A proof of Stanley's open problem

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References

- R. P. Stanley, “Open problem”, International Conference on Formal Power Series and Algebraic Combinatorics (Vadstena 2003), June 23 - 27, 2003, available from http://www-math.mit.edu/∼rstan/trans.html.
- M. Ishikawa, “Minor summation formula and a proof of Stanley’s open problem”, `arXiv:math.CO/0408204`.
- Masao Ishikawa, Hiroyuki Tagawa, Soichi Okada and Jiang Zeng, “Generalizations of Cuachy’s determinant and Schur’s Pfaffian”, `arXiv:math.CO/0411280`.
- M. Ishikawa and Jiang Zeng, “The Andrews-Stanley partition function and Al-Salam-Chihara polynomials”, `arXiv:math.CO/0506128`.

The ring of symmetric functions

The ring Λ of symmetric functions in countably many variables x_1, x_2, $\ldots$ is defined by the inverse limit. (See Macdonald's book "Symmetric functions and Hall polynomials, 2nd Edition", Oxford University Press, I, 2.)

Here we use the convention that $f(x)$ stands for a symmetric function in countably many variables $x = (x_1, x_2, \ldots)$, whereas $f(X)$ stands for a symmetric function in finitely many variables $X = (x_1, \ldots, x_n)$.

The Schur functions

For $X = (x_1, \ldots, x_n)$ and a partition λ such that $\ell(\lambda) \leq n$, let

$$s_\lambda(X) = \frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}}.$$

$s_\lambda(X)$ is called the Schur function corresponding to λ.

Power Sum Symmetric Functions

Let r denote a positive integer.

$$p_r(X) = x_1^r + x_2^r + \cdots + x_n^r$$

is called the rth power sum symmetric function.

$$p_1(X) = x_1 + x_2 + \cdots + x_n$$

$$p_2(X) = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$p_3(X) = x_1^3 + x_2^3 + \cdots + x_n^3$$

The four parameter weight

Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ, which is equal to $a^5 b^4 c^3 d^2$.

a	b	a	b	a
c	d	c	d	
a	b	a	b	
c				

An open problem by Richard Stanley

In FPSAC'03 R.P. Stanley gave the following conjecture in the open problem session:

Theorem

Let

$$z = \sum_{\lambda} \omega(\lambda) s_\lambda(x),$$

where the sum runs over all partitions λ.

Then we have

$$\log z - \sum_{n\ge1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n\ge1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

Strategy of the proof

1. Step1. Express $\omega(\lambda)$ and z by a single Pfaffian.

   Use the minor summation formula of Pfaffians.

2. Step2. Express z by a single determinant.

   Use the homogenious version of Okada's gereralization of Schur's Pfaffian.

3. Step3. Show that

$$\log z - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

   Use Stembridge's criterion.

The goal of the proof

Put

$$w = \log z - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2$$

and use the following Stembridge's criterion to w.

Proposition (Stembridge)

Let $f(x_1, x_2, \dots)$ be a symmetric function with infinite variables. Then

$$f \in \mathbb{Q}[p_1, p_3, p_5, \dots]$$

if and only if

$$f(t, -t, x_1, x_2, \dots) = f(x_1, x_2, \dots).$$

The aim of Step1

Can we write z by a Pfaffian?

Theorem A

Let n be a positive integer. Let

$$z_n = \sum_{\ell(\lambda) \le 2n} \omega(\lambda) s_\lambda(X_{2n})$$

be the sum restricted to $2n$ variables. Then we have

$$z_n = \frac{1}{\prod_{1 \le i < j \le 2n} (x_i - x_j)} (abcd)^{-\binom{n}{2}} \operatorname{Pf} (p_{ij})_{1 \le i < j \le 2n},$$

where

$$p_{ij} = \frac{\det \begin{pmatrix} x_i + a x_i^2 & 1 - a(b+c)x_i - abc x_i^3 \\ x_j + a x_j^2 & 1 - a(b+c)x_j - abc x_j^3 \end{pmatrix}}{(1 - ab x_i^2)(1 - ab x_j^2)(1 - abcd x_i^2 x_j^2)}.$$

The key idea to prove Theorem A

Can we write the four parameter weight $\omega(\lambda)$ by a Pfaffian?

Notation

Let m, n and r be integers such that $r \leq m, n$. Let A be an m by n matrix. For any index sets

$$I = \{i_1, \ldots, i_r\}_< \subseteq [m],$$
$$J = \{j_1, \ldots, j_r\}_< \subseteq [n],$$

let $\Delta^I_J(A)$ denote the submatrix obtained by selecting the rows indexed by I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write $\Delta_J(A)$ for $\Delta^{[m]}_J(A)$.

Notation

Fix a positive integer n.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition such that $\ell(\lambda) \leq n$,

then we put

$$l = (l_1, \ldots, l_n) = \lambda + \delta_n = (\lambda_1 + n - 1, \ldots, \lambda_n),$$

where $\delta_n = (n - 1, n - 2, \ldots, 1, 0)$,

and we write

$$I_n(\lambda) = \{l_n, l_{n-1}, \ldots, l_1\}.$$

We regard this set as a set of row/column indices.

Example

If $n = 6$ and $\boldsymbol{\lambda} = (5, 4, 4, 1, 0, 0)$, then

$$\boldsymbol{l} = \boldsymbol{\lambda} + \boldsymbol{\delta}_6 = (10, 8, 7, 3, 1, 0),$$

and

$$I_6(\boldsymbol{\lambda}) = \{0, 1, 3, 7, 8, 10\}.$$

Fact:

$$s_\lambda(X) = \frac{\det\left(\Delta_{I_n(\lambda)}(T)\right)_{1\leq i,j\leq n}}{\det(x_i^{j-1})_{1\leq i,j\leq n}}.$$

From now we restrict our attention to the case where n is even so that n will be replaced by $2n$ hereafter.

Theorem

Define a skew-symmetric array $A = (\alpha_{ij})_{0 \leq i,j}$ by

$$\alpha_{ij} = a^{\lceil (j-1)/2 \rceil} b^{\lfloor (j-1)/2 \rfloor} c^{\lceil i/2 \rceil} d^{\lfloor i/2 \rfloor}$$

for $i < j$.

Then we have

$$\mathrm{Pf}\left[A^{I_{2n}(\lambda)}_{I_{2n}(\lambda)}\right] = (abcd)^{\binom{n}{2}} \omega(\lambda).$$

Example

$A = (\alpha_{ij})_{0 \le i,j}$:

$$\begin{bmatrix} 0 & 1 & a & ab & a^2b & a^2b^2 & \dots \\ -1 & 0 & ac & abc & a^2bc & a^2b^2c & \dots \\ -a & -ac & 0 & abcd & a^2bcd & a^2b^2cd & \dots \\ -ab & -abc & -abcd & 0 & a^2bc^2d & a^2b^2c^2d & \dots \\ -a^2b & -a^2bc & -a^2bcd & -a^2bc^2d & 0 & a^2b^2c^2d^2 & \dots \\ -a^2b^2 & -a^2b^2c & -a^2b^2cd & -a^2b^2c^2d & -a^2b^2c^2d^2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The idea of the proof of Theorem A

- Write the Schur function $s_\lambda(\boldsymbol{X}_{2n})$ by the quotient of determinants. (The denominator is the Vandermonde determinant.)
- Write the weight $\omega(\boldsymbol{\lambda})$ by the Pfaffian.
- Take the product of the Pfaffian and the determinant, and sum up over all columns.

Theorem (Minor summation formula)

Let n and N be non-negative integers such that $2n \leq N$. Let $T = (t_{ij})_{1\leq i\leq 2n, 1\leq j\leq N}$ be a $2n$ by N rectangular matrix, and let $A = (a_{ij})_{1\leq i,j\leq N}$ be a skew-symmetric matrix of size N. Then

$$\sum_{I\in\binom{[N]}{2n}} \mathrm{Pf}\left(\Delta_I^I(A)\right)\det\left(\Delta_I(T)\right) = \mathrm{Pf}\left(T A\, {}^t T\right).$$

If we put $Q = (Q_{ij})_{1\leq i,j\leq 2n} = T A\, {}^t T$, then its entries are given by

$$Q_{ij} = \sum_{1\leq k<l\leq N} a_{kl} \det\left(\Delta_{kl}^{ij}(T)\right),$$

$(1 \leq i, j \leq 2n)$. Here we write $\Delta_{kl}^{ij}(T)$ for

$$\Delta_{\{kl\}}^{\{ij\}}(T) = \begin{vmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl} \end{vmatrix}.$$

The aim of Step2

Can we express the Pfaffian by a determinant?

Schur's Pfaffian

$$\operatorname{Pf}\left[\frac{x_i - x_j}{x_i + x_j}\right]_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j}.$$

(I. Schur, "Über die Darstellung der symmetrischen und der alternirenden Gruppe durch gebrochene lineare Substitutionen", J. Reine Angew. Math. **139** (1911), 155–250.)

A generalization

M. Ishikawa, S. Okada, H. Tagawa and J. Zeng "Generalizations of Cauchy's determinant and Schur's Pfaffian", `arXiv:math.CO/0411280`.

We gathered more generalizations of Cauchy's determinant and Schur's Pfaffian and their applications.

A homogeneous generalized Vandermonde determinants

Let $X = (x_1, \cdots, x_r)$, $Y = (y_1, \cdots, y_r)$, $A = (a_1, \cdots, a_r)$ and $B = (b_1, \cdots, b_r)$ be four vectors of variables of length r. For nonnegative integers p and q with $p + q = r$, define a generalized Vandermonde matrix $U^{p,q}(X, Y; A, B)$ by the $r \times r$ matrix with ith row

$$(a_i x_i^{p-1}, a_i x_i^{p-2} y_i, \cdots, a_i y_i^{p-1}, b_i x_i^{q-1}, b_i x_i^{p-2} y_i, \cdots, b_i y_i^{q-1}).$$

In this talk we restrict our attention to the case where $p = q = n$. Thus we write $U^n(X, Y; A, B)$ for $U^{n,n}(X, Y; A, B)$.

Example

When $n = 1$,

$$U^1(X, Y; A, B) = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

When $n = 2$,

$$U^2(X, Y; A, B) = \begin{bmatrix} a_1x_1 & a_1y_1 & b_1x_1 & b_1y_1 \\ a_2x_2 & a_2y_2 & b_2x_2 & b_2y_2 \\ a_3x_3 & a_3y_3 & b_3x_3 & b_3y_3 \\ a_4x_4 & a_4y_4 & b_4x_4 & b_4y_4 \end{bmatrix}.$$

When $n = 3$, $U^3(X, Y; A, B)$ is

$$\begin{bmatrix} a_1x_1^2 & a_1x_1y_1 & a_1y_1^2 & b_1x_1^2 & b_1x_1y_1 & b_1y_1^2 \\ a_2x_2^2 & a_2x_2y_2 & a_2y_2^2 & b_2x_2^2 & b_2x_2y_2 & b_2y_2^2 \\ a_3x_3^2 & a_3x_3y_3 & a_3y_3^2 & b_3x_3^2 & b_3x_3y_3 & b_3y_3^2 \\ a_4x_4^2 & a_4x_4y_4 & a_4y_4^2 & b_4x_4^2 & b_4x_4y_4 & b_4y_4^2 \\ a_5x_5^2 & a_5x_5y_5 & a_5y_5^2 & b_5x_5^2 & b_5x_5y_5 & b_5y_5^2 \\ a_6x_6^2 & a_6x_6y_6 & a_6y_6^2 & b_6x_6^2 & b_6x_6y_6 & b_6y_6^2 \end{bmatrix}.$$

A generalized Schur's Pfaffian

Theorem (A homogeneous version, a special case)

For six vectors of variables

$$X = (x_1, \cdots, x_{2n}),\ Y = (y_1, \cdots, y_{2n}),\ A = (a_1, \cdots, a_{2n}),$$
$$B = (b_1, \cdots, b_{2n}),\ C = (c_1, \cdots, c_{2n}),\ D = (d_1, \cdots, d_{2n}),$$

we have

$$\operatorname*{Pf}_{1 \le i < j \le 2n} \left[\frac{\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \cdot \begin{vmatrix} c_i & d_i \\ c_j & d_j \end{vmatrix}}{\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}} \right] = \frac{\det U^n(X, Y; A, B) \det U^n(X, Y; C, D)}{\prod_{1 \le i < j \le 2n} \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}}.$$

Application to this problem

Corollary

For three vectors of variables

$$X_{2n} = (x_1, \dots, x_{2n}), \ A_{2n} = (a_1, \dots, a_{2n}), \ B_{2n} = (b_1, \dots, b_{2n})$$

we have

$$\operatorname*{Pf}_{1 \le i < j \le 2n} \left[\frac{a_i b_j - a_j b_i}{1 - t x_i x_j} \right] = (-1)^{\binom{n}{2}} t^{\binom{n}{2}} \frac{\det U^n(X_{2n}, 1 + tX_{2n}^2; A_{2n}, B_{2n})}{\prod_{1 \le i < j \le 2n}(1 - t x_i x_j)},$$

where $X_{2n}^2 = (x_1^2, \dots, x_{2n}^2)$ and $1 + tX_{2n} = (1 + x_1^2, \dots, 1 + x_{2n}^2)$.

Answer to the question in Step2:

Theorem B

Let $X = (x_1, \dots, x_{2n})$ be a $2n$-tuple of variables. Then

$$z_n(X_{2n}) = (-1)^{\binom{n}{2}}$$
$$\times \frac{\det U^n(X^2, \mathbf{1} + abcdX^4; X + aX^2, \mathbf{1} - a(b+c)X^2 - abcX^3)}{\prod_{i=1}^{2n}(1 - abx_i^2) \prod_{1 \le i < j \le 2n}(x_i - x_j)(1 - abcdx_i^2 x_j^2)},$$

where $X^2 = (x_1^2, \dots, x_{2n}^2)$, $\mathbf{1} + abcdX^4 = (1 + abcdx_1^4, \dots, 1 + abcdx_{2n}^4)$, $X + aX^2 = (x_1 + ax_1^2, \dots, x_{2n} + ax_{2n}^2)$ and $\mathbf{1} - a(b+c)X^2 - abcX^3 = (1 - a(b+c)x_1^2 - abcx_1^3, \dots, 1 - a(b+c)x_{2n}^2 - abcx_{2n}^3)$.

Example

When $n = 2$,

$$U^2(X^2, \mathbf{1} + abcdX^4; X + aX^2, \mathbf{1} - a(b+c)X^2 - abcX^3)$$

looks as follows:

$$\begin{bmatrix} a_1x_1^2 & a_1(1 + abcdx_1^2) & b_1x_1^2 & b_1(1 + abcdx_1^2) \\ a_2x_2^2 & a_2(1 + abcdx_2^2) & b_2x_2^2 & b_2(1 + abcdx_2^2) \\ a_3x_3^2 & a_3(1 + abcdx_3^2) & b_3x_3^2 & b_3(1 + abcdx_3^2) \\ a_4x_4^2 & a_4(1 + abcdx_4^2) & b_4x_4^2 & b_4(1 + abcdx_4^2) \end{bmatrix}$$

where $a_i = x_i + ax_i^2$ and $b_i = 1 - a(b+c)x_i^2 - abcx_i^3$.

The aim of Step3

Prove Stanley's open problem by evaluating the determinant obtained in Theorem B (Use Stembridge's criterion).

Criterion

Proposition (Stembridge)

Let $f(x_1, x_2, \dots)$ be a symmetric function with infinite variables. Then

$$f \in \mathbb{Q}[p_1, p_3, p_5, \dots]$$

if and only if

$$f(t, -t, x_1, x_2, \dots) = f(x_1, x_2, \dots).$$

See Stanley's book "Enumerative Combinatorics II", p.p. 450, Exercise 7.7, or Stembridge's paper "Enriched P-partitions", Trans. Amer. Math. Soc. **349** (1997), 763–788.

Method of the proof

Put

$$w_n(X_{2n}) = \log z_n(X_{2n}) - \sum_{k\geq 1} \frac{1}{2k} a^k (b^k - c^k) p_{2k}(X_{2n}) - \sum_{k\geq 1} \frac{1}{4k} a^k b^k c^k d^k p_{2k}(X_{2n})^2.$$

Our goal is to show

$$w_{n+1}(t, -t, X_{2n}) = w_n(X_{2n}).$$

The end of the proof

To finish the proof, it is enough to show tha following:

Let $X = X_{2n} = (x_1, \ldots, x_{2n})$ be a $2n$-tuple of variables.

Put

$$f_n(X_{2n}) = \det U^n(X^2, 1 + abcdX^4; X + aX^2, 1 - a(b+c)X^2 - abcX^3).$$

Then $f_n(X_{2n})$ satisfies

$$f_{n+1}(t, -t, X_{2n}) = (-1)^n \cdot 2t(1 - abt^2)(1 - act^2) \times \prod_{i=1}^{2n}(t^2 - x_i^2) \prod_{i=1}^{2n}(1 - abcdt^2x_i^2) \cdot f_n(X_{2n}).$$

Corollaries and conjectures

The Big Schur functions

Let $S_\lambda(x;t) = \det(q_{\lambda_i - i + j}(x;t))$ denote the big Schur function corresponding to the partition λ.

Corollary

Let

$$Z(x;t) = \sum_\lambda \omega(\lambda) S_\lambda(x;t),$$

Here the sum runs over all partitions λ.

Then we have

$$\log Z(x;t) - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n)(1 - t^{2n}) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n (1 - t^{2n})^2 p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

Certain symmetric functions related to the Macdonald polynomials

Definition

Define $T_\lambda(x;q,t)$ by

$$T_\lambda(x;q,t) = \det\left(Q_{(\lambda_i-i+j)}(x;q,t)\right)_{1\le i,j\le \ell(\lambda)},$$

where $Q_\lambda(x;q,t)$ stands for the Macdonald polynomial corresponding to the partition λ, and $Q_{(r)}(x;q,t)$ is the one corresponding to the one row partition (r) (See Macdonald's book, IV, sec.4).

Corollary

Let

$$Z(x;q,t)=\sum_{\lambda}\omega(\lambda)T_{\lambda}(x;q,t),$$

Here the sum runs over all partitions λ.

Then we have

$$\log Z(x;q,t)-\sum_{n\geq 1}\frac{1}{2n}a^n(b^n-c^n)\frac{1-t^{2n}}{1-q^{2n}}p_{2n}$$
$$-\sum_{n\geq 1}\frac{1}{4n}a^n b^n c^n d^n\frac{(1-t^{2n})^2}{(1-q^{2n})^2}p_{2n}^2\in\mathbb{Q}[[p_1,p_3,p_5,\dots]].$$

Conjectures

Conjecture

Let

$$w(x;t) = \sum_{\lambda} \omega(\lambda) P_{\lambda}(x;t),$$

where $P_{\lambda}(x;t)$ denote the Hall-Littlewood function corresponding to the partition λ, and the sum runs over all partitions λ. Then

$$\log w(x;-1) + \sum_{n\geq 1 \text{ odd}} \frac{1}{2n} a^n c^n p_{2n}$$

$$+ \sum_{n\geq 2 \text{ even}} \frac{1}{2n} a^{\frac{n}{2}} c^{\frac{n}{2}} (a^{\frac{n}{2}} c^{\frac{n}{2}} - 2 b^{\frac{n}{2}} d^{\frac{n}{2}}) p_{2n} \in \mathbb{Q}[[p_1, p_3, p_5 \ldots]].$$

would hold.

Conjecture

Let

$$w(x;q,t)=\sum_{\lambda}\omega(\lambda)P_{\lambda}(x;q,t).$$

where $P_{\lambda}(x;q,t)$ denote the Macdonald polynomial corresponding to the partition λ, and the sum runs over all partitions λ. Then

$$\log w(x;q,-1)+\sum_{n\geq 1\text{ odd}}\frac{1}{2n}a^{n}c^{n}p_{2n}$$

$$+\sum_{n\geq 2\text{ even}}\frac{1}{2n}a^{\frac{n}{2}}c^{\frac{n}{2}}(a^{\frac{n}{2}}c^{\frac{n}{2}}-2b^{\frac{n}{2}}d^{\frac{n}{2}})p_{2n}\in\mathbb{Q}(q)[[p_1,p_3,p_5,\dots]]$$

would hold.

Further results afterward

M. Ishikawa and Jiang Zeng, "The Andrews-Stanley partition function and Al-Salam-Chihara polynomials", `arXiv:math.CO/0506128`.

A generalization of the main result by G.E. Andrews in "On a Partition Function of Richard Stanley", a weighted sum of Schur's P-funstions and Q-functions.

Thank you!