Littlewood's (Cauchy's) formulae of Schur functions and constant term expressions for the refined enumeration problems of TSSCPPs

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Introduction

Abstract

We consider several enumeration problems of TSSCPPs (totally symmetric self-complementary plane partitions) and establish certain bijections with (domino) plane partitions under some conditions. We show that the enumaration of the (domino) plane partitions is closely related to Littlewood's formulae or Cauchy's formulae of Schur functions.

Plan of My Talk

1. Plane partitions
2. Schur functions
3. RCSPPs (Restricted column-strict plane partitions)
4. Twisted Bender-Knuth involutions
5. RCSDPPs (Restricted column-strict Domino plane partitions) with all rows of even lenth
6. Twisted domino plane partitions
7. RCSDPPs (Restricted column-strict Domino plane partitions) with all columns of even lenth
8. RCSPPs (Restricted column-strict plane partitions) with restricted row length

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Bijections

TSSCPPs	RCSPPs		RCSPPs
	RCSPPs invariant under $\widetilde{\rho}$	Twisted Domino PPs	RCSDPPs with all columns of even length
	RCSPPs invariant under $\widetilde{\gamma}$		RCSDPPs with all rows of even length

Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{array}{cccccc} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ \vdots & & & & & \\ 0 & 0 & 0 & \ddots & & \end{array}$$

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Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

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Example of plane partitions

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- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: 1
- Plane partitions of 2:

  2   1 1   1/1

- Plane partitions of 3:

  3   1 1 1   1/1/1   2 1   2/1   1 1/1

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- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\begin{array}{|c|}\hline 1\\\hline\end{array}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 1\\\cline{1-1}\end{array}$$

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Ferrers graph

Definition

The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \left\{ (i, j, k) : k \leq \pi_{ij} \right\}$$

Example

Ferrers graph

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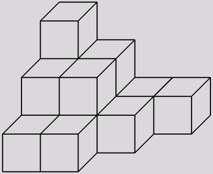

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Definition

In the paper “Self-complementary totally symmetric plane partitions” (*J. Combin. Theory Ser. A* **42**, (1986), 277–292), W.H. Mills, D.P. Robbins and H. Rumsey have defined totally symmetric self-complementary plane partitions (TSSCPPs).

A plane partition is said to be *totally symmetric self-complementary plane parition of size* $2n$ if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.

We denote the set of all self-complementary totally symmetric plane partitions of size $2n$ by $\mathscr{S}_n$.

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Column-strictness

Definition

A plane partition is said to be *column-strict* if it is strictly decreasing in coulumns.

Example

$\pi =$

5	5	4	3	3	3	1
4	4	2	2	1	1	
3	2	1	1			
1	1					

is a column-strict plane partition.

We write $\boldsymbol{x}^{\pi} = x_1^6 x_2^3 x_3^4 x_4^3 x_5^2$, where $\boldsymbol{x} = (x_1, x_2, \dots)$ is a tuple of variables.

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Schur functions

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Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an n-tuple of variables.

The *Schur function $s_\lambda(x)$* is, by definition,

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where the sum runs over all column-strict plane partitions of shape λ and each part $\leq n$.

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- Schur functions are symmetric functions.
- $s_\lambda(x) = \frac{\det(x_i^{\lambda_j+n-j})_{1\le i,j\le n}}{\det(x_i^{n-j})_{1\le i,j\le n}}$
- Schur functions are known as the irreducible characters of the general linear groups.

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An Example of Schur functions

Example

If $\lambda = (22)$ and $\boldsymbol{x} = (x_1, x_2, x_3)$, then the followings are column-strict plane partitions with all parts ≤ 3.

2	2
1	1

3	2
1	1

3	3
1	1

3	2
2	1

3	3
2	1

3	3
2	2

Hence we have

$$s_{(2^2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

Littlewood type identities

Littlewood's identity

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be n-tuple of variables. Then

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) = \prod_{i=1}^{n} (1 - x_i)^{-1} \prod_{1 \le i < j \le n} (1 - x_i x_j)^{-1},$$

where the sum runs over all partitions λ such that $\ell(\lambda) \le n$.

A Littlewood type identity (the bounded version)

$$\sum_{\substack{\lambda \\ \lambda_1 \le k}} s_\lambda(\boldsymbol{x}) = \frac{\det\left(x_i^{j-1} - x_i^{k+2n-j}\right)_{1 \le i,j \le n}}{\prod_{i=1}^{n}(1 - x_i) \prod_{1 \le i < j \le n}(x_j - x_i)(1 - x_i x_j)},$$

where the sum runs over all partitions λ contained in the rectangle $n \times k$.

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Caushy type identities

The Caushy identity

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be n-tuples of variables.

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1},$$

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A Cauchy type identity

An easy consequence of the above identity is the following:

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$$\sum_{(\lambda,\mu)} s_\lambda(\boldsymbol{x}) s_\mu(\boldsymbol{y}) = \prod_{i=1}^{n} (1 - x_i)^{-1} \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1},$$

where the sum runs over all pair (λ, μ) of partitions such that $\lambda \supseteq \mu$ and $\lambda \setminus \mu$ is a horizontal strip.

Caushy type identities

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A Cauchy type identity

An easy consequence of the above identity is the following:

$$\sum_{(\lambda,\mu)} s_\lambda(\boldsymbol{x}) s_\mu(\boldsymbol{y}) = \prod_{j=1}^{n} (1 + y_j) \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1},$$

where the sum runs over all pair (λ, μ) of partitions such that $\lambda \subseteq \mu$ and $\mu \setminus \lambda$ is a vertical strip.

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

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$\mathscr{P}_1$ consists of the single element $\emptyset$.

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A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

$\mathscr{P}_2$ consists of the following 2 elements:

$$\emptyset \qquad \boxed{1}$$

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

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Example

$\mathscr{P}_3$ consists of the followng 7 elements:

$$\emptyset \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&1\\\hline 1\\\cline{1-1}\end{array}$$

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$\emptyset$ $\boxed{1}$ $\boxed{1}\boxed{1}$ $\boxed{2}$ $\boxed{2}\boxed{1}$ $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1} \end{array}$

More General Definition

Definition

Let $\mathscr{P}_{n,m}$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $m + n - j$.

(C3) c has at most n columns.

Example

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(C3) c has at most n columns.

Example

$\mathscr{P}_{0,4}$ consists of the followng 1 element:

$$\emptyset$$

More General Definition

Definition

Let $\mathscr{P}_{n,m}$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $m + n - j$.

(C3) c has at most n columns.

Example

$\mathscr{P}_{1,3}$ consists of the followng 8 elements:

$$\emptyset \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$$

More General Definition

Example

$\mathscr{P}_{2,2}$ consists of the followng 25 elements:

$$\emptyset \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&1\\\hline 1\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2&2\\\hline 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&2\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&1\\\hline 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 3&2\\\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&2\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&2\\\hline 2&1\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&2\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3&2\\\hline 2&1\\\hline 1\\\cline{1-1}\end{array}$$

$\mathscr{P}_{3,1} = \mathscr{P}_{4,0}$ consists of 42 elements.

Another bijection

Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

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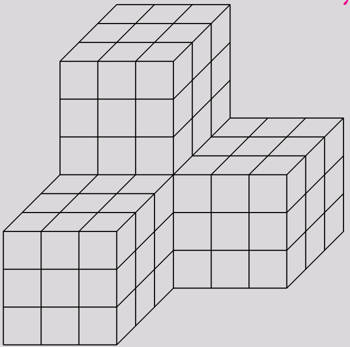
Example

π_1 $n = 3$



$\longrightarrow \quad \emptyset$

Another bijection

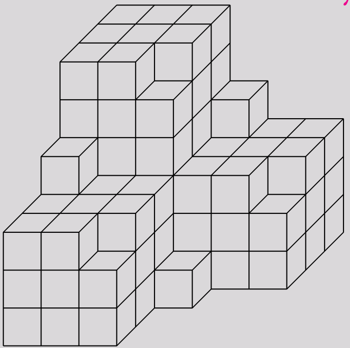
Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

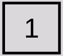
Example

π_2

$n = 3$



$\longrightarrow$



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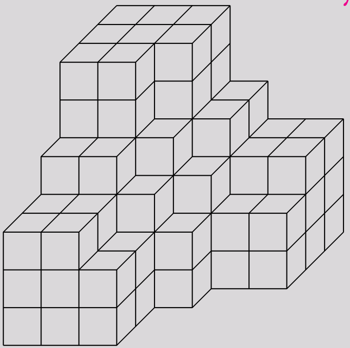
Example

π_3 $\qquad n = 3$



$\longrightarrow$

1	1

Another bijection

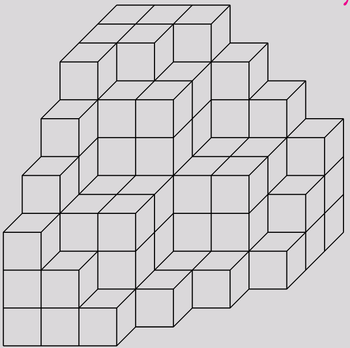
Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

$n = 3$

π_4



$\longrightarrow$ $\boxed{2}$

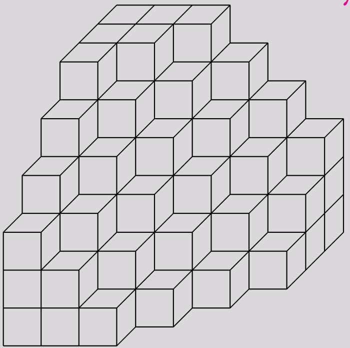
Another bijection

Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_5 $\qquad$ $n = 3$



$\longrightarrow$

2	1

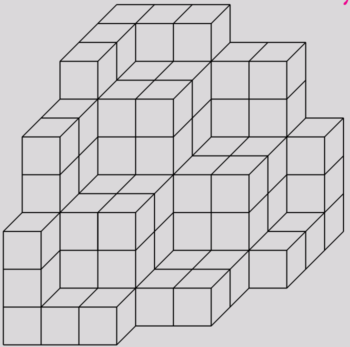
Another bijection

Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_6 $\qquad n = 3$



$\longrightarrow$

2
1

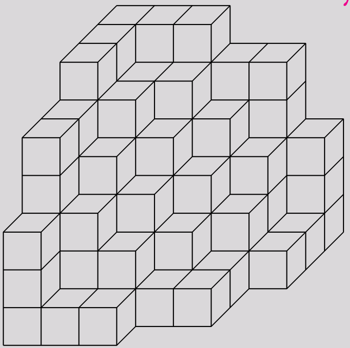
Another bijection

Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_7 $\qquad$ $n = 3$



$\longrightarrow$

2	1
1	

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.

Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, Saturated parts

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 1$, $\overline{U}_1(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

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Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 2$, $\overline{U}_2(c) = 5$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 3$, $\overline{U}_3(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 4$, $\overline{U}_4(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 5$, $\overline{U}_5(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 6$, $\overline{U}_6(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Mills-Robbins-Rumsey statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \leq i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 7$, $\overline{U}_7(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The Bender-Knuth involution

The Bender-Knuth involution

A classical method to prove that a Schur function is symmetric is to define involutions f_k on column-strict plane partitions c which swaps the number of k's and $(k-1)$'s, for each k. Consider the parts of c equal to k or $k-1$. If both of k and $k-1$ appear in the same column, we say k and $k-1$ paired. The other unpaired k's and $k-1$'s are swaped in each row.

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Example

f_2 acts on the following column-strict plane partitions:

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f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	2	2
4	4	3	2	2	2	1	1		
3	2	1	1						
2	1								

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5	5	4	3	3	3	3	2	2	2
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3	2	1	1						
2	1								

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f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	1	1
4	4	3	2	2	1	1	1		
3	2	2	1						
1	1								

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Example

f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	2	2
4	4	3	2	2	2	1	1		
3	2	1	1						
2	1								

The Bender-Knuth involution

Remark

f_2 gives a proof of

$$s_\lambda(x_2, x_1, x_3, \dots, x_n) = s_\lambda(x_1, x_2, x_3, \dots, x_n).$$

Hence $s_\lambda(x_1, x_2, \dots, x_n)$ is a symmetric function.

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps k's and $(k-1)$'s where we ignore saturated $(k-1)$ when we perform a swap.

Example

Twisted Bender-Knuth involution

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If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps k's and $(k-1)$'s where we ignore saturated $(k-1)$ when we perform a swap.

Example

$n = 7$ Apply $\widetilde{\pi}_3$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$ Apply $\widetilde{\pi}_3$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_3(c) \in \mathscr{P}_3$.

5	5	4	3	2
4	4	3	1	
3	3	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

We define an involution $\widetilde{\pi}_1$ on $\mathscr{P}_n$ similarly assuming the outside of the shape is filled with 0.

Example

Twisted Bender-Knuth involution

Definition

We define an involution $\widetilde{\pi}_1$ on $\mathscr{P}_n$ similarly assuming the outside of the shape is filled with 0.

Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	3	2
4	4	3	2	1
3	1			
1				

Twisted Bender-Knuth involution

Definition

We define an involution $\widetilde{\pi}_1$ on $\mathscr{P}_n$ similarly assuming the outside of the shape is filled with 0.

Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	3	2	1
4	4	3	2		
3	1	1			

Flips in words of RCSPP

Definition

We define involutions on $\mathscr{P}_n$

$$\widetilde{\rho} = \widetilde{\pi}_2 \widetilde{\pi}_4 \widetilde{\pi}_6 \cdots ,$$
$$\widetilde{\gamma} = \widetilde{\pi}_1 \widetilde{\pi}_3 \widetilde{\pi}_5 \cdots ,$$

and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Flips in words of RCSPP

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$$\widetilde{\rho} = \widetilde{\pi}_2\widetilde{\pi}_4\widetilde{\pi}_6\cdots,$$
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and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Conjecture 4 (Conjecture 4 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, $0 \leq r \leq n$ be integers. Then the number of elements c in $\mathscr{P}_n$ with $\widetilde{\rho}(c) = c$ and $\overline{U}_1(c) = r$ would be the same as the number of n by n alternating sign matrices a invariant under the half turn in their own planes (that is $a_{ij} = a_{n+1-i,n+1-j}$ for $1 \leq i, j \leq n$) and satisfying $a_{1,r} = 1$.

Flips in words of RCSPP

Definition

We define involutions on $\mathscr{P}_n$

$$\widetilde{\rho} = \widetilde{\pi}_2\widetilde{\pi}_4\widetilde{\pi}_6\cdots,$$
$$\widetilde{\gamma} = \widetilde{\pi}_1\widetilde{\pi}_3\widetilde{\pi}_5\cdots,$$

and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Conjecture 6 (Conjecture 6 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 3$ an odd integer and i, $0 \leq i \leq n-1$ be an integer. Then the number of c in $\mathscr{P}_n$ with $\gamma(c) = c$ and $\overline{U}_2(c) = i$ would be the same as the number of n by n alternating sign matrices with $a_{i1} = 1$ and which are invariant under the vertical flip (that is $a_{ij} = a_{i,n+1-j}$ for $1 \leq i, j \leq n$).

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_1^{\widetilde{\rho}} = \{\emptyset\}$

Invariants under $\widetilde{\rho}$

Example

$$\mathscr{P}_2^{\widetilde{\rho}} = \left\{ \emptyset, \boxed{1} \right\}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_3^{\widetilde{\rho}}$ is composed of the following 3 RCSPPs:

$\emptyset$

2
1

2	1
1	

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_4^{\widetilde{\rho}}$ is composed of the following 10 elements:

$\emptyset$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 2 & 1 & \color{red}{1} \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & \color{red}{2} \\ \hline 1 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 2 & \color{red}{2} & \color{red}{1} \\ \hline 1 & 1 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|}\hline \color{red}{3} \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline \color{red}{3} \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline \color{red}{3} & \color{red}{2} \\ \hline 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline \color{red}{3} & \color{red}{2} & \color{red}{1} \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_5^{\widetilde{\rho}}$ has 25 elements, and $\mathscr{P}_6^{\widetilde{\rho}}$ has 140 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

1	1

3	2	1
1		

3	3	1
2	2	
1		

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

The following $c \in \mathscr{P}_{11}$ is invariant under $\widetilde{\gamma}$:

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Remove all 1's from $c \in \mathscr{P}_{11}^{\widetilde{\gamma}}$.

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Then we obtain a PP in which each row has even length.

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Identify 3 with 2, 5 with 4, and 7 with 6.

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

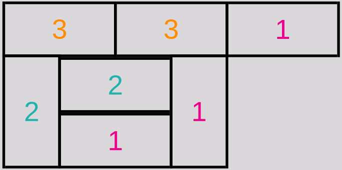
Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Repace 3 and 2 by dominos containing 1, 5 and 4 by dominos containing 2, 7 and 6 by dominos containing 3.



Column-strict domino plane partitions

Definition

Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{D}_{n,m}$ denote the set of column-strict domino plane partitions $d = (d_{ij})_{1 \leq i,j}$ such that

(D1) d has at most n columns;

(D2) each number in a domino which cross the jth column does not exceed $\lceil (n+m-j)/2 \rceil$.

If a number in a domino which cross the jth column of c is equal to $\lceil (n+m-j)/2 \rceil$, we call it a *saturated part*. Let $\mathscr{D}^{\mathrm{R}}_{n,m}$ (resp. $\mathscr{D}^{\mathrm{C}}_{n,m}$) denote the set of all $d \in \mathscr{D}_{n,m}$ which satisfy the condition that

(D3) each row (resp. column) of d has even length.

When $m = 0$, we write $\mathscr{D}_n$ for $\mathscr{D}_{n,0}$, $\mathscr{D}^{\mathrm{R}}_n$ for $\mathscr{D}^{\mathrm{R}}_{n,0}$ and $\mathscr{D}^{\mathrm{C}}_n$ for $\mathscr{D}^{\mathrm{C}}_{n,0}$.

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Column-strict domino plane partitions

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If a number in a domino which cross the jth column of c is equal to $\lceil (n + m - j)/2 \rceil$, we call it a *saturated part*. Let $\mathscr{D}^{R}_{n,m}$ (resp. $\mathscr{D}^{C}_{n,m}$) denote the set of all $d \in \mathscr{D}_{n,m}$ which satisfy the condition that

(D3) each row (resp. column) of d has even length.

When $m = 0$, we write $\mathscr{D}_n$ for $\mathscr{D}_{n,0}$, $\mathscr{D}^{R}_n$ for $\mathscr{D}^{R}_{n,0}$ and $\mathscr{D}^{C}_n$ for $\mathscr{D}^{C}_{n,0}$.

Column-strict domino plane partitions

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Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{D}_{n,m}$ denote the set of column-strict domino plane partitions $d = (d_{ij})_{1 \leq i,j}$ such that

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A bijection

Theorem

Let n be a positive integer. Let τ_{2n+1} denote our bijection of $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ onto $\mathscr{D}^{\mathrm{R}}_{2n-1}$. Then we have $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$.

Example

A bijection

Theorem

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Example

$\mathscr{D}^{\mathrm{R}}_{1} = \{\emptyset\}$ is the set of column-strict domino plane partitions with all columns ≤ 0.

A bijection

Theorem

Let n be a positive integer. Let τ_{2n+1} denote our bijection of $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ onto $\mathscr{D}^{\mathrm{R}}_{2n-1}$. Then we have $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$.

Example

$\mathscr{D}^{\mathrm{R}}_{3}$ is composed of the following 3 elements:

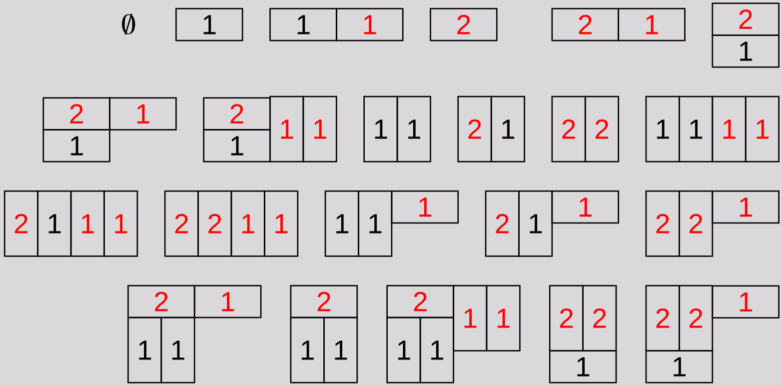
$$\emptyset, \qquad \boxed{1}\,, \qquad \boxed{1}\boxed{1}\,.$$

This is the set of column-strict domino plane partitions with the first and second columns ≤ 1, other columns ≤ 0 and each row of even length.

Example

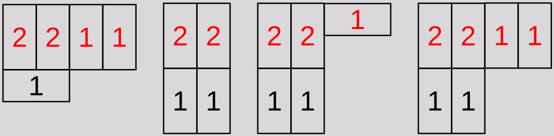
Example

$\mathscr{D}_5^R$ is the set of column-strict domino plane partitions with the 1st and 2nd columns ≤ 2, the 3rd and 4th columns ≤ 1, other columns ≤ 0 and each row of even length (26 elements):



Example

Example



$\mathscr{D}_7^{\mathrm{R}}$ is the set of column-strict domino plane partitions with the 1st and 2nd columns ≤ 3, the 3rd and 4th columns ≤ 2, the 5rd and 6th columns ≤ 1, other columns ≤ 0 and each row of even length (646 elements).

Statistics on Domino plane partitions

Definition

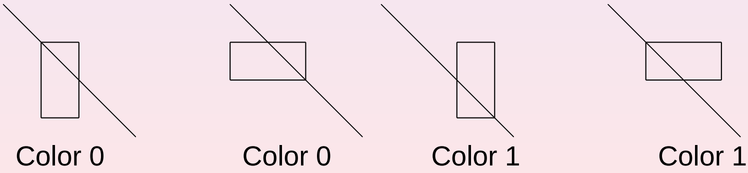
For $d \in \mathscr{D}_{n,m}$ and a positive integer $r \geq 1$, let $\overline{U}_r(d)$ denote the number of parts equal to r plus the number of saturated parts less than r.

Example

Theorem (Stanton-White, Carré-Leclerc)

We can define a map which associate a pair of column-strict plane partitions in $\mathscr{P}_{n,m}$ with a domino plane partition in $\mathscr{D}_{n,m}$.

Let Φ denote the map which associate the pair (c_0, c_1) of column-strict plane partitions with a column-strict domino plane partition d.



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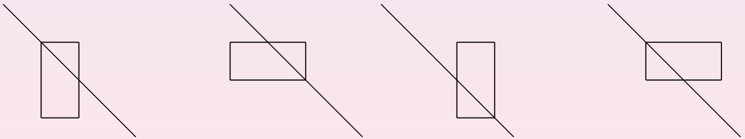

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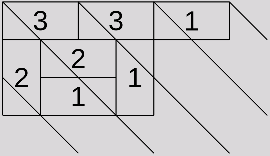
Domino plane partition

Example

For example, we associate the column-strict domino plane partition

$d =$



the pair

$c_0 =$

1	1

$c_1 =$

3	3	1
2	2	

of plane partitions.

Conditions on shape

Theorem

Let d be a column-strict domino plane partition, and let $(c_0, c_1) = \Phi(d)$. Then

(i) All columns of d have even length if, and only if, $\mathrm{sh}\, c_1 \subseteq \mathrm{sh}\, c_0$ and $\mathrm{sh}\, c_0 \setminus \mathrm{sh}\, c_1$ is a vertical strip.

(ii) All rows of d have even length if, and only if, $\mathrm{sh}\, c_0 \subseteq \mathrm{sh}\, c_1$ and $\mathrm{sh}\, c_1 \setminus \mathrm{sh}\, c_0$ is a horizontal strip.

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From RCSPPs to lattce paths

Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \leq y \leq x\}$ *be the vertex set, and direct an edge from u to v whenever* $v - u = (1, -1)$ *or* $(0, -1)$.
Let $u_j = (n - j, n - j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \ldots, n$, *and let* $\boldsymbol{u} = (u_1, \ldots, u_n)$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n)$. *We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with n-tuples of nonintersecting D-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$.

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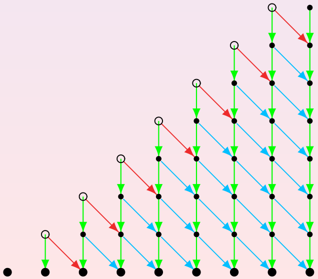


Example of lattice paths

Example

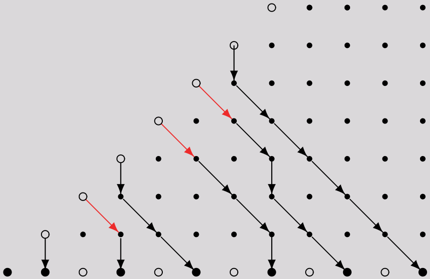
$n = 7$, $c \in \mathscr{P}_7$: RCSPP

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Example of lattice paths

Example

Lattice paths



Weight of each edge

Definition

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1. We assign the weight

$$\begin{cases} \prod_{k=j}^{n} t_k \cdot x_j & \text{if } j = i, \\ t_j x_j & \text{if } j < i, \end{cases}$$

to the horizontal edge from $u = (i, j)$ to $v = (i+1, j-1)$.

2. We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j-1)$.

We write

$$t^{\overline{U}(c)} x^c = t_1^{\overline{U}_1(c)} \cdots t_n^{\overline{U}_n(c)} x_1^{\sharp \text{ 1's in } c} \cdots x_n^{\sharp\, n\text{'s in } c}.$$

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Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c\in\mathscr{P}_n \\ \mathrm{sh}\,c=\lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(n-i)}_{\lambda_j-j+i}(t_1x_1,\dots,t_{n-i-1}x_{n-i-1},T_{n-i}x_{n-i})\right)_{1\leq i,j\leq n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

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$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \le n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_n \\ \mathrm{sh}\, c = \lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, T_{n-i} x_{n-i})\right)_{1 \le i,j \le n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	$t_1 x_1$	$t_1^2 t_2 t_3 x_1^2$	$t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

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$\emptyset$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1\\\hline 1 & \\\cline{1-1}\end{array}$
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

A determinantal expression

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{\mathrm{R}}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^{o}_{n}(t) = (R^{o}_{ij})_{0\leq i,j\leq n-1}$ be the $n \times n$ matrix where

$$R^{o}_{ij} = \binom{i+j-1}{2i-j} + \left(\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right)t + \binom{i+j-1}{2i-j}t^{2}$$

with the convention that $R^{o}_{0,0} = R^{o}_{0,1} = 1$. Then we obtain

$$\sum_{c\in\mathscr{P}^{\widetilde{\gamma}}_{2n+1}} t^{\overline{U}_1(c)} = \det R^{o}_{n}(t).$$

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Let $n \geq 2$ be a positive integer. Let $R^{\mathrm{o}}_n(t) = (R^{\mathrm{o}}_{i,j})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

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with the convention that $R^{\mathrm{o}}_{0,0} = R^{\mathrm{o}}_{0,1} = 1$. Then we obtain

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Let $n \geq 2$ be a positive integer. Let $R^{o}_{n}(t) = (R^{o}_{i,j})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

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with the convention that $R^{\mathrm{o}}_{0,0} = R^{\mathrm{o}}_{0,1} = 1$. Then we obtain

$$\sum_{c \in \mathscr{P}^{\gamma}_{2n+1}} t^{\overline{U}_2(c)} = \det R^{\mathrm{o}}_n(t).$$

The determinants

Example

If $n = 2$, then $\sum_{c \in \mathscr{P}_5^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 \\ 0 & 1 + t + t^2 \end{pmatrix}$$

which is equal to $1 + t + t^2$.

The determinants

Example

If $n = 3$, then $\sum_{c \in \mathscr{P}_7^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+t+t^2 & 1+2t+t^2 \\ 0 & t & 3+4t+3t^2 \end{pmatrix}$$

which is equal to $3 + 6t + 8t^2 + 6t^3 + 3t^4$.

The determinants

Example

If $n = 4$, then $\sum_{c \in \mathscr{P}_7^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1+t+t^2 & 1+2t+t^2 & t \\ 0 & t & 3+4t+3t^2 & 4+7t+4t^2 \\ 0 & 0 & 1+4t+t^2 & 10+15t+10t^2 \end{pmatrix}$$

which is equal to $26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$.

A constant term expression for the determinant

Theorem

Let $n \geq 2$ be a positive integer, and r be a positive integer such that $1 \leq r \leq n$. Then the generating function $\sum_{b \in \mathscr{P}_{2n-1}^{\widetilde{\gamma}}} t^{\overline{U}_r(b)}$ is given by

$$\mathrm{CT}_{\boldsymbol{x}}\mathrm{CT}_{\boldsymbol{y}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{1 \leq i < j \leq n} \left(1 - \frac{y_i}{y_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right)$$
$$\times \prod_{j=2}^{n} \left(1 + \frac{1}{y_j}\right)^{j-2} \left(1 + \frac{t}{y_j}\right) \prod_{j=1}^{n} (1 + y_j) \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}.$$

Generalized domino plane partitions

Generalized domino plane partitions

A *domino* is a special kind of skew shape consists of two squares. A 1×2 domino is called a *horizontal domino* while a 2×1 domino is called a *vertical domino*. A *generalized domino plane partition of shape λ* consists of a tiling of the shape λ by means of ordinary 1×1 squares or dominoes, and a filling of each square or domino with a positive integer so that the integers are weakly decreasing along either rows or columns. Further we call it a *domino plane partition* if the shape λ is tiled with only dominoes.

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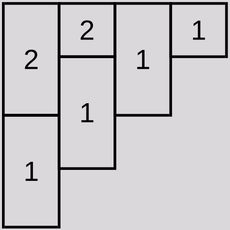
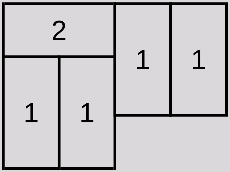
Generalized domino plane partitions

Example

The left-below is a column-strict generalized domino plane partition of shape $(4, 3, 2, 1)$, and the right-below is a column-strict domino plane partition of shape $(4, 4, 2)$.





Twisted domino plane partitions

Definition

Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{P}_{n,m}^{\mathrm{HTS}}$ denote the set of column-strict generalized domino plane partitions c subject to the constraints that

(E1) c has at most n columns;

(E2) each part in the jth column does not exceed $\lceil (n+m-j)/2 \rceil$;

(E3) A domino containing $\lceil (n+m-j)/2 \rceil$ must not cross the jth column for any j such that $n+m-j$ is odd.

(E4) A single box can appear only when it contains $\lceil (n+m-j)/2 \rceil$ and it is in the jth column such that $n+m-j$ is odd.

We call an element in $\mathscr{P}_{n,m}^{\mathrm{HTS}}$ a *twisted domino plane partition*, and we simply write $\mathscr{P}_{n}^{\mathrm{HTS}}$ for $\mathscr{P}_{n,0}^{\mathrm{HTS}}$.

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(E1) c has at most n columns;

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(E3) A domino containing $\lceil (n+m-j)/2 \rceil$ must not cross the jth column for any j such that $n+m-j$ is odd.

(E4) A single box can appear only when it contains $\lceil (n+m-j)/2 \rceil$ and it is in the jth column such that $n+m-j$ is odd.

We call an element in $\mathscr{P}^{\mathrm{HTS}}_{n,m}$ a *twisted domino plane partition*, and we simply write $\mathscr{P}^{\mathrm{HTS}}_{n}$ for $\mathscr{P}^{\mathrm{HTS}}_{n,0}$.

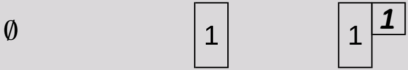
Twisted domino plane partitions

Example

$\mathscr{P}_1^{\mathrm{HTS}} = \{\emptyset\}$

$\mathscr{P}_2^{\mathrm{HTS}} = \left\{\emptyset, \boxed{\mathbf{\mathit{1}}}\right\}$

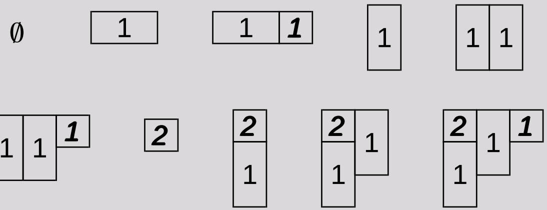
$\mathscr{P}_3^{\mathrm{HTS}}$ is composed of the following 3 elements:



Twisted domino plane partitions

Example

$\mathscr{P}_4^{\mathrm{HTS}}$ is composed of the following 10 elements:



$\mathscr{P}_5^{\mathrm{HTS}}$ has 25 elements and $\mathscr{P}_6^{\mathrm{HTS}}$ has 140 elements.

Twisted domino PPs and RCSDPPs with all columns of even length

Conjecture

For a positive integer n, there would be a bijection between $\mathscr{P}_n^{\mathrm{HTS}}$ (the set of twisted domono PPs) and $\mathscr{D}_n^{\mathrm{C}}$ (the set of restricted column-strict domino PPs with all columns of even length) which has the following property;

1. the numeber of 1's is kept invariant;
2. the number of columns is kept invariant.

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Twisted domino PPs and RCSDPPs with all columns of even length

General Conjecture

For a positive integer n, there would be a bijection between $\mathscr{P}^{\mathrm{HTS}}_{n,m}$ (the set of twisted domono PPs) and $\mathscr{D}^{\mathrm{C}}_{n,m}$ (the set of restricted column-strict domino PPs with all columns of even length) which has the following property;

1. the numeber of 1's is kept invariant;
2. the number of columns of each PP is kept invariant.

Twisted domino PPs and RCSDPPs with all columns of even length

General Conjecture

For a positive integer n, there would be a bijection between $\mathscr{P}^{\mathrm{HTS}}_{n,m}$ (the set of twisted domono PPs) and $\mathscr{D}^{\mathrm{C}}_{n,m}$ (the set of restricted column-strict domino PPs with all columns of even length) which has the following property;

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Twisted domino PPs and RCSDPPs with all columns of even length

General Conjecture

For a positive integer n, there would be a bijection between $\mathscr{P}^{\mathrm{HTS}}_{n,m}$ (the set of twisted domono PPs) and $\mathscr{D}^{\mathrm{C}}_{n,m}$ (the set of restricted column-strict domino PPs with all columns of even length) which has the following property;

1. the numeber of 1's is kept invariant;
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RCSDPPs with all columns of even length

Example

$\mathscr{D}_1^{\mathrm{C}} = \{\emptyset\}$

$\mathscr{D}_2^{\mathrm{C}} = \left\{\emptyset, \boxed{\mathbf{1}}\right\}$

$\mathscr{D}_3^{\mathrm{C}}$ has the following 3 elements:

$$\emptyset, \quad \boxed{1}\,, \quad \boxed{1}\boxed{1}\,.$$

RCSDPPs with all columns of even length

Example

$\mathscr{D}_4^{\mathrm{C}}$ has the following 10 elements:

$$\emptyset, \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \quad \begin{array}{|c|}\hline 2 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

$\mathscr{D}_5^{\mathrm{C}}$ has 25 elements, $\mathscr{D}_6^{\mathrm{C}}$ has 140 elements, and $\mathscr{D}_7^{\mathrm{C}}$ has 588 elements.

A determinantal expression

Theorem

Let n be a positive integer and let r be a integer such that $0 \leq r \leq n$.

A determinantal expression

Theorem

Let n be a positive integer and let r be a integer such that $0 \le r \le n$. If n is even, let $C_n^{\mathrm{e}}(t) = (C_{i,j}^{\mathrm{e}})_{0 \le i,j \le n/2-1}$ be the $n/2 \times n/2$ matrix where

$$C_{ij}^{\mathrm{e}} = \left\{2\binom{i+j-2}{2i-j-1} + \binom{i+j-2}{2i-j}\right\}(1+t^2) + \left\{2\binom{i+j-2}{2i-j-2} + \binom{i+j-2}{2i-j-1} + 2\binom{i+j-2}{2i-j} + \binom{i+j-2}{2i-j+1}\right\} t$$

with the convention that $C_{0,0}^{\mathrm{e}} = 1+t$, $C_{0,1}^{\mathrm{e}} = t$ and $C_{1,0}^{\mathrm{e}} = 0$.

Then we obtain

$$\sum_{d \in \mathcal{D}_n^{\mathrm{C}}} t^{\overline{U}_r(d)} = \det C_n^{\mathrm{e}}(t).$$

A determinantal expression

Theorem

Let n be a positive integer and let r be a integer such that $0 \le r \le n$. If n is even, let $C_n^{\mathrm{e}}(t) = (C_{i,j}^{\mathrm{e}})_{0 \le i,j \le n/2-1}$ be the $n/2 \times n/2$ matrix where

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with the convention that $C_{0,0}^{\mathrm{e}} = 1 + t$, $C_{0,1}^{\mathrm{e}} = t$ and $C_{1,0}^{\mathrm{e}} = 0$.
Then we obtain

$$\sum_{d \in \mathscr{D}_n^{\mathrm{C}}} t^{\overline{U}_r(d)} = \det C_n^{\mathrm{e}}(t).$$

A determinantal expression

Theorem

Let n be a positive integer and let r be a integer such that $0 \le r \le n$. If n is odd, let $C_n^{o}(t) = (C_{i,j}^{o})_{0 \le i,j \le (n-1)/2}$ be the $(n+1)/2 \times (n+1)/2$ matrix where

$$C_{ij}^{o} = \left\{ 2\binom{i+j-3}{2i-j-2} + \binom{i+j-3}{2i-j-1} \right\}(1+t^2) + \left\{ 2\binom{i+j-3}{2i-j-3} + \binom{i+j-3}{2i-j-2} + 2\binom{i+j-3}{2i-j-1} + \binom{i+j-3}{2i-j} \right\} t$$

with the convention that $C_{0,0}^{o} = 1$, $C_{0,1}^{o} = C_{0,2}^{o} = C_{2,0}^{o} = 0$, $C_{1,0}^{o} = 1 + t$ and $C_{1,1}^{o} = 1 + t + t^2$. Then we obtain

$$\sum_{d \in \mathscr{D}_n^{C}} t^{\overline{U}_r(d)} = \det C_n^{o}(t).$$

A determinantal expression

Theorem

Let n be a positive integer and let r be a integer such that $0 \le r \le n$. If n is odd, let $C_n^{\mathrm{o}}(t) = (C_{i,j}^{\mathrm{o}})_{0 \le i,j \le (n-1)/2}$ be the $(n+1)/2 \times (n+1)/2$ matrix where

$$C_{ij}^{\mathrm{o}} = \left\{ 2\binom{i+j-3}{2i-j-2} + \binom{i+j-3}{2i-j-1} \right\} (1+t^2) + \left\{ 2\binom{i+j-3}{2i-j-3} + \binom{i+j-3}{2i-j-2} + 2\binom{i+j-3}{2i-j-1} + \binom{i+j-3}{2i-j} \right\} t$$

with the convention that $C_{0,0}^{\mathrm{o}} = 1$, $C_{0,1}^{\mathrm{o}} = C_{0,2}^{\mathrm{o}} = C_{2,0}^{\mathrm{o}} = 0$, $C_{1,0}^{\mathrm{o}} = 1 + t$ and $C_{1,1}^{\mathrm{o}} = 1 + t + t^2$. Then we obtain

$$\sum_{d \in \mathscr{D}_n^{\mathrm{C}}} t^{\overline{U}_r(d)} = \det C_n^{\mathrm{o}}(t).$$

A constant term expression for the determinant

Theorem

Let $n \geq 2$ be a positive integer, and r be a positive integer such that $1 \leq r \leq n$. Then the generating function $\sum_{b \in \mathscr{P}^{\tilde{\gamma}}_{2n-1}} t^{\overline{U}_r(b)}$ is given by

$$\mathrm{CT}_{\boldsymbol{x}}\mathrm{CT}_{\boldsymbol{y}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{1 \leq i < j \leq n} \left(1 - \frac{y_i}{y_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right) \times \prod_{j=2}^{n} \left(1 + \frac{1}{y_j}\right)^{j-2} \left(1 + \frac{t}{y_j}\right) \prod_{i=1}^{n} (1 - x_i)^{-1} \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}.$$

Monotone triangle conjecture

Definition

Let $\mathscr{A}_n^k$ denote the set of $n \times n$ alternating sign matrices $a = (a_{ij})_{1 \le i,j \le n}$ such that

- $a_{ij} = 0$ if $i - j > k$.

Example

Monotone triangle conjecture

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- $a_{ij} = 0$ if $i - j > k$.

Example

$n = 3$, $k = 0$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The generating function is 1.

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- $a_{ij} = 0$ if $i - j > k$.

Example

$n = 3$, $k = 1$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The generating function is $2 + 2t + t^2$.

Monotone triangle conjecture

Definition

Let $\mathscr{A}_n^k$ denote the set of $n \times n$ alternating sign matrices $a = (a_{ij})_{1 \le i,j \le n}$ such that

- $a_{ij} = 0$ if $i - j > k$.

Example

$n = 3, k = 2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The generating function is $2 + 3t + 2t^2$.

$\mathscr{P}^k_{n,m}$

Definition

Let $\mathscr{P}^k_{n,m}$ denote the set of RCSPPs $c \in \mathscr{P}_{n,m}$ such that

- c has at most k rows.

We write $\mathscr{P}^k_n$ for $\mathscr{P}^k_{n,0}$.

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Example

If $n = 3$ and $k = 0$, $\mathscr{P}^0_3$ consists of the single PP:

$$\emptyset.$$

$\sum_{c\in\mathscr{P}^0_3} t^{\overline{U}_r(c)} = 1$

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Example

If $n = 3$ and $k = 1$, $\mathscr{P}^1_3$ consists of the following 5 PPs:

$\emptyset$	1	1 1	2	2 1

$\sum_{c\in\mathscr{P}^1_3} t^{\overline{U}_r(c)} = 2 + 2t + t^2$

$\mathscr{P}^k_{n,m}$

Definition

Let $\mathscr{P}^k_{n,m}$ denote the set of RCSPPs $c \in \mathscr{P}_{n,m}$ such that

- c has at most k rows.

We write $\mathscr{P}^k_n$ for $\mathscr{P}^k_{n,0}$.

Example

If $n = 3$ and $k = 2$, $\mathscr{B}^2_3$ consists of the followng 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$\sum_{c\in\mathscr{P}^2_3} t^{\overline{U}_r(c)} = 2 + 3t + 2t^2$

The Mills-Robins-Rumsey conjecture in words of RCSPPs

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let n, k and r be integers such that $n \geq 2$, $0 \leq k \leq n-1$ and $0 \leq r \leq n$. Then the number of c in $\mathscr{P}_n^k$ with $\overline{U}_r(c) = j$ would be the same as the number of alternating sign matrices $a = (a_{ij})_{1 \leq i,j \leq n} \in \mathscr{A}_n^k$ such that $a_{1j} = 1$.

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A Pfaffian formula

Theorem

Let $n \geq 2$ be a positive integer, and k be a positive integer such that $1 \leq k \leq n$. If r is a positive integer such that $1 \leq r \leq n$, then the generating function for all plane partitions $c \in \mathscr{P}_n^k$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_n^k} t^{\overline{U}_r(c)} = \lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \mathrm{Pf} \begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \overline{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix}.$$

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Definition

For positive integers n and N, let $B_n^N(t) = (b_{ij}(t))_{0 \leq i \leq n-1,\ 0 \leq j \leq n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1} t & \text{otherwise.} \end{cases}$$

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Theorem

Let $n \geq 2$ be a positive integer, and k be a positive integer such that $1 \leq k \leq n$. If r is a positive integer such that $1 \leq r \leq n$, then the generating function for all plane partitions $c \in \mathscr{P}_n^k$ with the weight $t^{\overline{U}_r(c)}$ is given by

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Definition

For positive integers n, let $J_n = (\delta_{i,n+1-j})_{1 \leq i,j \leq n}$ be the $n \times n$ anti-diagonal matrix.

A Pfaffian formula

Theorem

Let $n \geq 2$ be a positive integer, and k be a positive integer such that $1 \leq k \leq n$. If r is a positive integer such that $1 \leq r \leq n$, then the generating function for all plane partitions $c \in \mathscr{P}_n^k$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_n^k} t^{\overline{U}_r(c)} = \lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \mathrm{Pf} \begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \overline{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix}.$$

Definition

$\overline{L}_n^{(m,k)}(\varepsilon) = (\overline{l}_{ij}^{(m,k)}(\varepsilon))_{1 \leq i,j \leq n}$ (k is even)

$$\overline{l}_{ij}^{(m,k)}(\varepsilon) = \begin{cases} (-1)^{j-i-1}\varepsilon & \text{if } 1 \leq i < j \leq n \text{ and } i \leq m+k, \\ (-1)^{j-i-1} & \text{if } m+k < i < j \leq n. \end{cases}$$

A Pfaffian formula

Theorem

Let $n \geq 2$ be a positive integer, and k be a positive integer such that $1 \leq k \leq n$. If r is a positive integer such that $1 \leq r \leq n$, then the generating function for all plane partitions $c \in \mathscr{P}_n^k$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_n^k} t^{\overline{U}_r(c)} = \lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \mathrm{Pf} \begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \overline{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix}.$$

Definition

$\overline{L}_n^{(m,k)}(\varepsilon) = (\overline{l}_{ij}^{(m,k)}(\varepsilon))_{1 \leq i,j \leq n}$ (k is odd)

$$\overline{l}_{ij}^{(m,k)}(\varepsilon) = \begin{cases} (-1)^{j-i-1}\varepsilon & \text{if } 1 \leq i < j \leq m+k, \\ (-1)^{j-i-1} & \text{if } 1 \leq i < j \leq n \text{ and } m+k < j. \end{cases}$$

A constant term identity

Theorem

Let n be a positive integer. If $0 \leq k \leq n-1$ and $1 \leq r \leq n$, then $\sum_{c \in \mathscr{P}_n^k} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right)$$
$$\times \frac{\det(x_i^{j-1} - x_i^{k+2n-j})_{1 \leq i,j \leq n}}{\prod_{i=1}^{n}(1-x_i) \prod_{1 \leq i < j \leq n}(x_j - x_i)(1 - x_i x_j)}.$$

Example of $n = 3$

Example

If $n = 3$ and $k = 0$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times \frac{\det\begin{pmatrix} 1-x_1^{5} & x_1-x_1^{4} & x_1^2-x_1^{3} \\ 1-x_2^{5} & x_2-x_1^{4} & x_2^2-x_2^{3} \\ 1-x_3^{5} & x_3-x_1^{4} & x_3^2-x_3^{3} \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to 1.

Example of $n = 3$

Example

If $n = 3$ and $k = 1$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times \frac{\det\begin{pmatrix} 1-x_1^{6} & x_1-x_1^{5} & x_1^2-x_1^{5} \\ 1-x_2^{6} & x_2-x_1^{5} & x_2^2-x_2^{5} \\ 1-x_3^{6} & x_3-x_1^{5} & x_3^2-x_3^{5} \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2 + 2t + t^2$.

Example of $n = 3$

Example

If $n = 3$ and $k = 2$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$

$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$

$$\times \frac{\det\begin{pmatrix} 1-x_1^{7} & x_1-x_1^{6} & x_1^2-x_1^{5} \\ 1-x_2^{7} & x_2-x_1^{6} & x_2^2-x_2^{5} \\ 1-x_3^{7} & x_3-x_1^{6} & x_3^2-x_3^{5} \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2 + 3t + 2t^2$.

References

Main papers

1. M. Ishikawa, “On refined enumerations of totally symmetric self-complementary plane partitions I”, `arXiv:math.CO/0602068`.
2. M. Ishikawa, “On refined enumerations of totally symmetric self-complementary plane partitions II”, `arXiv:math.CO/0606082`.

The end

Thank you!