q\right\}. \end{aligned} \tag{5.14}$$

Setting

$$X = 1 + \frac{cq^n}{1-q}, \quad Y = \frac{c}{1 - q + cq^n}, \quad Z = \frac{cq^n}{1-q},$$

then $1-cq^i[n-i]_q = X(1-Yq^i)$ and $-cq^i[n-i]_q = Z(1-q^{i-n})$. Hence, in (5.14) making the following substitutions:

$$\prod_{i=0}^{k-1}\left\{1-cq^i[n-i]_q\right\} = X^k(Y;q)_k,$$
$$\prod_{i=1}^{m}\left\{1-cq^i[n-i]_q\right\} = X^m(Yq;q)_m,$$
$$\prod_{i=k}^{m-1}\left\{-cq^i[n-i]_q\right\} = (-1)^m Z^{m-k} q^{\binom{m}{2}-mn}\frac{(q^{n-m+1};q)_m}{(q^{-n};q)_k},$$

and writing the q-binomial coefficients as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (-1)^k q^{kn-k(k-1)/2}\frac{(q^{-n};q)_k}{(q;q)_k}, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = (-1)^m q^{mn-m(m-1)/2}\frac{(q^{-n};q)_m}{(q;q)_m},$$

we see, after simplifying, that identity (5.14) is equivalent to the special case $Y = c/(1-q+cq^n)$ of the identity:

$$\sum_{k=0}^{m}\frac{(Y;q)_k}{(q;q)_k}Y^{m-k} = \frac{(Yq;q)_m}{(q;q)_m},$$

which can be easily verified by induction. □

Proof of Theorem 5.3. We proceed by induction on n. When $n=1$, by a direct computation we obtain $\det P_1 = a$, $\det P_1^1 = \det P_1^2 = a^2\, y\, [2]_{t,u}$ and $\det P_1^3 = 0$. This shows the theorem is true when $n=1$. Let n be an integer ≥ 2. Assume the theorem is true for $n-1$.

(i) We get

$$\det P_n = \det\left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1}\end{array}\right) = \det P_{n-1}\cdot\det\left(\widehat{P}_{n-1} - X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}\right)$$

and

$$\det P_n^k = \det\left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1}^k\end{array}\right) = \det P_{n-1}\cdot\det\left(\widehat{P}_{n-1}^k - X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}\right).$$

(ii) By direct computation we can see that the (i,j)th entry of $X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ $(1 \leq i,j \leq n+1)$ is equal to

$$\begin{cases} \frac{\det P_{n-1}^j}{\det P_{n-1}} & \text{if } i=1, \\ 0 & \text{otherwise.}\end{cases}$$

By the induction hypothesis, the $(1,j)$th entry of $X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ equals

$$\begin{cases} a\,x^{\binom{j-1}{2}-\binom{n-1}{2}}\,y^{\binom{n+1-j}{2}}\,[n]_{t,u}\begin{bmatrix} n\\ j-1\end{bmatrix}_{x,y} & \text{if } 1\le j\le n-1,\\ a\,y\,[n-1]_{x,y}\,[n]_{t,u} & \text{if } j=n,\\ 0 & \text{if } j=n+1.\end{cases} \tag{5.15}$$

(iii) Put $W_{n-1}^{k}=\widehat{P}_{n-1}^{k}-X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ and $W_{n-1}=\widehat{P}_{n-1}-X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$. Then, by (i), we have $\frac{\det P_n^k}{\det P_{n-1}}=\det W_{n-1}^k$ and $\frac{\det P_n}{\det P_{n-1}}=\det W_{n-1}$. By (5.9) and (5.15), we can see that the $(1,j)$th entry of W_{n-1}^k is

$$-a\,x^{\binom{j-1}{2}-\binom{n-1}{2}}\,y^{\binom{n+1-j}{2}}[n]_{t,u}\begin{bmatrix} n\\ j-1\end{bmatrix}_{x,y}$$

for $1\le j\le n$, and the $(1,n+1)$th entry is 0 (the top row does not depend on k). It is also easy to see that the $(1,j)$th entry of W_{n-1} is

$$-a\,x^{\binom{j-1}{2}-\binom{n-1}{2}}\,y^{\binom{n+1-j}{2}}[n]_{t,u}\begin{bmatrix} n\\ j-1\end{bmatrix}_{x,y}$$

for $1\le j\le n+1$.

(iv) We claim that

$$\det W_{n-1}=(-1)^{n-1}ax^{n-1}[n]_{t,u}\prod_{i=1}^{n-1}(1-ax^{i}[n-i]_{x,y}).$$

In fact, the (i,j)th entry of W_{n-1} is

$$\begin{cases} -a\,y^{\frac{(n-j)(n-j+1)}{2}}\,x^{\frac{(j-1)(j-2)}{2}-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\begin{bmatrix} n\\ j-1\end{bmatrix}_{x,y} & \text{if } i=1 \text{ and } 1\le j\le n+1,\\ 1-ax^{j-1}[n+1-j]_{x,y} & \text{if } i=j+1 \text{ and } 1\le j\le n,\\ -ax^{j-2}[n+2-j]_{x,y} & \text{if } i=j \text{ and } 2\le j\le n+1,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, if we expand $\det W_{n-1}$ along the top row, then we obtain

$$\begin{aligned}\det W_{n-1}&=-ax^{-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\\ &\quad\times\sum_{j=1}^{n+1}(-1)^{j+1}y^{\frac{(n-j)(n-j+1)}{2}}x^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} n\\ j-1\end{bmatrix}_{x,y}\det W_{n-1}(1;j).\end{aligned}$$

If we use

$$\det W_{n-1}(1;j)=\prod_{\nu=0}^{j-2}(1-ax^{\nu}[n-\nu]_{x,y})\prod_{\nu=j-1}^{n-1}(-ax^{\nu}[n-\nu]_{x,y}),$$

then we obtain

$$\det W_{n-1} = -ax^{-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\sum_{j=1}^{n+1}(-1)^{j+1}y^{\frac{(n-j)(n-j+1)}{2}}x^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} n \\ j-1 \end{bmatrix}_{x,y}$$

$$\times\prod_{\nu=0}^{j-2}(1-ax^{\nu}[n-\nu]_{x,y})\prod_{\nu=j-1}^{n-1}(-ax^{\nu}[n-\nu]_{x,y})$$

$$= (-1)^{n-1}ax^{n-1}[n]_{t,u}\prod_{i=1}^{n-1}(1-ax^{i}[n-i]_{x,y})$$

by (5.13). Thus, by (i), we conclude that

$$\frac{\det P_n}{\det P_{n-1}} = \det W_{n-1} = (-1)^{n-1}ax^{n-1}[n]_{t,u}\prod_{i=1}^{n-1}(1-ax^{i}[n-i]_{x,y}). \tag{5.16}$$

(v) We claim that

$$\frac{\det P_n^k}{\det P_n} = \frac{\det W_{n-1}^k}{\det W_{n-1}} = a\,x^{\frac{(k-1)(k-2)}{2}-\frac{n(n-1)}{2}}\,y^{\frac{(n+1-k)(n+2-k)}{2}}\,[n+1]_{t,u}\begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y}$$

for $1 \le k \le n$. Because the rightmost column of $\widehat{P}_{n-1}^k$ is the kth column of U_n, we have the $(i, n+1)$th entry of $\widehat{P}_{n-1}^k$ is

$$\begin{cases} -ay^n[n+1]_{t,u} & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

when $k = 1$,

$$\begin{cases} -ay^{n+2-k}x^{k-2}[n+1]_{t,u} & \text{if } i = k, \\ -ay^{n+1-k}x^{k-1}[n+1]_{t,u} & \text{if } i = k+1, \\ 0 & \text{otherwise,} \end{cases}$$

when $2 \le k \le n$,

$$\begin{cases} -ayx^{n-1}[n+1]_{t,u} & \text{if } i = n+1, \\ 0 & \text{otherwise,} \end{cases}$$

when $k = n+1$, and all zero when $k = n+2$. By the induction hypothesis, the $(1, n+1)$th entry of $X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ is $\frac{\det P_{n-1}^{n+1}}{\det P_{n-1}} = 0$. Thus the $(n+1)$th column of $W_{n-1}^k = \widehat{P}_{n-1}^k - X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ equals the $(n+1)$th column of $\widehat{P}_{n-1}^k$.

(a) When $k = 1$, we expand $\det W_{n-1}^1$ along the $(n+1)$th column, then, by direct computation, we obtain

$$\det W_{n-1}^1 = (-1)^{n+3}(-ay^n[n+1]_{t,u})\det W_{n-1}(2; n+1)$$

By expanding $\det W_{n-1}^1(2; n+1)$ along the top row we obtain

$$\det W_{n-1}(2; n+1) = \left(-a\,y^{\frac{n(n-1)}{2}}\,x^{-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\right)\prod_{\nu=1}^{n-1}(1-ax^{\nu}[n-\nu]_{x,y}).$$

Thus we conclude that

$$\det W_{n-1}^1 = (-1)^{n-1} a^2 y^{\frac{n(n+1)}{2}} x^{-\frac{(n-1)(n-2)}{2}} [n]_{t,u} \times [n+1]_{t,u} \prod_{\nu=1}^{n-1} (1 - ax^\nu [n-\nu]_{x,y}). \tag{5.17}$$

By (5.16), this implies

$$\frac{\det W_{n-1}^1}{\det W_{n-1}} = ay^{\frac{n(n+1)}{2}} x^{-\frac{n(n-1)}{2}} [n+1]_{t,u}$$

which is the desired identity.

(b) When $2 \le k \le n$, we expand $\det W_{n-1}^k$ along the $(n+1)$th column, then we obtain

$$\begin{aligned} \det W_{n-1}^k &= (-1)^{k+n+1} (-ay^{n+2-k} x^{k-2} [n+1]_{t,u}) \det W_{n-1}(k; n+1) \\ &\quad + (-1)^{k+n+2} (-ay^{n+1-k} x^{k-1} [n+1]_{t,u}) \det W_{n-1}(k+1; n+1). \end{aligned} \tag{5.18}$$

By expanding along the top row, we obtain

$$\begin{aligned} \det W_{n-1}(k; n+1) &= -ax^{-\frac{(n-1)(n-2)}{2}} [n]_{t,u} \sum_{j=1}^{k-1} (-1)^{j+1} y^{\frac{(n-j)(n-j+1)}{2}} x^{\frac{(j-1)(j-2)}{2}} \\ &\quad \times \begin{bmatrix} n \\ j-1 \end{bmatrix}_{x,y} \det W_{n-1}(1, k; j, n+1), \\ \det W_{n-1}(k+1; n+1) &= -ax^{-\frac{(n-1)(n-2)}{2}} [n]_{t,u} \sum_{j=1}^{k} (-1)^{j+1} y^{\frac{(n-j)(n-j+1)}{2}} x^{\frac{(j-1)(j-2)}{2}} \\ &\quad \times \begin{bmatrix} n \\ j-1 \end{bmatrix}_{x,y} \det W_{n-1}(1, k+1; j, n+1), \end{aligned}$$

where $W_{n-1}(1, k; j, n+1) = W_{n-1}(k; n+1)(1; j)$ and $W_{n-1}(1, k+1; j, n+1) = W_{n-1}(k+1; n+1)(1; j)$. If we use

$$\begin{aligned} \det W_{n-1}(1, k; j, n+1) &= \prod_{\nu=0}^{j-2} (1 - ax^\nu [n-\nu]_{x,y}) \prod_{\nu=j-1}^{k-3} (-ax^\nu [n-\nu]_{x,y}) \\ &\quad \times \prod_{\nu=k-1}^{n-1} (1 - ax^\nu [n-\nu]_{x,y}), \\ \det W_{n-1}(1, k+1; j, n+1) &= \prod_{\nu=0}^{j-2} (1 - ax^\nu [n-\nu]_{x,y}) \prod_{\nu=j-1}^{k-2} (-ax^\nu [n-\nu]_{x,y}) \\ &\quad \times \prod_{\nu=k}^{n-1} (1 - ax^\nu [n-\nu]_{x,y}), \end{aligned}$$

then, by (5.13), we obtain

$$\det W_{n-1}(k;n+1) = (-1)^{k-1} a x^{\frac{(k-2)(k-3)}{2} - \frac{(n-1)(n-2)}{2}} y^{\frac{(n-k+2)(n-k+1)}{2}}$$
$$\times [n]_{t,u} \begin{bmatrix} n \\ k-2 \end{bmatrix}_{x,y} \prod_{\nu=1}^{n-1} (1 - a x^{\nu} [n-\nu]_{x,y}),$$
$$\det W_{n-1}(k+1;n+1) = (-1)^{k} a x^{\frac{(k-1)(k-2)}{2} - \frac{(n-1)(n-2)}{2}} y^{\frac{(n-k)(n-k+1)}{2}}$$
$$\times [n]_{t,u} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{x,y} \prod_{\nu=1}^{n-1} (1 - a x^{\nu} [n-\nu]_{x,y}).$$

Thus, from (5.18), we conclude that

$$\begin{aligned}\det W_{n-1}^{k} =&(-1)^{n-1} a^2 x^{-\frac{(n-1)(n-2)}{2} + \frac{(k-1)(k-2)}{2}} y^{\frac{(n+1-k)(n+2-k)}{2}} [n]_{t,u} [n+1]_{t,u} \\ &\times \left(y^{n-k+2} \begin{bmatrix} n \\ k-2 \end{bmatrix}_{x,y} + x^{k-1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{x,y} \right) \prod_{\nu=1}^{n-1} (1 - a x^{\nu} [n-\nu]_{x,y}) \\ &= (-1)^{n-1} a^2 x^{-\frac{(n-1)(n-2)}{2} + \frac{(k-1)(k-2)}{2}} y^{\frac{(n+1-k)(n+2-k)}{2}} \\ &\times [n]_{t,u} [n+1]_{t,u} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y} \prod_{\nu=1}^{n-1} (1 - a x^{\nu} [n-\nu]_{x,y}).\end{aligned}$$

Using (5.16), we obtain

$$\frac{\det W_{n-1}^{k}}{\det W_{n-1}} = a x^{-\frac{n(n-1)}{2} + \frac{(k-1)(k-2)}{2}} y^{\frac{(n+1-k)(n+2-k)}{2}} [n+1]_{t,u} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y},$$

which is the desired identity.

(c) When $k = n+1$, we also expand $\det W_{n-1}^{k}$ along the $(n+1)$th column and repeat the same argument. It is not hard to obtain

$$\frac{\det W_{n-1}^{n+1}}{\det W_{n-1}} = a y [n+1]_{t,u} [n]_{x,y}.$$

The details are left to the reader.

(d) When $k = n+2$, $\det \widehat{P}_{n-1}^{k}$ vanishes since all the entries of the last column of $\det \widehat{P}_{n-1}^{k}$ are zero.

This proves the theorem is true for n. By induction we conclude that the theorem is true for all $n \geq 1$. This completes the proof. □

Since $\det(P_1) = a$, Theorem 5.1 (5.5) follows easily from (5.10). □

5.2. **Proof of Theorem 5.2.** Let $F = \{F_n\}_{n=1}^{\infty}$ be a sequence of non-zero functions in finitely many variables $t_1, t_2, \ldots$. We use the convention that $F_n! = \prod_{k=1}^{n} F_k$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{cases} \frac{F_n!}{F_k! F_{n-k}!}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

We prove Theorem 5.2 (5.6) by considering the following matrix $N_n(t,a)$, which generalize the matrix N_n (set $\lambda=1$ and $F_n=[n]_{t,u}$ to obtain N_n). Let $N_n(\lambda,a)$ be the matrix defined inductively as follows:

$$N_0(\lambda,a)=(\lambda)$$

and

$$N_n(\lambda,a)=\left(\begin{array}{c|c} N_{n-1}(\lambda,a) & \overline{N}_{n-1}(\lambda,a) \\ \hline O_{n+1,\widehat{n-1}} & \widehat{N}_{n-1}(\lambda,a) \end{array}\right) \tag{5.19}$$

where $\widehat{N}_{n-1}(\lambda,a)$ is the $(n+1)\times(n+1)$ matrix defined by

$$\widehat{N}_{n-1}(\lambda,a)=\left(\lambda\delta_{ij}-ax^{i-1}[n+1-i]_{xy}(\delta_{ij}+\delta_{i+1,j})\right)_{1\le i,j\le n+1} \tag{5.20}$$

and $\overline{N}_{n-1}(\lambda,a)$ is the $\widehat{n-1}\times(n+1)$ matrix

$$\left(\begin{array}{c} O_{\widehat{n-2},n+1} \\ \hline \check{N}_{n-1} \end{array}\right)$$

with the $n\times(n+1)$ matrix

$$\check{N}_{n-1}=\left(-ay^{n-i}F_n\cdot(\delta_{ij}+\delta_{i+1,j})\right)_{1\le i\le n,\,1\le j\le n+1}. \tag{5.21}$$

For instance, we get

$$N_2(\lambda,a)=\begin{pmatrix} \lambda & -aF_1 & -aF_1 & 0 & 0 & 0 \\ 0 & \lambda-a & -a & -ayF_2 & -ayF_2 & 0 \\ 0 & 0 & \lambda & 0 & -aF_2 & -aF_2 \\ 0 & 0 & 0 & \lambda-a(1+xy) & -a(1+xy) & 0 \\ 0 & 0 & 0 & 0 & \lambda-ax & -ax \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Let $\dot{N}_n(\lambda,a)$ denote the matrix obtained from $N_n(\lambda,a)$ by deleting the $\widehat{n}$th row and the first column. Then the following theorem is sufficient to prove our result. Here our strategy is as follows. We regard $\det N_n(\lambda,a)$ as a polynomial in λ and find all linear factors. Finally we check the leading coefficient in the both sides.

Theorem 5.5. *We have*

$$\det\dot{N}_n(\lambda,a)=(-1)^{n(n-1)/2}a^n\,F_n!\,\lambda^n\prod_{m=1}^{n-1}\prod_{k=1}^{n-m}\left(\lambda-ax^{k-1}[m]_{xy}\right). \tag{5.22}$$

Then by setting $\lambda=1$ and $F_n=[n]_{t,u}$ we obtain Theorem 5.2 (5.6).

For instance, we have

$$\det \dot{N}_1(\lambda, a) = \det \begin{pmatrix} -aF_1 & -aF_1 \\ \lambda - a & -a \end{pmatrix} = a\, F_1\, \lambda$$

and

$$\begin{aligned} \det \dot{N}_2(\lambda, a) = \det & \begin{pmatrix} -aF_1 & -aF_1 & 0 & 0 & 0 \\ \lambda - a & -a & -ayF_2 & -ayF_2 & 0 \\ 0 & \lambda & 0 & -aF_2 & -aF_2 \\ 0 & 0 & \lambda - a(1+xy) & -a(1+xy) & 0 \\ 0 & 0 & 0 & \lambda - ax & -ax \end{pmatrix} \\ & = -a^2\, F_1 F_2\, \lambda^2 (\lambda - a). \end{aligned}$$

Fix positive integers m and k. Define the row vectors $\boldsymbol{X}_n^{m,k}$ of degree $\widehat{n}$ as follows: For $1 \leq i \leq n+1$ and $1 \leq j \leq i$, the $\left(\frac{i(i-1)}{2} + j\right)$th entry of $\boldsymbol{X}_n^{m,k}$ is equal to

$$X_{i,j}^{m,k} = (-1)^{i+m+k} x^{-(m+k-1)(i-m-k)+\binom{j-k}{2}} y^{\binom{i-m-k}{2}} \frac{F_{i-m-k}!}{[i-m-k]_{xy}!} \begin{bmatrix} i-1 \\ m+k-1 \end{bmatrix}_F \Bigg] \begin{matrix} m \\ m+k-j \end{matrix} \Bigg[_{x,y}. \tag{5.23}$$

Here we use the convention that $F_n! = [n]_{xy}! = 1$ if $n \leq 0$. For example, if $n = 3$, $m = k = 1$, then

$$\boldsymbol{X}_3^{1,1} = \left(0, 1, 1, -\frac{F_2}{x}, -\frac{F_2}{x}, 0, \frac{F_2 F_3\, y}{x^2 [2]_{xy}!}, \frac{F_2 F_3\, y}{x^2 [2]_{xy}!}, 0, 0\right).$$

Lemma 5.6. *Let n be a positive integer. Let m and k be positive integers such that $0 \leq m \leq n-1$ and $1 \leq k \leq n-m$. Then we have*

$$\boldsymbol{X}_n^{m,k}\, N_n(\lambda, a) = (\lambda - a x^{k-1} [m]_{xy})\, \boldsymbol{X}_n^{m,k}. \tag{5.24}$$

Before we proceed to the proof of the lemma, we see it in an example. If $n = 2$ and $m = k = 1$, then we have

$$\left(0, 1, 1, -\frac{F_2}{x}, -\frac{F_2}{x}, 0\right) \begin{pmatrix} \lambda & -F_1 & -F_1 & 0 & 0 & 0 \\ 0 & \lambda - a & -a & -ayF_2 & -ayF_2 & 0 \\ 0 & 0 & \lambda & 0 & -aF_2 & -aF_2 \\ 0 & 0 & 0 & \lambda - a(1+xy) & -a(1+xy) & 0 \\ 0 & 0 & 0 & 0 & \lambda - ax & -ax \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$
$$= \left(0, \lambda - a, \lambda - a, -\frac{F_2}{x}(\lambda - a), -\frac{F_2}{x}(\lambda - a), 0\right).$$

Proof of Lemma 5.6. We proceed by induction on n. When $n = 0$ or $n = 1$, our claim is easy to check by direct computation. Assume (5.24) is true up to $n - 1$. Then the first $\widehat{n-1}$ entries of $\boldsymbol{X}_n^{m,k} N_n(\lambda, a)$ agree with those of $(\lambda - x^{k-1}[m]_{xy})\boldsymbol{X}_n^{m,k}$ by the induction hypothesis. So we have to check the last $n + 1$ entries. In fact we verify the following three cases.

(i) If $i = n + 1$ and $j = 1$, then the $\left(\frac{n(n+1)}{2} + 1\right)$th entry of $\boldsymbol{X}_n^{m,k} N_n(\lambda, a)$ is equal to

$$\begin{aligned}
&(-ay^{n-1} F_n)X_{n,1}^{m,k} + (\lambda - a[n]_{xy})X_{n+1,1}^{m,k} \\
=&(-1)^{n+m+k} x^{-(m+k-1)(n-m-k)+\binom{1-k}{2}} y^{\binom{n-m-k}{2}} \\
&\quad \times \frac{F_{n-m-k}!}{[n-m-k]_{xy}!} \begin{bmatrix} n-1 \\ m+k-1 \end{bmatrix}_F \Big] \begin{matrix} m \\ m+k-1 \end{matrix} \Big[_{x,y} (-ay^{n-1} F_n) \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{1-k}{2}} y^{\binom{n-m-k+1}{2}} \\
&\quad \times \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \Big] \begin{matrix} m \\ m+k-1 \end{matrix} \Big[_{x,y} (\lambda - a[n]_{xy}).
\end{aligned}$$

Note that the coefficient $\Big] \begin{matrix} m \\ m+k-1 \end{matrix} \Big[_{x,y}$ becomes zero from definition unless $k = 1$. When $k = 1$, then this equals

$$\begin{aligned}
&(-1)^{n+m} x^{-m(n-m)} y^{\binom{n-m}{2}} \frac{F_{n-m}!}{[n-m]_{xy}!} \begin{bmatrix} n \\ m \end{bmatrix}_F \Big] \begin{matrix} m \\ m \end{matrix} \Big[_{x,y} \\
&\times \left\{ ax^m y^m [n-m]_{xy} + (\lambda - a[n]_{xy}) \right\}.
\end{aligned}$$

Thus, by direct computation, one can easily check that this sum equals

$$\begin{aligned}
&(-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{1-k}{2}} y^{\binom{n-m-k+1}{2}} \\
&\quad \times \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \Big] \begin{matrix} m \\ m+k-1 \end{matrix} \Big[_{x,y} (\lambda - ax^{k-1}[m]_{xy}).
\end{aligned}$$

(ii) If $i = n+1$ and $2 \leq j \leq n$, then the $\left(\frac{n(n+1)}{2} + j\right)$th entry of $\boldsymbol{X}_n^{m,k} N_n(\lambda, a)$ is equal to

$$
\begin{aligned}
&(-ay^{n-j+1}F_n)X_{n,j-1}^{m,k} + (-ay^{n-j}F_n)X_{n,j}^{m,k} \\
&\qquad + \left(-ax^{j-2}[n-j+2]_{xy}\right) X_{n+1,j-1}^{m,k} + \left(\lambda - ax^{j-1}[n-j+1]_{xy}\right) X_{n+1,j}^{m,k} \\
&= (-1)^{n+m+k} x^{-(m+k-1)(n-m-k)+\binom{j-k-1}{2}} y^{\binom{n-m-k}{2}} \frac{F_{n-m-k}!}{[n-m-k]_{xy}!} \\
&\qquad \times \begin{bmatrix} n-1 \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j+1 \end{matrix}\right[_{x,y} \left(-ay^{n-j+1}F_n\right) \\
&+ (-1)^{n+m+k} x^{-(m+k-1)(n-m-k)+\binom{j-k}{2}} y^{\binom{n-m-k}{2}} \frac{F_{n-m-k}!}{[n-m-k]_{xy}!} \\
&\qquad \times \begin{bmatrix} n-1 \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \left(-ay^{n-j}F_n\right) \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k-1}{2}} y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\qquad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j+1 \end{matrix}\right[_{x,y} \left(-ax^{j-2}[n-j+2]_{xy}\right) \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}} y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\qquad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \left(\lambda - ax^{j-1}[n-j+1]_{xy}\right).
\end{aligned}
$$

This equals

$$
\begin{aligned}
&(-1)^{n+m+k} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}}\, y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \frac{[m]_{xy} - x^{j-k}[m+k-j]_{xy}}{[m+k-j+1]_{xy}} \\
&\quad \times x^{m+k-1-(j-k-1)} y^{-(n-m-k)} \left(-a y^{n-j+1}\right) [n-m-k+1]_{xy} \\
&+ (-1)^{n+m+k} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}}\, y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \\
&\quad \times x^{m+k-1} y^{-(n-m-k)} \left(-a y^{n-j}\right) [n-m-k+1]_{xy} \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}}\, y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \frac{[m]_{xy} - x^{j-k}[m+k-j]_{xy}}{[m+k-j+1]_{xy}} \\
&\quad \times x^{-(j-k-1)} \left(-a x^{j-2}[n-j+2]_{xy}\right) \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}}\, y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \left(\lambda - a x^{j-1}[n-j+1]_{xy}\right)
\end{aligned}
$$

if $j-m \le k \le j$, and 0 otherwise. Thus, this becomes

$$
\begin{aligned}
&(-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}}\, y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n+1-m-k]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-j \end{matrix}\right[_{x,y} \\
&\quad \times \Bigg\{ a\, x^{m+2k-j} y^{m+k-j+1} \frac{[m]_{xy} - x^{j-k}[m+k-j]_{xy}}{[m+k-j+1]_{xy}} [n-m-k+1]_{xy} \\
&\qquad\qquad + a\, x^{m+k-1} y^{m+k-j} [n-m-k+1]_{xy} \\
&\qquad\qquad - a\, x^{k-1} \frac{[m]_{xy} - x^{j-k}[m+k-j]_{xy}}{[m+k-j+1]_{xy}} [n-j+2]_{xy} \\
&\qquad\qquad + \lambda - a\, x^{j-1}[n-j+1]_{xy} \Bigg\},
\end{aligned}
$$

which equals

$$
\begin{aligned}
&(-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}} y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n+1-m-k]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left] \begin{matrix} m \\ m+k-j \end{matrix} \right[_{x,y} \\
&\quad \times \Bigg\{ \lambda - a \Bigg(-x^{m+2k-j} y^{m+k-j+1} \frac{[m]_{xy} - x^{j-k}[m+k-j]_{xy}}{[m+k-j+1]_{xy}} [n-m-k+1]_{xy} \\
&\qquad - x^{m+k-1} y^{m+k-j} [n-m-k+1]_{xy} \\
&\qquad + x^{k-1} \frac{[m]_{xy} - x^{j-k}[m+k-j]_{xy}}{[m+k-j+1]_{xy}} [n-j+2]_{xy} + x^{j-1}[n-j+1]_{xy} \Bigg) \Bigg\}.
\end{aligned}
$$

This is written as

$$
\begin{aligned}
&(-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}} y^{\binom{n-m-k+1}{2}} \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \\
&\quad \times \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left] \begin{matrix} m \\ m+k-j \end{matrix} \right[_{x,y} \times \Bigg[\lambda - \frac{a}{[m+k-j+1]_{xy}} \\
&\quad \times \Big\{ - x^{m+2k-j} y^{m+k-j+1} \left([m]_{xy} - x^{j-k}[m+k-j]_{xy} \right) [n-m-k+1]_{xy} \\
&\qquad - x^{m+k-1} y^{m+k-j} [m+k-j+1]_{xy} [n-m-k+1]_{xy} \\
&\qquad + x^{k-1} \left([m]_{xy} - x^{j-k}[m+k-j]_{xy} \right) [n-j+2]_{xy} \\
&\qquad + x^{j-1} [m+k-j+1]_{xy} [n-j+1]_{xy} \Big\} \Bigg].
\end{aligned}
$$

By simple computation, we obtain

$$
\begin{aligned}
&- x^{m+2k-j} y^{m+k-j+1} \left([m]_{xy} - x^{j-k}[m+k-j]_{xy} \right) [n-m-k+1]_{xy} \\
&\qquad - x^{m+k-1} y^{m+k-j} [m+k-j+1]_{xy} [n-m-k+1]_{xy} \\
&\qquad + x^{k-1} \left([m]_{xy} - x^{j-k}[m+k-j]_{xy} \right) [n-j+2]_{xy} \\
&\qquad + x^{j-1} [m+k-j+1]_{xy} [n-j+1]_{xy} = x^{k-1} [m]_{xy} [m+k-j+1]_{xy}.
\end{aligned}
$$

The following identity follows. By direct computation, one can easily check this equals

$$
\begin{aligned}
&(-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}} y^{\binom{n-m-k+1}{2}} \\
&\quad \times \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left] \begin{matrix} m \\ m+k-j \end{matrix} \right[_{x,y} \left(\lambda - a\, x^{k-1} [m]_{xy} \right).
\end{aligned}
$$

(iii) If $i = j = n+1$, then the $\frac{(n+1)(n+2)}{2}$th entry of $\boldsymbol{X}_n^{m,k} N_n(\lambda, a)$ is equal to

$$
\begin{aligned}
&(-aF_n) X^{m,k}_{n,n} + \left(-ax^{n-1}\right) X^{m,k}_{n+1,n} + \lambda X^{m,k}_{n+1,n+1} \\
&= (-1)^{n+m+k} x^{-(m+k-1)(n-m-k)+\binom{n-k}{2}} y^{\binom{n-m-k}{2}} \\
&\qquad \times \frac{F_{n-m-k}!}{[n-m-k]_{xy}!} \begin{bmatrix} n-1 \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-n \end{matrix}\right[_{x,y} (-aF_n) \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{n-k}{2}} y^{\binom{n-m-k}{2}} \\
&\qquad \times \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-n \end{matrix}\right[_{x,y} \left(-ax^{n-1}\right) \\
&+ (-1)^{n+m+k+1} x^{-(m+k-1)(n-m-k+1)+\binom{n-k+1}{2}} y^{\binom{n-m-k+1}{2}} \\
&\qquad \times \frac{F_{n-m-k+1}!}{[n-m-k+1]_{xy}!} \begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F \left]\begin{matrix} m \\ m+k-n-1 \end{matrix}\right[_{x,y} \lambda.
\end{aligned}
$$

From the assumption that $m+k-n \leq 0$, we see the third term always vanishs. When $m+k-n = 0$, one can easily check the first term and the second term kill each other. One can easily check this is always equal to zero.

Thus this completes the proof of our lemma. □

Corollary 5.7. *Let n be a positive integer. Then there exists a polynomial $\varphi(\lambda)$ such that*

$$
\det \dot{N}_n(\lambda, a) = \varphi(t) t^n \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} \left(\lambda - ax^{k-1}[m]_{xy}\right).
$$

Proof. Let $\dot{\boldsymbol{X}}_n^{m,k}$ (resp. $\ddot{\boldsymbol{X}}_n^{m,k}$) denote the vector of degree $\widehat{n}-1$ obtained from $\boldsymbol{X}_n^{m,k}$ by deleting the last (resp. first) entry. Note that we have shown, in the proof of Lemma 5.6, that the last entry of $\boldsymbol{X}_n^{m,k}$ is always zero. Thus, by (5.24), we obtain

$$
\dot{\boldsymbol{X}}_n^{m,k} \dot{N}_n(\lambda, a) = \left(\lambda - ax^{k-1}[m]_{xy}\right) \ddot{\boldsymbol{X}}_n^{m,k}.
$$

By substituting $\lambda = ax^{k-1}[m]_{xy}$ into this identity we obtain

$$
\dot{\boldsymbol{X}}_n^{m,k} \dot{N}_n(ax^{k-1}[m]_{xy}, a) = \mathbf{0} \tag{5.25}
$$

for $0 \leq m \leq n-1$ and $1 \leq k \leq n-m$. If $1 \leq m \leq n-1$ and $1 \leq k \leq n-m$, then $\dot{\boldsymbol{X}}_n^{m,k}$ is non-zero vector so that (5.25) implies $\dot{N}_n(ax^{k-1}[m]_{xy}, a)$ is singular, i.e. $\det \dot{N}_n(ax^{k-1}[m]_{xy}, a) = 0$. Since $\det \dot{N}_n(\lambda, a)$ is a polynomial of λ, $\det \dot{N}_n(ax^{k-1}[m]_{xy}, a)$ is divisible by $\lambda - ax^{k-1}[m]_{xy}$. If $m = 0$, then $\dot{\boldsymbol{X}}_n^{0,k}$, $1 \leq k \leq n$, are evidently linearly independent so that (5.25) implies $\det \dot{N}_n(\lambda, a)$ is divisible by λ^n. This immediately imply the corollary. □

Now we are in position to complete the proof of Theorem 5.5.

Proof of Theorem 5.5. To complete the proof of Theorem 5.5, we need to show that the degree of $\det \dot{N}_n(\lambda, a)$ is $\frac{n(n+1)}{2}$ as a polynomial in λ, and the leading coefficient of $\det \dot{N}_n(\lambda, a)$ is equal to $(-1)^{n(n-1)/2} a^n F_n!$. Let $\mathrm{K}_n = \widehat{n}-1$ which is the degree of the matrix

$\dot{N}_n(\lambda, a)$. Let $\dot{b}_{ij}$ denote the (i,j)th entry of $\dot{N}_n(\lambda, a)$. By the definition of determinants we have

$$\det \dot{N}_n(\lambda, a) = \sum_{\pi \in S_{\mathrm{K}_n}} \operatorname{sgn} \pi \, \dot{b}_{\pi(1)1} \dot{b}_{\pi(2)2} \cdots \dot{b}_{\pi(\mathrm{K}_n)\mathrm{K}_n}.$$

We use the two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \dots & \mathrm{K}_n \\ \pi(1) & \pi(2) & \dots & \pi(\mathrm{K}_n) \end{pmatrix}$$

to express a permutation π of $[\mathrm{K}_n]$. For each j, if $\pi(j) = j+1$, then the entry $\dot{b}_{\pi(j)j}$ is of degree 1 as a polynomial in λ, and otherwise it is a constant. Thus $\det \dot{N}_n(\lambda, a)$ is apparently of at most $\mathrm{K}_n - 1 = \frac{(n+1)(n+2)}{2} - 2$ degree as a polynomial in λ. For example $\dot{N}_3(\lambda, a)$ looks as follows.

$$\left(\begin{array}{cc|ccc|cccc}
-aF_1 & \boldsymbol{-aF_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\boldsymbol{\lambda - a} & -a & -ayF_2 & -ayF_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & -aF_2 & \boldsymbol{-aF_2} & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & \boldsymbol{\lambda - a[2]_{xy}} & -a[2]_{xy} & 0 & -ay^2F_3 & -ay^2F_3 & 0 & 0 \\
0 & 0 & 0 & \boldsymbol{\lambda - ax} & -ax & 0 & -ayF_3 & -ayF_3 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & -aF_3 & \boldsymbol{-aF_3} \\ \hline
0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda - a[3]_{xy}} & -a[3]_{xy} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda - ax[2]_{xy}} & -ax[2]_{xy} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda - ax^2} & -ax^2
\end{array}\right),$$

Our first claim is that $\det \dot{N}_n(\lambda, a)$ is a polynomial of degree $\frac{n(n+1)}{2}$. For each $i = 1, \dots, n$, let Col_i denote the set of columns $j = \frac{i(i+1)}{2}, \frac{i(i+1)}{2} + 1, \dots, \frac{(i+1)(i+2)}{2} - 1$. Note that Col_i includes $i+1$ columns. We claim that $\pi(j) = j+1$ can happen at most i column indices j in each block Col_i. Otherwise $\dot{b}_{\pi(1)1} \dot{b}_{\pi(2)2} \cdots \dot{b}_{\pi(\mathrm{K}_n)\mathrm{K}_n}$ vanishes. In fact, assume that $\pi(j) = j+1$ for all j in a certain block Col_i. Then this must be the case for the block Col_{i+1}. There is no other choice if we assume $\dot{b}_{\pi(1)1} \dot{b}_{\pi(2)2} \cdots \dot{b}_{\pi(\mathrm{K}_n)\mathrm{K}_n}$ is nonzero. And this must be also the case for the block Col_{i+2}, and so on. Finally we have to take $\pi(j) = j+1$ for all j in the block Col_n, but this is impossible. Thus we reach a contradiction. We conclude that the degree of $\det \dot{N}_n(\lambda, a)$ is at most $\frac{n(n+1)}{2}$. In fact there is a permutation which realize this degree, i.e.

$$\pi = \left(\begin{array}{cc|ccc|cc|ccccc} 1 & 2 & 3 & 4 & 5 & \dots & \dots & \mathrm{K}_{n-1}+1 & \mathrm{K}_{n-1}+2 & \dots & \mathrm{K}_n - 1 & \mathrm{K}_n \\ 2 & 1 & 4 & 5 & 3 & \dots & \dots & \mathrm{K}_{n-1}+2 & \mathrm{K}_{n-1}+3 & \dots & \mathrm{K}_n & \mathrm{K}_{n-1}+1 \end{array}\right).$$

It is easy to see that this π is the only permutation with which $\dot{b}_{\pi(1)1} \dot{b}_{\pi(2)2} \cdots \dot{b}_{\pi(\mathrm{K}_n)\mathrm{K}_n}$ does not vanish and of degree $\frac{n(n+1)}{2}$. Thus we conclude that the leading coefficient of $\det \dot{N}_n(\lambda, a)$ equals

$$\operatorname{sgn} \pi \cdot (-1)^n \, a^n \, F_n!.$$

This immediately implies the resulting identity (5.22). $\square$

Remark 5.8. *One may notice that M_n in* (5.7) *and N_n in* (5.19) *are in a similar form, but our methods to evaluate them are far from parallel. It seems that the first method does not work with the matrix N_n since we can't guess the entries of $CA^{-1}N$ as we did in*

(5.11). *Meanwhile, the second method does not work with the matrix M_n at this point since even if we generalize M_n to $M_n(\lambda, a)$, we don't know the general form of the eigenvectors of $M_n(\lambda, a)$. The reader can find the general guidance about matrix evaluation in* [9]. *We may say that the second proof follows this general philosophy.*

6. Concluding remarks

We record some open problems in this section. First of all, it is desirable to find a simpler proof of Theorem 2.10. Secondly, regarding Steingrísson's original problems in [18] there are still three conjectures remaining open, that we recall as follows. For an ordered partition $\pi = B_1/B_2/\ldots/B_k$ in $\mathcal{OP}_n^k$ we say that i is a *block descent* in π if $B_i > B_{i+1}$; the *block major index* of π, denoted $\mathrm{bmaj}(\pi)$, is the sum of the block descents in π. Consider the ordered partition $\pi = 6\ 8/5/1\ 4\ 7/3\ 9/2$, since $\{6,8\} > \{5\}$ and $\{3,9\} > \{2\}$, so 1 and 4 are the two block descents in π and $\mathrm{bmaj}\,\pi = 1+4 = 5$. Then Steingrímsson presented the following conjecture in [18].

Conjecture 6.1 (Steingrímsson)**.** *The following statistics would be Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak} + \mathrm{bMaj}, \quad \mathrm{lmak} + \mathrm{bMaj}, \quad \mathrm{cmajLSB} := \mathrm{lsb} + \mathrm{cbMaj} + \binom{k}{2},$$

where $\mathrm{cbMaj} = \binom{k}{2} - \mathrm{bMaj}$.

Consider the following two generating functions of ordered partitions with $k \geq 0$ blocks:

$$\xi_k(a;x,y) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{mak} + \mathrm{bMaj})\pi}\, y^{\mathrm{cmajLSB}\ \pi}\, a^{|\pi|}, \tag{6.1}$$

$$\eta_k(a;x,y) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{lmak} + \mathrm{bMaj})\pi}\, y^{\mathrm{cmajLSB}\ \pi}\, a^{|\pi|}. \tag{6.2}$$

Then we expect the following more general conjecture would hold:

Conjecture 6.2. *For $k \geq 0$, the following identities would hold:*

$$\xi_k(a;x,y) = \frac{a^k\,(xy)^{\binom{k}{2}}[k]_{x,y}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{6.3}$$

$$\eta_k(a;x,y) = \frac{a^k (xy)^{\binom{k}{2}}\,[k]_{x,y}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}. \tag{6.4}$$

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