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Schur Function Identities and Hook Length Posets

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Abstract. In this paper we find new classes of posets which generalize the d-complete posets. In fact the d-complete posets are classified into 15 irreducible classes in the paper "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets" (*J. Algebraic Combin.* **9** (1999), 61 – 94) by R. A. Proctor. Here we present six new classes of posets of hook-length property which generalize the 15 irreducible classes. Our method to prove the hook-length property is based on R. P. Stanley's (P,ω)-partitions and Schur function identities.

Résumé. Dans cet article nous trouvons des nouvelles classes de posets qui généralisent les posets d-complets. En fait, les posets d-completes sont classés en 15 classes irréductibles dans l'article "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets" (*J. Algebraic Combin.* **9** (1999), 61 – 94) par R. A. Proctor. Dans cet article nous présentons six nouvelles classes de posets ayant la propriété de longueur de crochet, qui généralisent les 15 classes irréductibles. Notre méthode pour prouver la propriété de longueur de crochet est basée sur les (P,ω)-partitions de R. P. Stanley et identités de fonctions de Schur.

Introduction

In [2] R. A. Proctor defined d-complete posets, which include shapes, shifted shapes and trees, by certain local structural conditions and showed that arbitrary connected d-complete poset is decomposed into a slant sum of irreducible ones. He also classified 15 exhaustive classes of irreducible d-complete components and described all of the members of each class. In this paper we define six types of posets, and these six types generalize the 15 types of irreducible d-complete posets. First we enumerate eight product formulas involving the Schur functions, which will be applied to obtain the hook formulas of the new posets, which we call "leaf posets".

Schur Function Identities

In this section we state eight Cauchy type identities of the Schur functions, which will be applied in the following sections. The *Schur function* $s_\lambda(x_1,x_2,\dots,x_n)$ of variables $x_1,x_2,\dots,x_n$ with respect to a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ is defined to be

$$s_\lambda(x_1,x_2,\dots,x_n)=\frac{\det(x_i^{\lambda_j+n-j})_{1\le i,j\le n}}{\det(x_i^{n-j})_{1\le i,j\le n}}.$$

For a positive integer m, we write $X_m=(x_1,x_2,\dots,x_m)$, $Y_m=(y_1,y_2,\dots,y_m)$ and $Z_m=(z_1,z_2,\dots,z_m)$ in short. Let $\mathscr{P}$ denote the set of all partitions. If $\lambda=(\lambda_1,\lambda_2,\dots)$ is a partition and a and b are positive integers such that $a\le b$, then we write $\lambda[a,b]$, in short, for the partition $(\lambda_a,\lambda_{a+1},\dots,\lambda_b)$. If $X_m=(x_1,x_2,\dots,x_m)$ is an m-tuple of variables, then we use the notation $\|X_m\|:=\prod_{i=1}^m x_i$ for brevity. We proved the following variants of the Cauchy identity.

Theorem 2.1. Let m be a positive integer. (i) If $m\ge 1$, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)\in\mathscr{P}} w^{\lambda_m}s_\lambda(X_m)s_\lambda(Y_m)=\frac{1-\|X_m\|\|Y_m\|}{(1-w\|X_m\|\|Y_m\|)\prod_{i,j=1}^m(1-x_iy_j)}.$$

(ii) If $m\ge 2$, and $v=1$ or 2, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)\in\mathscr{P}} z_v^{|\lambda|-\lambda_m-\lambda_{m-1}}w^{\lambda_m}s_\lambda(X_m)s_{\lambda[1,m-1]}(Y_{m-1})s_{\lambda[m-1,m]}(Z_2)$$
$$=\frac{\prod_{k=1}^{m-1}(1-z_v^{m-2}y_k\|X_m\|\|Y_{m-1}\|\|Z_2\|)}{(1-wz_v^{m-2}\|X_m\|\|Y_{m-1}\|\|Z_2\|)\prod_{i=1}^m\prod_{j=1}^{m-1}(1-x_iy_jz_v)\prod_{k=1}^m(1-z_v^{m-3}x_k^{-1}\|X_m\|\|Y_{m-1}\|\|Z_2\|)}.$$

(iii) If $v=1$ or 2, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_6)\in\mathscr{P}} z_v^{\lambda_1+\lambda_2}w^{\lambda_6}s_{\lambda[1,3]}(X_3)s_{\lambda[3,4]}(Z_2)s_{\lambda[4,6]}(X_3)s_{\lambda[1,5]}(Y_5)s_{\lambda[5,6]}(Z_2)$$
$$=\frac{1}{(1-wz_v^2\|X_3\|^2\|Y_5\|\|Z_2\|^2)\prod_{i=1}^3\prod_{j=1}^5(1-x_iy_jz_v)}$$
$$\times\frac{\prod_{k=1}^5(1-z_v^2y_k\|X_3\|^2\|Y_5\|\|Z_2\|^2)}{\prod_{k=1}^3(1-z_vx_k^{-1}\|X_3\|^2\|Y_5\|\|Z_2\|^2)\prod_{1\le i<j\le 5}(1-z_vy_i^{-1}y_j^{-1}\|X_3\|\|Y_5\|\|Z_2\|)}.$$

(iv) If $v=1$ or 2, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2r})\in\mathscr{P}} z_v^{\lambda_1}w^{\lambda_{2r}}s_\lambda(X_{2r})\prod_{i=1}^r s_{\lambda[2i-1,2i]}(Y_2)\prod_{i=1}^{r-1}s_{\lambda[2i,2i+1]}(Z_2)$$
$$=\frac{\prod_{i=1}^2(1-z_vz_i\|X_{2r}\|\|Y_2\|^r\|Z_2\|^{r-1})}{(1-wz_v\|X_{2r}\|\|Y_2\|^r\|Z_2\|^{r-1})\prod_{i=1}^{2r}\prod_{j=1}^2(1-x_iy_jz_v)\prod_{1\le i<j\le 2r}(1-x_ix_j\|Y_2\|\|Z_2\|)}.$$

(v) If $v=1$ or 2, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2r+1})\in\mathscr{P}} z_v^{\lambda_1}w^{\lambda_{2r+1}}s_\lambda(X_{2r+1})\prod_{i=1}^r s_{\lambda[2i-1,2i]}(Y_2)\prod_{i=1}^{r}s_{\lambda[2i,2i+1]}(Z_2)$$
$$=\frac{\prod_{i=1}^2(1-z_vy_i\|X_{2r+1}\|\|Y_2\|^r\|Z_2\|^{r})}{(1-wz_v\|X_{2r+1}\|\|Y_2\|^r\|Z_2\|^{r})\prod_{i=1}^{2r+1}\prod_{j=1}^2(1-x_iz_jz_v)\prod_{1\le i<j\le 2r+1}(1-x_ix_j\|Y_2\|\|Z_2\|)}.$$

(vi) If $r\ge 2$, $v\in\{s,t\}\subseteq\{1,2,3\}$ and $s\ne t$, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2r})\in\mathscr{P}} x_v^{\lambda_1}w^{\lambda_{2r}}s_{\lambda[1,2r-1]}(Y_{2r-1})s_{\lambda[2r-2,2r]}(X_3)\prod_{i=1}^r s_{\lambda[2i-1,2i]}(Z_2)\prod_{i=1}^{r-2}s_{\lambda[2i,2i+1]}(x_s,x_t)$$
$$=\frac{1}{(1-w(x_sx_t)^{r-2}x_v\|X_3\|\|Y_{2r-1}\|\|Z_2\|^r)\prod_{i=1}^{2r-1}\prod_{j=1}^2(1-x_vy_iz_j)\prod_{1\le i<j\le 2r-1}(1-x_sx_ty_iy_j\|Z_2\|)}$$
$$\times\frac{\prod_{k=1}^{2r-1}(1-(x_sx_t)^{r-2}x_vy_k\|X_3\|\|Y_{2r-1}\|\|Z_2\|^r)}{\prod_{k=1}^2(1-(x_sx_t)^{r-2}z_k\|X_3\|\|Y_{2r-1}\|\|Z_2\|^{r-1})\prod_{k=1}^{2r-1}(1-(x_sx_t)^{r-3}x_vy_k^{-1}\|X_3\|\|Y_{2r-1}\|\|Z_2\|^{r-1})}.$$

(vii) If $r\ge 1$, $v\in\{1,2\}$ and $1\le s\ne t\le 3$, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2r+1})\in\mathscr{P}} z_v^{\lambda_1}w^{\lambda_{2r+1}}s_{\lambda[1,2r]}(Y_{2r})s_{\lambda[2r-1,2r+1]}(X_3)\prod_{i=1}^r s_{\lambda[2i,2i+1]}(Z_2)\prod_{i=1}^{r-1}s_{\lambda[2i-1,2i]}(x_s,x_t)$$
$$=\frac{1}{(1-w(x_sx_t)^{r-1}z_v\|X_3\|\|Y_{2r}\|\|Z_2\|^r)\prod_{i=s,t}\prod_{j=1}^{2r}(1-x_iy_jz_v)\prod_{1\le i<j\le 2r}(1-x_sx_ty_iy_j\|Z_2\|)}$$
$$\times\frac{\prod_{k=1}^{2r}(1-(x_sx_t)^{r-1}y_kz_v\|X_3\|\|Y_{2r}\|\|Z_2\|^r)}{\prod_{k=s,t}(1-(x_sx_t)^{r-2}x_k\|X_3\|\|Y_{2r}\|\|Z_2\|^r)\prod_{k=1}^{2r}(1-(x_sx_t)^{r-2}y_k^{-1}z_v\|X_3\|\|Y_{2r}\|\|Z_2\|^{r-1})}.$$

(viii) If $v=1,2,3$ or 4, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_6)\in\mathscr{P}} y_v^{\lambda_1}w^{\lambda_6}s_{\lambda[1,3]}(X_3)s_{\lambda[3,4]}(Z_2)s_{\lambda[4,6]}(X_3)s_{\lambda[1,2]}(Z_2)s_{\lambda[2,5]}(Y_4)s_{\lambda[5,6]}(Z_2)$$
$$=\frac{1}{(1-wy_v\|X_3\|^2\|Y_4\|\|Z_2\|^3)\prod_{\substack{1\le k\le 4\\ k\ne v}}\prod_{j=1}^2(1-y_k^{-1}z_j\|X_3\|\|Y_4\|\|Z_2\|)}$$
$$\times\frac{\prod_{1\le k\le 4}(1-y_ky_v\|X_3\|^2\|Y_4\|\|Z_2\|^3)}{\prod_{k=1}^3(1-x_k\|X_3\|\|Y_4\|\|Z_2\|^2)\prod_{i=1}^3\prod_{j=1}^2(1-x_iy_vz_j)\prod_{\substack{1\le k\le 4\\ k\ne v}}\prod_{j=1}^3(1-x_j^{-1}y_ky_v\|X_3\|\|Z_2\|)}.$$

Hook Length Posets

The aim of this section is to define six new classes of hook length posets which include any irreducible d-complete poset. We call these classes basic leaf posets. Let P be a partilly ordered set (poset). A P-partition is a map φ from P to $\{0,1,2,\dots\}$ satisfying that $\varphi(x)\ge\varphi(y)$ if $x<y$ in P, i.e. φ is order reversing map. We denote the set of all P-partitions by $\mathscr{A}(P)$. We say that P is a *hook-length poset* if there exists a map h from P to $\{1,2,\dots\}$ satisfying

$$\sum_{\varphi\in\mathscr{A}(P)} q^{\sum_{x\in P}\varphi(x)}=\prod_{x\in P}\frac{1}{1-q^{h(x)}}.$$

It is well known that shapes, shifted shapes and trees are hook length posets. From now on, we denote the set of the strictly decreasing sequences of nonnegative integers by $\mathscr{S}$. Basic leaf posets are defined as follows:

Definition 3.1. (i) Let $m\ge 2$ be an integer, and let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$ and $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_m)$ be elements of $\mathscr{S}$. Let δ and f be nonnegative integers which satisfy $f\ge\delta\ge 0$. Then, a *ginkgo* $\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta)$ is a poset defined by the diagram in Figure 1. In the diagram c_δ denotes the chain of length δ.

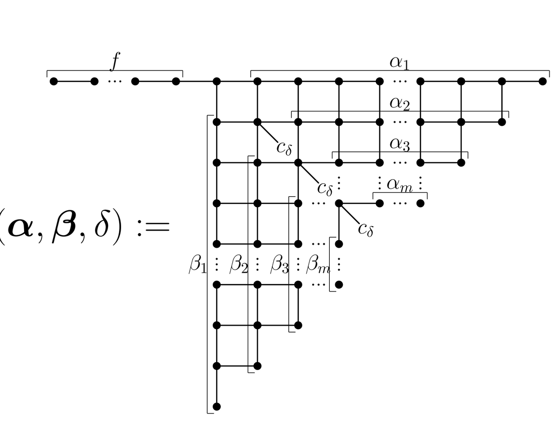
$\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta):=$ 

Figure 1: A gingko

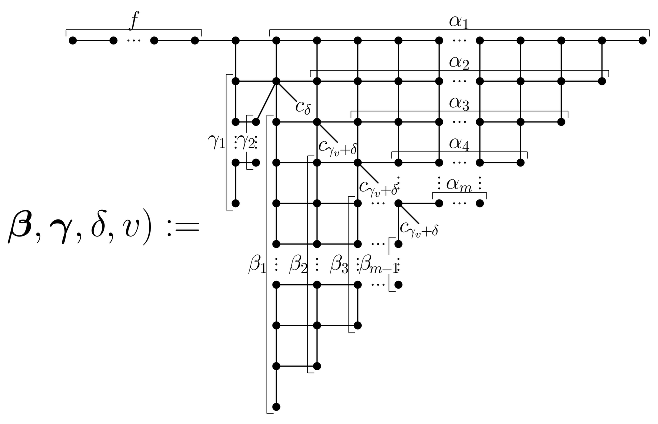
$\mathfrak{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=$ 

Figure 2: A bamboo

(ii) Let $m\ge 3$ be an integer, and let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_{m-1})$, $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be elements of $\mathscr{S}$. Let δ and f be nonnegative integers which satisfy $f\ge\beta_1+\delta\ge 0$. For $v=1,2$, we define a poset $\mathfrak{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$ called a *bamboo* by the diagram of Figure 2.
(iii) Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5)$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be elements of $\mathscr{S}$. Let δ and f be nonnegative integers which satisfy $f\ge\beta_1+\delta\ge 0$. For $v=1,2$, an *ivy* $\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$ is a poset defined by the diagram of Figure 3.

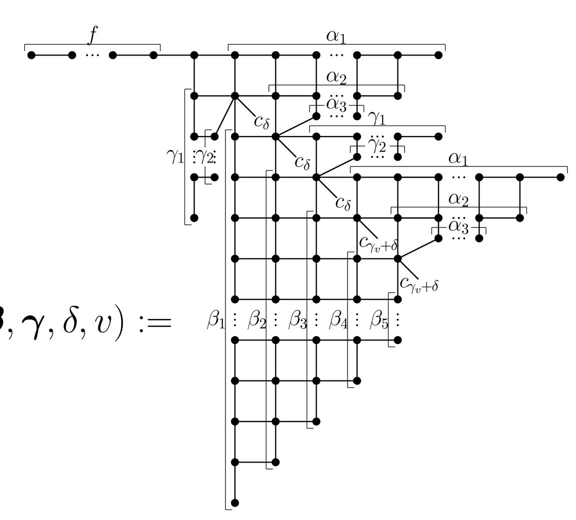
$\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=$ 

Figure 3: An ivy

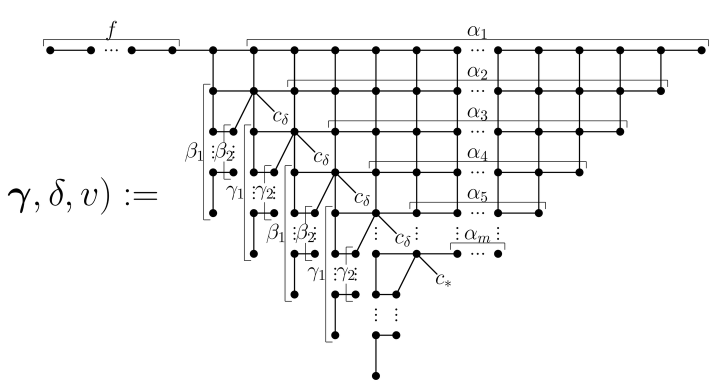
$\mathfrak{W}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=$ 

Figure 4: A wisteria

(iv) Let $m\ge 4$ be a positive integer, and let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2)$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be elements of $\mathscr{S}$. Let δ and f be nonnegative integers which satisfy $f\ge\gamma_1+\delta\ge 0$. Assume $v=1$ or 2. We define a poset $\mathfrak{W}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$ called a *wisteria* by the diagram of Figure 4. In the diagram, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ appear alternately in the place under the left and c_* equals $c_{\gamma_v+\delta}$ (resp. $c_{\beta_v+\delta}$) if m is even (resp. m is odd).
(v) Let $m\ge 4$ be a positive integer, let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_{m-1})$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be elements of $\mathscr{S}$. Let δ and f be nonnegative integers which satisfy $f\ge\beta_1+\delta\ge 0$. Fix positive integers s, t which satisfy $1\le s<t\le 3$, and let $v\in\{s,t\}$ if m is even, and let $v\in\{1,2\}$ if m is odd. Write $\tilde{\alpha}:=(\alpha_s,\alpha_t)$. Then, a poset $\mathfrak{F}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,s,t,v)$ called a *fir* is defined in the diagram of Figure 5. In the diagram, $\boldsymbol{\gamma}$ and $\tilde{\alpha}$ appear alternatively in the place following upper right α, and c_* equals $c_{\alpha_v+\delta}$ (resp. $c_{\gamma_v+\delta}$) if m is even (resp. m is odd).

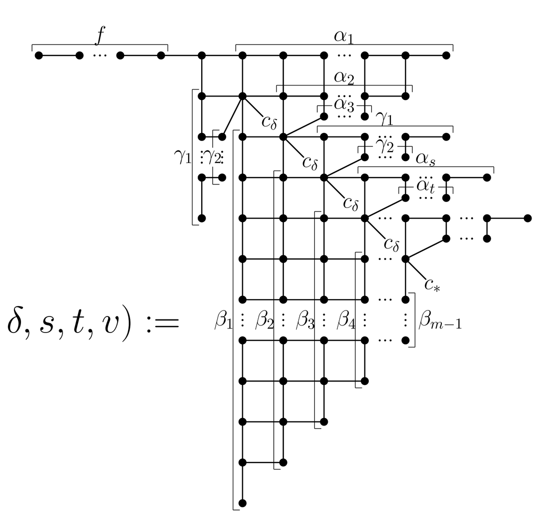
$\mathfrak{F}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,s,t,v):=$ 

Figure 5: A fir

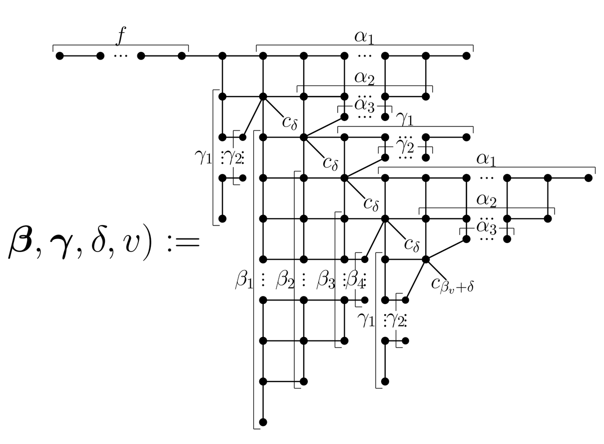
$\mathfrak{C}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=$ 

Figure 6: A chrysanthemum

(vi) Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4)$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be elements of $\mathscr{S}$. Let δ and f be nonnegative integers which satisfy $f\ge\beta_1+\delta\ge 0$. For $v=1,2,3,4$, this *chrysanthemum* $\mathfrak{C}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$ is a poset defined by the diagram in Figure 6. We call these new classes of posets, i.e. ginkgoes, bamboos, ivies, wisterias, firs and chrysantemums, *basic leaf posets*.
By applying our Cauchy type identities described in Theorem 2.1, lattice path method and R. P. Stanley's (P,ω)-partitions, we obtained the following.

Corollary 3.2. Any basic leaf poset is a hook length poset. In particular, any d-complete poset is a hook length poset since it can be realized as a leaf poset.

A general leaf poset defined from the basic ones by using an operation called "joint sum", which is a similar operation called "slant sum" introduced in [2] in order to combine two irreducible d-complete posets to generate a general d-complete poset. We conclude that any leaf poset is a hook length poset.

References

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[2] R. A. Proctor, "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets", *J. Algebraic Combin.* **9** (1999), 61 – 94.