The Andrews-Stanley partition function and Al-Salam-Chihara polynomials

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Basic hypergeometric series

We shall define an ${}_r\phi_s$ basic hypergeometric series by

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n $$

with $\binom{n}{2}=\frac{n(n-1)}{2}$, where $q\neq 0$ when $r>s+1$.

We shall use the compact notations

$$ (a_1,\ldots,a_m;q)_n = (a_1,q)_n\cdots(a_n,q)_n $$
$$ (a_1,\ldots,a_m;q)_\infty = (a_1,q)_\infty\cdots(a_n,q)_\infty $$

for the product of q-shifted factorials:

$$ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). $$

The generating function of partitions

Theorem (Euler)

For $|q| < 1$,

$$\sum_{\lambda} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \qquad \left(\sum_{\lambda} q^{|\lambda|} = \prod_{n=1}^{N} \frac{1}{1-q^n} \right)$$

where the sum runs over all partitions λ (where each part of λ is $\leq N$).

More generally,

$$\sum_{\lambda} z^{\ell(\lambda)} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-zq^n} \qquad \left(\sum_{\lambda} z^{\ell(\lambda)} q^{|\lambda|} = \prod_{n=1}^{N} \frac{1}{1-zq^n} \right)$$

where the sum runs over all partitions λ (where each part of λ is $\leq N$).

Andrews' Theorem

Theorem (G.E.Andrews)

Let $\overline{\omega}(\lambda) = z^{\mathcal{O}(\lambda)} y^{\mathcal{O}(\lambda')} q^{|\lambda|}$ where $\mathcal{O}(\lambda)$ denote the number of odd parts of λ.

$$\sum_{\lambda} \overline{\omega}(\lambda) = \frac{\sum_{j=1}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_j (-zy^{-1}q;q^4)_{N-j} (yq)^{2N-2j}}{(q^4;q^4)_N (z^2q^4;q^4)_N}$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N$.

$$\sum_{\lambda} \overline{\omega}(\lambda) = \frac{\sum_{j=1}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_{j+1} (-zy^{-1}q;q^4)_{N-j} (yq)^{2N-2j}}{(q^4;q^4)_N (z^2q^4;q^4)_{N+1}}$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N+1$. (G.E.Andrews, "On a partition function of Richard Stanley", Electron. J. Combin. 11(2) (2004) #1.)

The four parameter weight

Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5,4,4,1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ, which is equal to $a^5b^4c^3d^2$.

a	b	a	b	a
c	d	c	d	
a	b	a	b	
c				

Boulet's Theorem

Theorem (C.Boulet)

Let $q = abcd$. If $|a|, |b|, |c|, |d| < 1$, then

$$\sum_{\lambda} \omega(\lambda) = \frac{(-a;q)_\infty(-abc;q)_\infty}{(q;q)_\infty(ab;q)_\infty(ac;q)_\infty}$$

where the sum runs over all (ordinary) partitions λ, and

$$\sum_{\mu} \omega(\mu) = \frac{(-a;q)_\infty(-abc;q)_\infty}{(ab;q)_\infty},$$

where the sum runs over all strict partitions.

(C.Boulet, "A four parameter partition identity", arXiv:math.CO/0308012, to appear in Ramanujan J.)

Our generalization

Theorem

Let $q = abcd$. Then

$$\sum_{\lambda} \omega(\lambda) = \frac{(-a;q)_N}{(q;q)_N (ac;q)_N} \, {}_2\phi_1 \left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N+1} \end{matrix} ; q, -bq \right),$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N$.

$$\sum_{\lambda} \omega(\lambda) = \frac{(-a;q)_{N+1}}{(q;q)_N (ac;q)_{N+1}} \, {}_2\phi_1 \left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N} \end{matrix} ; q, -b \right),$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N+1$.

Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomial $Q_n(x) = Q_n(x;\alpha,\beta|q)$ is, by definition,

$$\begin{aligned} &Q_n(x;\alpha,\beta|q) \\ &= (\alpha u;q)_n u^{-n}\,{}_2\phi_1\left(\begin{matrix} q^{-n},\beta u^{-1} \\ \alpha^{-1}q^{-n+1}u^{-1}\end{matrix};q,\alpha^{-1}qu\right), \end{aligned}$$

where $x = \frac{u+u^{-1}}{2}$ (R. Koelof and R.F.Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue Delft University of Technology, Report no. 98-17 (1998), p.80).

Al-Salam-Chihara Recurrence relation

The Al-salam polynomials satisfy the three-term recurrence relation

$$2xQ_n(x) = Q_{n+1}(x) + (\alpha+\beta)q^n Q_n(x) \\ + (1-q^n)(1-\alpha\beta q^{n-1})Q_{n-1}(x),$$

with $Q_{-1}(x) = 0$, $Q_0(x) = 1$.

Associated Al-Salam-Chihara Recurrence relation

We also consider a more general recurrence relation:

$$2x\widetilde{Q}_n(x) = \widetilde{Q}_{n+1}(x) + t(\alpha+\beta)q^n \widetilde{Q}_n(x) \\ + (1-tq^n)(1-t\alpha\beta q^{n-1})\widetilde{Q}_{n-1}(x),$$

which we call the associated Al-Salam-Chihara recurrence relation.

Associated Askey-Wilson polynomials

M.E.H. Ismail and M. Rahman "The associated Askey-Wilson polynomials", Trans. Amer. Math. Soc. **328** (1991), 201 – 237.

Solutions of AASC Recurrence relation

Let

$$\widetilde{Q}_n^{(1)}(x) = u^{-n}(t\alpha u;q)_n {}_2\phi_1\left(\begin{matrix} t^{-1}q^{-n}, \beta u^{-1} \\ t^{-1}\alpha^{-1}q^{-n+1}u^{-1} \end{matrix}; q, \alpha^{-1}qu\right),$$

$$\widetilde{Q}_n^{(2)}(x) = u^n \frac{(tq;q)_n(t\alpha\beta;q)_n}{(t\beta uq;q)_n} {}_2\phi_1\left(\begin{matrix} tq^{n+1}, \alpha^{-1}qu \\ t\beta q^{n+1}u \end{matrix}; q, \alpha u\right).$$

Then $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are two linearly independent solutions of the above associated Al-Salam-Chihara recurrence relation.

Generating Function (ordinary partitions)

Let us consider

$$\Phi_N = \Phi_N(a, b, c, d; z) = \sum_{\substack{\lambda \\ \lambda_1 \le N}} \omega(\lambda) z^{\ell(\lambda)},$$

where the sum runs over all partitions λ such that each part of λ is less than or equal to N.

For example, the first few terms can be computed directly as follows:

$$\Phi_0 = 1,$$

$$\Phi_1 = \frac{1 + az}{1 - acz^2},$$

$$\Phi_2 = \frac{1 + a(1+b)z + abcz^2}{(1 - acz^2)(1 - qz^2)},$$

$$\Phi_3 = \frac{1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3}{(1 - z^2ac)(1 - z^2q)(1 - z^2acq)}.$$

Generating Function (strict partitions)

Let

$$\Psi_N = \Psi_N(a, b, c, d; z) = \sum \omega(\mu) z^{\ell(\mu)},$$

where the sum is over all strict partitions μ such that each part of μ is less than or equal to N.

Example

For example, we have

$$\begin{aligned}
\Psi_0 &= 1, \\
\Psi_1 &= 1 + az, \\
\Psi_2 &= 1 + a(1+b)z + abcz^2, \\
\Psi_3 &= 1 + a(1+b+ab)z \\
&\quad + abc(1+a+ad)z^2 + a^3bcdz^3.
\end{aligned}$$

Strict partitions with all parts ≤ 3

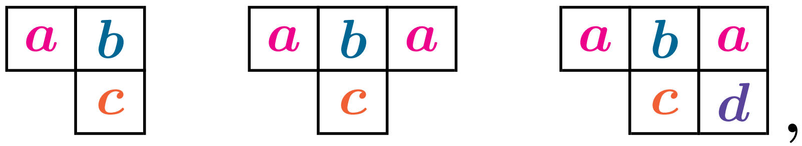
$N = 3$

$\ell(\mu) = 0$

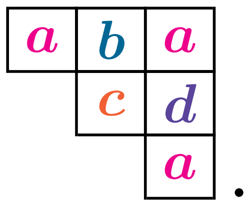
$$\emptyset,$$

$\ell(\mu) = 1$

$\ell(\mu) = 2$



$\ell(\mu) = 3$



Relation between Φ_N and Ψ_N

Theorem

$$\Phi_N(a,b,c,d;z) = \frac{\Psi_N(a,b,c,d;z)}{(z^2q;q)_{\lfloor N/2\rfloor}(z^2ac;q)_{\lceil N/2\rceil}}.$$

Thus we only need to consider the srict partitions case.

Recurrence equation satisfied by Ψ_N

Theorem

Let $\Psi_N = \Psi_N(a,b,c,d;z)$ be as above. Then we have

$$\Psi_{2N} = (1+b)\Psi_{2N-1} + (a^N b^N c^N d^{N-1} z^2 - b)\Psi_{2N-2},$$
$$\Psi_{2N+1} = (1+a)\Psi_{2N} + (a^{N+1} b^N c^N d^N z^2 - a)\Psi_{2N-1},$$

for any positive integer N.

Pfaffian expression for the weight $\omega(\mu)z^{\ell(\mu)}$

Theorem

Define a skew-symmetric array $A = (\alpha_{ij})_{0 \le i,j}$ by

$$\alpha_{ij} = \begin{cases} a^{\lceil j/2 \rceil} b^{\lfloor j/2 \rfloor} z & \text{if } i = 0, \\ a^{\lceil j/2 \rceil} b^{\lfloor j/2 \rfloor} c^{\lceil i/2 \rceil} d^{\lfloor i/2 \rfloor} z^2 & \text{if } i > 0. \end{cases}$$

for $i < j$. If $\mu = (\mu_1, \dots, \mu_{2n})$ is a strict partition such that $\mu_1 > \cdots > \mu_{2n} \ge 0$, then we put $I(\mu) = \{\mu_{2n}, \dots, \mu_1\}$. Then we have

$$\mathrm{Pf}\left[A^{I(\mu)}_{I(\mu)}\right] = \omega(\mu) z^{\ell(\mu)},$$

where $A^{I(\mu)}_{I(\mu)}$ denote the $2n \times 2n$ matrix obtained from A by choosing the rows and columns indexed by $I(\mu)$.

Minor summation formula of Pfaffians

Theorem (Minor summation formula)

Let $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$ be skew symmetric matrices of size n. Then

$$\sum_{t=0}^{\lfloor n/2\rfloor} z^t \sum_{I\in\binom{[n]}{2t}} \gamma^{|I|}\mathrm{Pf}\left(\Delta_I^I(A)\right)\mathrm{Pf}\left(\Delta_I^I(B)\right) = \mathrm{Pf}\begin{bmatrix} J_n\,{}^tAJ_n & J_n \\ -J_n & C \end{bmatrix},$$

where $|I| = \sum_{i\in I} i$ and $C = (C_{ij})_{1\le i,j\le n}$ is given by

$$C_{ij} = \gamma^{i+j} b_{ij} z$$

and $J_n = (\delta_{i,n+1-j})_{1\le i,j\le n}$ is the anti-diagonal matrix.

The sum of the weights $\omega(\mu) z^{\ell(\mu)}$

Let S_n denote the $n \times n$ skew-symmetric matrix whose (i, j)th entry is 1 for $0 \leq i < j \leq n$.

Theorem

Let N be a nonnegative integer.

$$\Psi_N(a, b, c, d; z) = \mathrm{Pf} \begin{bmatrix} S_{N+1} & J_{N+1} \\ -J_{N+1} & A \end{bmatrix},$$

where $A = (\alpha_{ij})_{0 \leq i < j \leq N}$ is the $N \times N$ skew-symmetric matrix whose (i, j)th entry α_{ij} is defined above.

Recurrence equations of X_N and Y_N

Theorem Set $q = abcd$ and put $X_N = \Psi_{2N}$ and $Y_N = \Psi_{2N+1}$.

Then X_N and Y_N satisfy

$$
\begin{aligned}
X_{N+1} &= \left\{1 + ab + a(1+bc)z^2q^N\right\} X_N \\
&\qquad - ab(1 - z^2q^N)(1 - acz^2q^{N-1})X_{N-1}, \\
Y_{N+1} &= \left\{1 + ab + abc(1+ad)z^2q^N\right\} Y_N \\
&\qquad - ab(1 - z^2q^N)(1 - acz^2q^N)Y_{N-1},
\end{aligned}
$$

where $X_0 = 1$, $Y_0 = 1 + az$, $X_1 = 1 + a(1+b)z + abcz^2$ and

$$
Y_1 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3.
$$

Reduction to AASC Recurrence equation

Corollary

If we put $X'_N = (ab)^{-\frac{N}{2}} X_N$ and $Y'_N = (ab)^{-\frac{N}{2}} Y_N$, then the above recurrence equation can be rewritten as

$$\left\{(ab)^{\frac{1}{2}} + (ab)^{-\frac{1}{2}}\right\} X'_N = X'_{N+1} - a^{\frac{1}{2}} b^{-\frac{1}{2}} (1 + bc) z^2 q^N X'_N + (1 - z^2 q^N)(1 - acz^2 q^{N-1}) X'_{N-1},$$

$$\left\{(ab)^{\frac{1}{2}} + (ab)^{-\frac{1}{2}}\right\} Y'_N = Y'_{N+1} - a^{\frac{1}{2}} b^{\frac{1}{2}} c (1 + ad) z^2 q^N Y'_N + (1 - z^2 q^N)(1 - a^2 b c^2 d z^2 q^{N-1}) Y'_{N-1}.$$

Solve these recurrence equations with the above initial conditions.

Solution for X_N ($X_N = \Psi_{2N}$)

$$X_N = \frac{(-az^2q, -abc; q)_\infty}{(-a, -abcz^2; q)_\infty} \Big\{ (s_0^X X_1 - s_1^X X_0)$$

$$\times (-abcz^2; q)_N \, {}_2\phi_1 \left(\begin{matrix} q^{-N}z^{-2}, -b^{-1} \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix} ; q, -c^{-1}q \right)$$

$$+ (r_1^X X_0 - r_0^X X_1)$$

$$\times (ab)^N \frac{(qz^2, acz^2; q)_N}{(-aqz^2; q)_N} \, {}_2\phi_1 \left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -aq^{N+1}z^2 \end{matrix} ; q, -abc \right) \Big\},$$

where

$$r_0^X = {}_2\phi_1\left(\begin{matrix} z^{-2}, -b^{-1} \\ -(abc)^{-1}z^{-2}q \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^X = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -az^2q \end{matrix}; q, -abc\right),$$

$$r_1^X = (1+abcz^2)\,{}_2\phi_1\left(\begin{matrix} z^{-2}q^{-1}, -b^{-1} \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_1^X = \frac{ab(1-z^2q)(1-acz^2)}{1+az^2q}\,{}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -az^2q^2 \end{matrix}; q, -abc\right).$$

Solution for Y_N ($Y_N = \Psi_{2N+1}$)

$$\begin{aligned} Y_N = \frac{(-a^2bcdz^2q, -abc; q)_\infty}{(-a^2bcd, -abcz^2; q)_\infty} &\Bigg\{ (s_0^Y Y_1 - s_1^Y Y_0) \\ &\times (-abcz^2; q)_N \, {}_2\phi_1 \left(\begin{matrix} q^{-N}z^{-2}, -acd \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix} ; q, -c^{-1}q \right) \\ &+ (r_1^Y Y_0 - r_0^Y Y_1) \\ &\times (ab)^N \frac{(qz^2, a^2bc^2dz^2; q)_N}{(-a^2bcdqz^2; q)_N} \, {}_2\phi_1 \left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -a^2bcdq^{N+1}z^2 \end{matrix} ; q, -abc \right) \Bigg\}, \end{aligned}$$

where

$$r_0^Y = {}_2\phi_1\left(\begin{matrix} z^{-2}, -acd \\ (-abc)^{-1}qz^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$r_1^Y = (1 + abcz^2)\,{}_2\phi_1\left(\begin{matrix} q^{-1}z^{-2}, -ac \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^Y = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -a^2bcdz^2q \end{matrix}; q, -abc\right),$$

$$s_1^Y = \frac{ab(1 - z^2q)(1 - a^2bc^2dz^2)}{1 + a^2bcdz^2q}\,{}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -a^2bcdz^2q^2 \end{matrix}; q, -abc\right).$$

Infinite sum of the weight $\omega(\mu) z^{\ell(\mu)}$

Set $q = abcd$. Let s_i^X, s_i^Y, X_i, Y_i $(i = 0, 1)$ be as in the above theorem. Then we have

$$\sum_{\mu} \omega(\mu) z^{\ell(\mu)} = \frac{(-abc, -az^2q; q)_\infty}{(ab; q)_\infty}(s_0^X X_1 - s_1^X X_0)$$
$$= \frac{(-abc, -a^2bcdz^2q; q)_\infty}{(ab; q)_\infty}(s_0^Y Y_1 - s_1^Y Y_0),$$

where the sum runs over all strict partitions μ.

Weighted sums of Schur's $P(Q)$-functions

We consider a weighted sum of Schur's P-functions and Q-functions

$$\xi_N(a, b, c, d; X_n) = \sum_{\substack{\mu \\ \mu_1 \leq N}} \omega(\mu) P_\mu(x_1, \dots, x_n),$$

$$\eta_N(a, b, c, d; X_n) = \sum_{\substack{\mu \\ \mu_1 \leq N}} \omega(\mu) Q_\mu(x_1, \dots, x_n),$$

where the sums run over all strict partitions μ such that each part of μ is less than or equal to N. More generally, we can unify these problems to finding the following sum:

$$\zeta_N(a, b, c, d; z; X_n) = \sum_{\substack{\mu \\ \mu_1 \leq N}} \omega(\mu) z^{\ell(\mu)} P_\mu(x_1, \dots, x_n),$$

where the sum runs over all strict partitions μ such that each part of μ is less than or equal to N.

Infinite Sum

Further, let us put

$$\begin{aligned}\zeta(a, b, c, d; z; X_n) &= \lim_{N \to \infty} \zeta_N(a, b, c, d; z; X_n) \\ &= \sum_{\mu} \omega(\mu) z^{\ell(\mu)} P_\mu(X_n),\end{aligned}$$

where the sum runs over all strict partitions μ. We also write

$$\xi(a, b, c, d; X_n) = \zeta(a, b, c, d; 1; X_n) = \sum_{\mu} \omega(\mu) P_\mu(X_n),$$

where the sum runs over all strict partitions μ.

Theorem

Let n be a positive integer. Then

$$\zeta(a,b,c,d;z;X_n) = \begin{cases} \operatorname{Pf}\left(\gamma_{ij}\right)_{1\le i<j\le n} / \operatorname{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even,} \\ \operatorname{Pf}\left(\gamma_{ij}\right)_{0\le i<j\le n} / \operatorname{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$\gamma_{ij} = \frac{x_i - x_j}{x_i + x_j} + u_{ij} z + v_{ij} z^2$$

with

$$u_{ij} = \frac{a \det \begin{pmatrix} x_i + b x_i^2 & 1 - ab x_i^2 \\ x_j + b x_j^2 & 1 - ab x_j^2 \end{pmatrix}}{(1 - ab x_i^2)(1 - ab x_j^2)},$$

$$v_{ij} = \frac{abc x_i x_j \det \begin{pmatrix} x_i + a x_i^2 & 1 - a(b+d) x_i^2 - abd x_i^3 \\ x_j + a x_j^2 & 1 - a(b+d) x_j^2 - abd x_j^3 \end{pmatrix}}{(1 - ab x_i^2)(1 - ab x_j^2)(1 - abcd x_i^2 x_j^2)},$$

if $1 \leq i, j \leq n$, and

$$\gamma_{0j} = 1 + \frac{a x_j (1 + b x_j)}{1 - a b x_j^2} z$$

if $1 \leq j \leq n$.

Especially, when $z = 1$, we have

$$\xi(a, b, c, d; X_n) = \begin{cases} \mathrm{Pf}\left(\widetilde{\gamma}_{ij}\right)_{1 \leq i < j \leq n} / \mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even,} \\ \mathrm{Pf}\left(\widetilde{\gamma}_{ij}\right)_{0 \leq i < j \leq n} / \mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$\widetilde{\gamma}_{ij} = \begin{cases} \frac{1 + a x_j}{1 - a b x_j^2} & \text{if } i = 0, \\ \frac{x_i - x_j}{x_i + x_j} + \widetilde{v}_{ij} & \text{if } 1 \leq i < j \leq n, \end{cases} \quad with$$

$$\widetilde{v}_{ij} = \frac{a \det \begin{pmatrix} x_i + b x_i^2 & 1 - b(a + c) x_i^2 - a b c x_i^3 \\ x_j + b x_j^2 & 1 - b(a + c) x_j^2 - a b c x_j^3 \end{pmatrix}}{(1 - a b x_i^2)(1 - a b x_j^2)(1 - a b c d x_i^2 x_j^2)}.$$