Selberg integrals and Catalan-Pfaffian Hankel determinants

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Definition

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the Catalan number, $M_n = \sum_{k=0}^{n}\binom{n}{2k}C_k$ the Motzkin number, $D_n = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}$ the central Delannoy number, $S_n = \sum_{k=0}^{n}\binom{n+k}{2k}C_k$ the Schröder number. Finally, the number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is known as a Narayana number, and

$$N_n(a) = \sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}a^k$$

is known as the nth Narayana polynomial.

The Selberg integral is introduced and proven by Atle Selberg [Selberg(1944)]:

$$\begin{aligned} S_n(\alpha,\beta,\gamma) &= \int_{[0,1]^n}\prod_{i=1}^{n}t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\le i<j\le n}|t_i-t_j|^{2\gamma}\,dt \\ &= \prod_{j=0}^{n-1}\frac{\Gamma(\alpha+j\gamma)\,\Gamma(\beta+j\gamma)\,\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\,\Gamma(1+\gamma)}. \end{aligned}$$

The following therem was conjectured in [Ishikawa et al.(2013)Ishikawa, Tagawa, and Zeng], and first proved in [Ishikawa and Koutschan(2012)].

Theorem

Let $n \ge 1$ be an integer. Then the following identities hold:

$$\begin{aligned} \mathrm{Pf}\Big((j-i)M_{i+j-3}\Big)_{1\le i,j\le 2n} &= \prod_{k=0}^{n-1}(4k+1), \\ \mathrm{Pf}\Big((j-i)D_{i+j-3}\Big)_{1\le i,j\le 2n} &= 2^{n^2-1}(2n-1)\prod_{k=1}^{n-1}(4k-1), \\ \mathrm{Pf}\Big((j-i)S_{i+j-2}\Big)_{1\le i,j\le 2n} &= 2^{n^2}\prod_{k=0}^{n-1}(4k+1), \\ \mathrm{Pf}\Big((j-i)N_{i+j-2}(a)\Big)_{1\le i,j\le 2n} &= a^{n^2}\prod_{k=0}^{n-1}(4k+1). \end{aligned}$$

Hence, we obtain the following identities:

$$\begin{aligned} \mathrm{Pf}\Big((j-i)M_{i+j-3}\Big)_{1\le i,j\le 2n} &= \frac{1}{(2\pi)^n n!}\int_{[-1,\,3]^n}\prod_{i<j}(x_i-x_j)^4\prod_i\sqrt{(x_i+1)(3-x_i)}\,dx, \\ \mathrm{Pf}\Big((j-i)D_{i+j-3}\Big)_{1\le i,j\le 2n} &= \frac{1}{\pi^n n!}\int_{[3-2\sqrt{2},\,3+2\sqrt{2}]^n}\frac{\prod_{i<j}(x_i-x_j)^4}{\prod_i\sqrt{6x_i-x_i^2-1}}\,dx, \\ \mathrm{Pf}\Big((j-i)S_{i+j-2}\Big)_{1\le i,j\le 2n} &= \frac{1}{(2\pi)^n n!}\int_{[3-2\sqrt{2},3+2\sqrt{2}]^n}\prod_{i<j}(x_i-x_j)^4\prod_i\sqrt{6x_i-x_i^2-1}\,dx. \end{aligned}$$

From Selberg's formula we derive readily the following special cases:

$$\begin{aligned} \int_{[a,b]^n}\prod_{i<j}(x_i-x_j)^4\prod_{i=1}^{n}\sqrt{(x_i-a)(b-x_i)}\,dx &= (b-a)^{2n^2}S_n\left(\frac{3}{2},\frac{3}{2},2\right), \\ \int_{[a,b]^n}\frac{\prod_{i<j}(x_i-x_j)^4}{\prod_{i=1}^{n}\sqrt{(x_i-a)(b-x_i)}}\,dx &= (b-a)^{2n(n-1)}S_n\left(\frac{1}{2},\frac{1}{2},2\right). \end{aligned}$$

Definition

In this section we use the standard q-exponential functions:

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \qquad E_q(x) = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}x^n}{(q;q)_n} = (-x;q)_\infty.$$

Al-Salam and Carlitz [Al-Salam and Carlitz(1965), Chihara(1978)] defined the sequences $\{U_n^{(a)}(y;q)\}$ $(a<0)$ and $\{V_n^{(a)}(x;q)\}$ of orthogonal polynomials by

$$\begin{aligned} \rho_a(x;q)e_q(xy) &= \sum_{n=0}^{\infty}U_n^{(a)}(y;q)\frac{x^n}{(q;q)_n}, \\ \frac{1}{\rho_a(x;q)}E_q(-xy) &= \sum_{n=0}^{\infty}V_n^{(a)}(y;q)\frac{(-1)^n q^{\frac{n(n-1)}{2}}x^n}{(q;q)_n}, \end{aligned}$$

where

$$\rho_a(x;q) = (x;q)_\infty(ax;q)_\infty = E_q(-x)E_q(-ax).$$

These sequences $\{U_n^{(a)}(y;q)\}$ and $\{V_n^{(a)}(x;q)\}$ are called the Al-Salam and Carlitz I polynomials and the Al-Salam and Carlitz II polynomials, respectively. The orthogonality relations of these polynomials are given by

$$\begin{aligned} \int_a^1 U_m^{(a)}(x;q)U_n^{(a)}(x;q)w_U^{(a)}(x;q)\,d_qx &= (1-q)(-a)^n q^{\frac{n(n-1)}{2}}(q;q)_n\delta_{m,n}, \\ \int_1^\infty V_m^{(a)}(x;q)V_n^{(a)}(x;q)w_V^{(a)}(x;q)\,d_qx &= (1-q)a^n q^{-n^2}(q;q)_n\delta_{m,n}, \end{aligned}$$

where the weight functions $w_U^{(a)}(x;q)$ and $w_V^{(a)}(x;q)$ are defined by

$$\begin{aligned} w_U^{(a)}(x;q) &= \frac{(qx;q)_\infty\left(\frac{qx}{a};q\right)_\infty}{(q;q)_\infty(aq;q)_\infty\left(\frac{q}{a};q\right)_\infty}, \\ w_V^{(a)}(x;q) &= \frac{(q;q)_\infty(aq;q)_\infty\left(\frac{q}{a};q\right)_\infty}{(x;q)'_\infty\left(\frac{x}{a};q\right)_\infty}. \end{aligned}$$

(See [Al-Salam and Carlitz(1965), Chihara(1978)].) Here $(x;q)'_\infty$ denotes the product except the term which equals 0. Note that these Jackson integrals are given by

$$\begin{aligned} \int_a^1 f(x)d_qx &= (1-q)\left\{\sum_{n=0}^{\infty}f(q^n)q^n - a\sum_{n=0}^{\infty}f(aq^n)q^n\right\}, \\ \int_1^\infty f(x)d_qx &= (1-q)\sum_{n=0}^{\infty}f(q^{-n})q^{-n}. \end{aligned}$$

The nth moments of the above measures are given by

$$\begin{aligned} \int_a^1 x^n w_U^{(a)}(x;q)d_qx &= (1-q)\,F_n^{(a)}(a;q), \\ \int_1^\infty x^n w_V^{(a)}(x;q)d_qx &= (1-q)\,G_n^{(a)}(a;q), \end{aligned}$$

where

$$F_n^{(a)}(a;q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^k, \qquad G_n^{(a)}(a;q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a^{k(k-n)}.$$

Here $\begin{bmatrix}n\\k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$ denotes the q-binomial coefficient.

Theorem

Let $F_n^{(a)}(a;q)$ and $G_n^{(a)}(a;q)$ be as above. Then we have

$$\begin{aligned} \mathrm{Pf}\Big((q^{i-1}-q^{j-1})F_{i+j-3}(a;q)\Big)_{1\le i,j\le 2n} &= a^{n(n-1)}q^{\frac{1}{6}n(n-1)(4n-5)}\prod_{k=1}^{n}(q;q)_{2k-1}, \\ \mathrm{Pf}\Big((q^{i-1}-q^{j-1})F_{i+j-2}(a;q)\Big)_{1\le i,j\le 2n} & \\ = a^{n(n-1)}q^{\frac{1}{6}n(n-1)(4n+1)}\prod_{k=1}^{n}(q;q)_{2k-1}&\sum_{k=0}^{n}q^{(n-k)(n-k-1)}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}a^k. \\ \mathrm{Pf}\Big((q^{i-1}-q^{j-1})G_{i+j-3}(a;q)\Big)_{1\le i,j\le 2n} &= a^{n(n-1)}q^{-n(n-1)(4n-5)/3}\prod_{k=1}^{n}(q;q)_{2k-1}, \\ \mathrm{Pf}\Big((q^{i-1}-q^{j-1})G_{i+j-2}(a;q)\Big)_{1\le i,j\le 2n} & \\ = a^{n(n-1)}q^{-\frac{2}{3}n(n-1)(2n-1)}\prod_{k=1}^{n}(q;q)_{2k-1}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}a^k. \end{aligned}$$

We use

$$\begin{aligned} \mathrm{Pf}\Big((q^{i-1}-q^{j-1})\,F_{i+j+r-2}^{(a)}(a;q)\Big)_{1\le i<j\le 2n} &= \frac{1}{n!}q^{n(n-1)}(1-q)^n \\ \times\int_{[a,1]^n}\prod_{i<j}\prod_{l=0}^{1}(x_i-q^lx_j)(x_i-q^{-l}x_j)&\prod_{i=1}^{n}x_i^{r+1}w_U^{(a)}(x_i;q)d_qx, \\ \mathrm{Pf}\Big((q^{i-1}-q^{j-1})\,G_{i+j+r-2}^{(a)}(a;q)\Big)_{1\le i<j\le 2n} &= \frac{1}{n!}q^{n(n-1)}(1-q)^n \\ \times\int_{[1,\infty)^n}\prod_{i<j}\prod_{l=0}^{1}(x_i-q^lx_j)(x_i-q^{-l}x_j)&\prod_{i=1}^{n}x_i^{r+1}w_V^{(a)}(x_i;q)d_qx. \end{aligned}$$

[Baker and Forrester(2000)] proved

$$\begin{aligned} \int_{[a,1]^n}\Delta_k^2(x)\prod_{i=1}^{n}w_U^{(a)}(x_i;q)d_q\,x &= (1-q)^n(-a)^{\frac{kn(n-1)}{2}}q^{k^2\binom{n}{3}-\frac{k(k-1)}{2}\binom{n}{2}}\prod_{i=1}^{n}\frac{(q;q)_{ki}}{(q;q)_k}, \\ \int_{[1,\infty]^n}\Delta_k^2(x)\prod_{i=1}^{n}w_V^{(a)}(x_i;q)\,d_qx &= (1-q)^n a^{\frac{kn(n-1)}{2}}q^{-2k^2\binom{n}{3}-k^2\binom{n}{2}}\prod_{i=1}^{n}\frac{(q;q)_{ki}}{(q;q)_k}, \end{aligned}$$

where

$$\Delta_k^2(x) = \prod_{i<j}\prod_{l=-k+1}^{k}(x_i-q^lx_j).$$

Let $a_n = \frac{1}{2n+1}\binom{3n}{n} = \frac{1}{3n+1}\binom{3n+1}{n}$. [Gessel and Xin(2006)] prove that $\det(a_{i+j-1})_{1\le i,j\le n}$ equals the number of $(2n+1)\times(2n+1)$ alternating sign matrices that are invariant under vertical reflection. [Krattenthaler(2005), Theorem 31] collects this type of Hankel determinants. We made the following cnjectures for Pfaffian analogues:

Conjecture (Gessel-Xin-Krattenthaler type)

If $a_n^{(1)} = \frac{1}{3n+1}\binom{3n+1}{n} = [x^n]\frac{{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{1}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(1)}\right)_{1\le i,j\le 2n} = \frac{1}{2^n}\prod_{k=0}^{n-1}\frac{(12k+6)!(4k+1)!(3k+2)!}{(8k+2)!(8k+5)!(3k+1)!}.$$

If $a_n^{(2)} = \frac{1}{3n+2}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{4}{3},\frac{2}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-2}^{(2)}\right)_{1\le i,j\le 2n} = 12^{-n}\prod_{k=0}^{n-1}\frac{(12k+10)!(4k+2)!(4k+1)}{(8k+3)!(8k+7)!(3k+2)(12k+5)}.$$

If $a_n^{(3)} = \frac{2}{3n+1}\binom{3n+1}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{7}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{5}{3},\frac{4}{3};\frac{5}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(3)}\right)_{1\le i,j\le 2n} = \left(\frac{4}{3}\right)^n\prod_{k=0}^{n-1}\frac{(12k+15)!(4k+5)!(2k+1)}{(8k+8)!(8k+11)!(12k+13)}.$$

If $a_n^{(4)} = \frac{2}{(3n+1)(3n+2)}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{5}{3},\frac{4}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-1}^{(4)}\right)_{1\le i,j\le 2n} = \left(\frac{2}{3}\right)^n(6n+1)!\prod_{k=0}^{n-1}\frac{(12k+6)!(4k+5)!(4k+3)}{(8k+5)!(8k+10)!(k+1)(3k+1)}.$$

If $a_n^{(5)} = \frac{9n+5}{(3n+1)(3n+2)}\binom{3n+2}{n+1} = [x^n]\frac{{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{5}{2};\frac{27}{4}x\right)}{{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{3}{2};\frac{27}{4}x\right)}$, then

$$\mathrm{Pf}\left((j-i)a_{i+j-2}^{(5)}\right)_{1\le i,j\le 2n} = 3^{-n}\prod_{k=0}^{n-1}\frac{(6k+6)!(2k)!}{(4k+1)!(4k+4)!(3k+2)}.$$

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