Refined Enumerations of Totally Symmetric Self-Complementary Plane Partitions and Constant Term Identities

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Introduction

Abstract

In this talk we give Pfaffian or determinant expressions, and constant term identities for the conjectures in the paper "Self-complementary totally symmetric plane partitions" (*J. Combin. Theory Ser. A* **42**, (1986), 277–292) by W.H. Mills, D.P. Robbins and H. Rumsey. We also settle a weak version of Conjecture 6 in the paper, i.e., the number of shifted plane partitions invariant under a certain involution is equal to the number of alternating sign matrices invariant under the vertical flip.

The conjectures on TSSCPPs

1. Conjecture 2 (The refined TSSCPP conjecture)
2. Conjecture 3 (The doubly refined TSSCPP conjecture)
3. Conjecture 7, 7' (Related to the monotone triangles)
4. Conjecture 4 (Related to half-turn symmetric ASMs)
5. Conjecture 6 (Related to vertical symmetric ASMs)

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Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{array}{cccccc} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \ddots & & \end{array}$$

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$$\begin{matrix} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ \vdots & & & & & \\ 0 & 0 & 0 & \ddots & & \end{matrix}$$

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Shape

Definition

Let $\pi = (\pi_{ij})_{i,j \geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
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Example of plane partitions

Example

- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\boxed{1}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 1\\\cline{1-1}\end{array}$$

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1

- Plane partitions of 2:

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1	1

1
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- Plane partitions of 3:

3

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1
1
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2	1

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Ferrers graph

Definition

The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \left\{(i, j, k) : k \leq \pi_{ij}\right\}$$

Example

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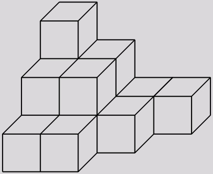


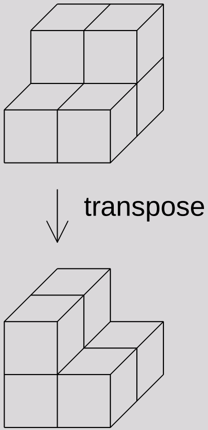
Symmetries of plane paritions

Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example



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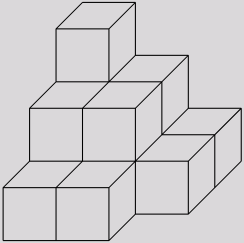
A symmetric PP



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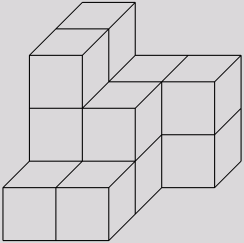

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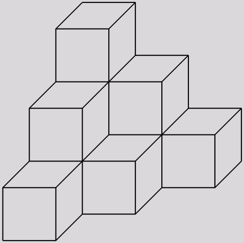
Example

A totally symmetric PP



Complement

Definition

Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r,s,t) = [r] \times [s] \times [t]$.

Define the *complement* π^c of π by
$\pi^c = \{ (r+1-i, s+1-j, t+1-k) \ : \ (i,j,k) \notin \pi \}$.

- π is said to be *(r,s,t)-self-complementary* if $\pi = \pi^c$. i.e. $(i,j,k) \in \pi \Leftrightarrow (r+1-i, s+1-j, t+1-k) \notin \pi$.

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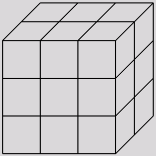

$B(2,3,3)$

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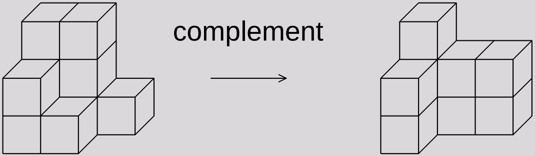

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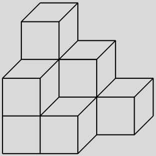

A $(2,3,3)$-self-complementary PP

Symmetry classes of plane partitions

Symmetry classes (Stanley)

The transformation ^{c} and the group S_3 generate a group T of order 12. The group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems.

Table

2	$B(r,r,t)$	*Symmetric*
		Cyclically symmetric
4	$B(r,r,r)$	*Totally symmetric*
		Self-complementary
6	$B(r,r,t)$	*Complement = transpose*
		Symmetric and self-complementary
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
		Cyclically symmetric and self-complementary
10	$B(r,r,r)$	*Totally symmetric and self-complementary*

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[illegible]	[illegible]	*Self-complementary*
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Table (R. P. Stanley, "Symmetries of Plane Partitions", *J. Combin. Theory Ser.* **A 43**, 103-113 (1986))

1	$B(r,s,t)$	Any
2	$B(r,r,t)$	*Symmetric*
3	$B(r,r,r)$	*Cyclically symmetric*
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Totally symmetric self-complementary plane partitions

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A plane partition is said to be *totally symmetric self-complementary plane parition of size $2n$* if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.

We denote the set of all self-complementary totally symmetric plane partitions of size $2n$ by $\mathscr{S}_n$.

Example

$\mathscr{S}_1$ consists of the single partition

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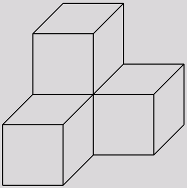
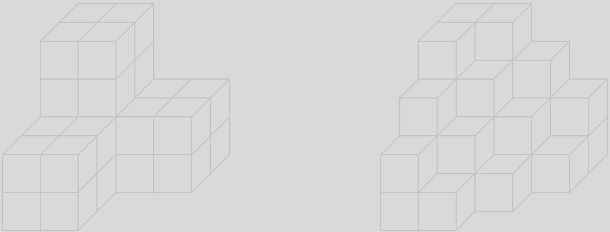


TSSCPPs of size 4

Example

$\mathscr{S}_2$ consists of the following two partitions:



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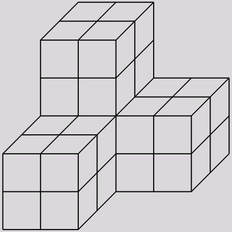
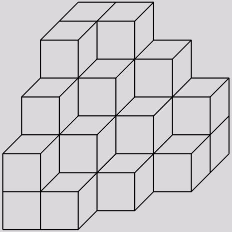
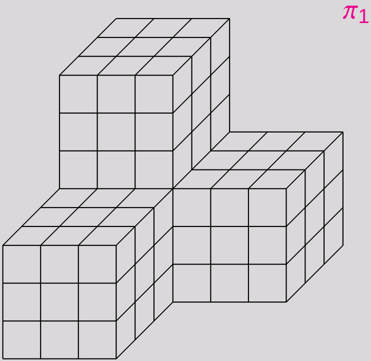


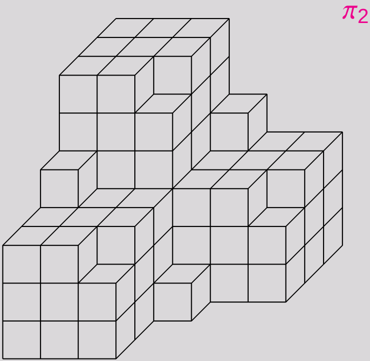


TSSCPPs of size 6

Example

$\mathscr{S}_3$ consists of the following seven partitions:





TSSCPPs of size 6

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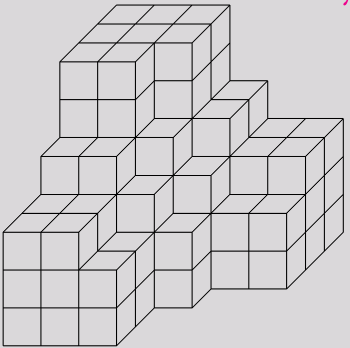
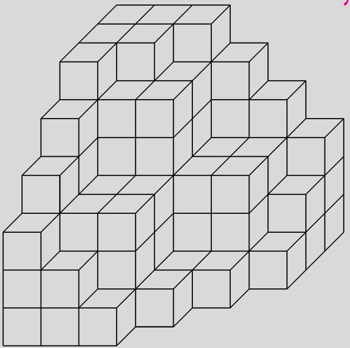


π_4



TSSCPPs of size 6

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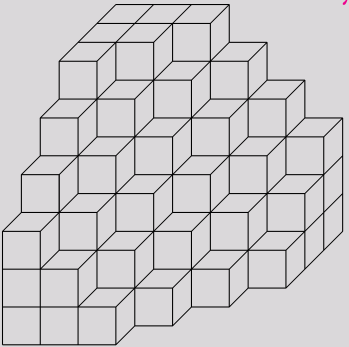

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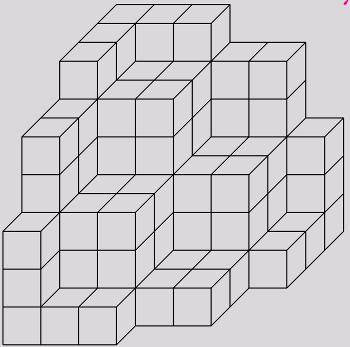

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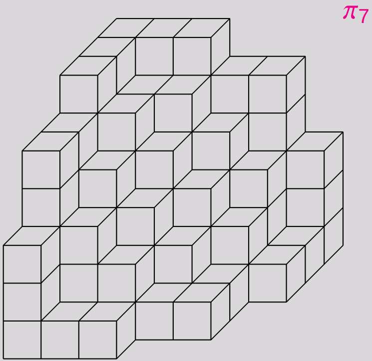

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \ldots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

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(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

$\mathscr{B}_1$ consists of the single PP $\emptyset$.

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

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We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

$\mathscr{B}_2$ consists of the following 2 PPs:

$\boxed{2} \qquad \boxed{1}$

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

$\mathscr{B}_3$ consists of the followng 7 PPs

$$\begin{array}{cc} 3 & 3 \\ & 3 \end{array} \quad \begin{array}{cc} 3 & 3 \\ & 2 \end{array} \quad \begin{array}{cc} 3 & 3 \\ & 1 \end{array} \quad \begin{array}{cc} 3 & 2 \\ & 2 \end{array} \quad \begin{array}{cc} 3 & 2 \\ & 1 \end{array} \quad \begin{array}{cc} 2 & 2 \\ & 2 \end{array} \quad \begin{array}{cc} 2 & 2 \\ & 1 \end{array}$$

A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

A bijection

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Let n be a positive integer.
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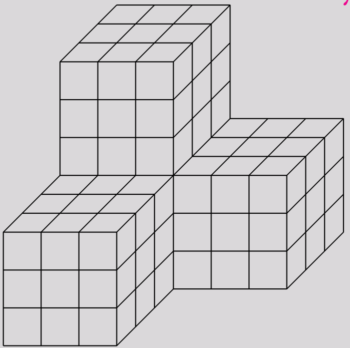
Example

π_1 $\qquad n = 3$



$\longrightarrow$

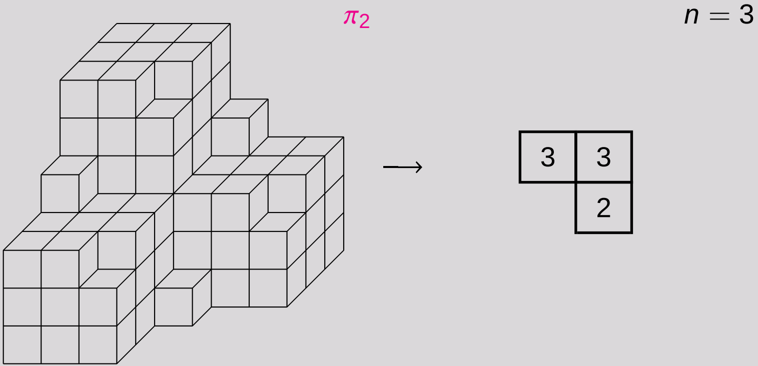
3	3
	3

A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example



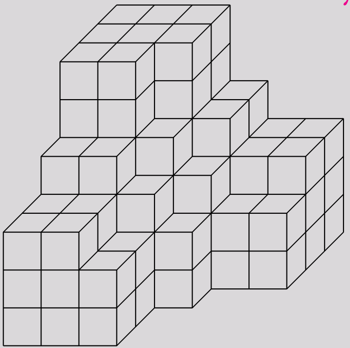
A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_3 $\qquad$ $n = 3$



$\longrightarrow$

3	3
	1

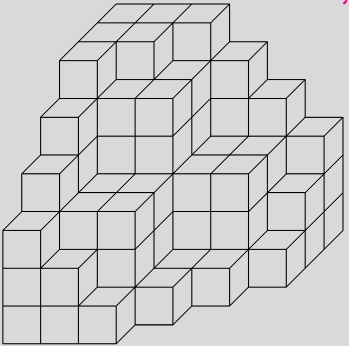
A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_4 $\qquad n = 3$



$\longrightarrow$

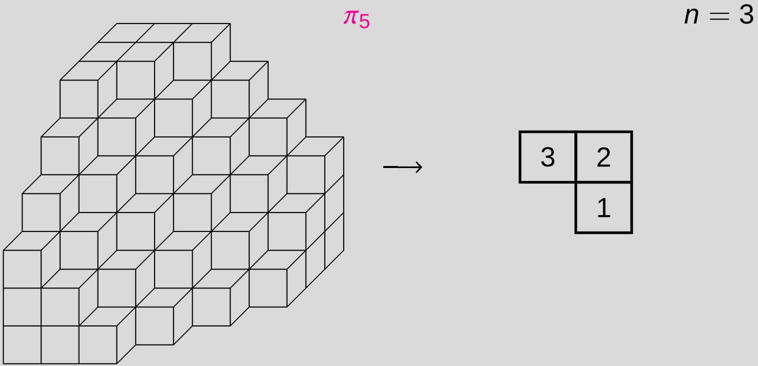
3	2
	2

A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

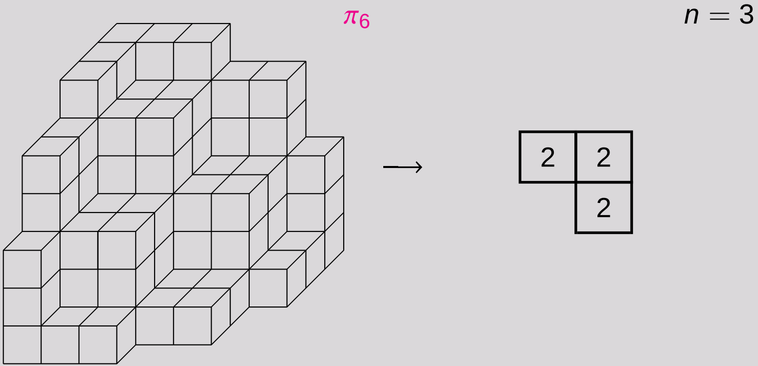


A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example



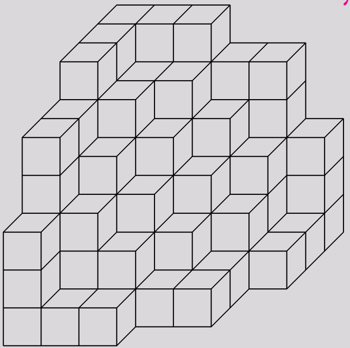
A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_7 $\qquad$ $n = 3$



$\longrightarrow$

2	2
	1

Statistics

Definition (Mills, Robbins and Rumsey)

Let $b = (b_{ij})_{1 \le i \le j \le n-1}$ be in $\mathscr{B}_n$ and $k = 1, \ldots, n$,

Let

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Here We set $b_{tn} = n - t$ for all $t = l, \ldots, n - 1$ by convention, and $\chi\{\ldots\}$ has value 1 when the statement "…" is true and 0 otherwise.

Statistics

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Let $b = (b_{ij})_{1 \le i \le j \le n-1}$ be in $\mathscr{B}_n$ and $k = 1, \dots, n$,

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Here We set $b_{tn} = n - t$ for all $t = l, \dots, n-1$ by convention, and $\chi\{\dots\}$ has value 1 when the statement "$\dots$" is true and 0 otherwise.

Statistics

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Statistics

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Here We set $b_{tn} = n - t$ for all $t = l, ..., n - 1$ by convention, and $\chi\{\dots\}$ has value 1 when the statement "…" is true and 0 otherwise.

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 1, \quad U_1(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 1, \quad U_1(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 2, \quad U_2(b) = 1$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 3, \quad U_3(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 4, \quad U_4(b) = 2$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 5, \quad U_5(b) = 2$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 6, \quad U_6(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 7, \quad U_7(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

The refined TSSCPP conjecture

Conjecture (Conjecture 2 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $0 \leq r \leq n-1$ and $1 \leq k \leq n$. Then the number of elements b of $\mathscr{B}_n$ such that $U_k(b) = r$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ such that $a_{1,r+1} = 1$.

The refined TSSCPP conjecture

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Example

$n = 3$, $b \in \mathscr{B}_3$

b	3 3 / 3	3 3 / 2	3 3 / 1	3 2 / 2	3 2 / 1	2 2 / 2	2 2 / 1
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

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Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_3} t^{U_k(b)} = 2 + 3t + 2t^2.$$

The refined enumeration of ASM

Zeilberger (1996), Kuperberg (1996)

The number of n by n alternating sign matrices $a = (a_{ij})$ such that $a_{1,r+1} = 1$ is equal to

$$\frac{\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}}{\binom{2n-2}{n-1}}A_{n-1} = \frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}}A_n.$$

Here A_n is

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

The doubly refined TSSCPP conjecture

Conjecture (Conjecture 3 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, s with $0 \leq r, s \leq n - 1$ be integers. Then the number of partitions in $\mathscr{B}_n$ with $U_1(b) = r$ and $U_2(b) = s$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ with

$$a_{1,r+l} = a_{n,n-s} = 1.$$

The doubly refined TSSCPP conjecture

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$$a_{1,r+l} = a_{n,n-s} = 1.$$

Example

$b \in \mathscr{B}_3$	$\begin{smallmatrix}3&3\\&3\end{smallmatrix}$	$\begin{smallmatrix}3&3\\&2\end{smallmatrix}$	$\begin{smallmatrix}3&3\\&1\end{smallmatrix}$	$\begin{smallmatrix}3&2\\&2\end{smallmatrix}$	$\begin{smallmatrix}3&2\\&1\end{smallmatrix}$	$\begin{smallmatrix}2&2\\&2\end{smallmatrix}$	$\begin{smallmatrix}2&2\\&1\end{smallmatrix}$
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

The doubly refined TSSCPP conjecture

Conjecture (Conjecture 3 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, s with $0 \leq r, s \leq n-1$ be integers. Then the number of partitions in $\mathscr{B}_n$ with $U_1(b) = r$ and $U_2(b) = s$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ with

$$a_{1,r+l} = a_{n,n-s} = 1.$$

Example

Thus we have

$$\sum_{b \in \mathscr{B}_3} t^{U_1(b)} u^{U_2(b)} = 1 + t + u + tu + t^2u + tu^2 + t^2u^2.$$

The doubly refined enumeration of ASM

Di Francesco and Zinn-Justin (2004)

The doubly-refined ASM number generating function is given by

$$A_n(t,u) = \frac{\{\omega^2(\omega+t)(\omega+u)\}^{n-1}}{3^{n(n-1)/2}} \times s^{(2n)}_{\delta(n-1,n-1)}\left(\frac{1+\omega t}{\omega+t}, \frac{1+\omega u}{\omega+u}, 1, \dots, 1\right)$$

Here $s^{(n)}_{\lambda}(x_1,\dots,x_n)$ stands for the Schur function in the n variables $x_1, \dots, x_n$, corresponding to the partition λ, and $\delta(n-1,n-1) = (n-1,n-1,n-2,n-2,\dots,1,1)$ and $\omega = e^{2i\pi/3}$. (The coefficient of $t^{j-1}s^{k-1}$ is the number of $n \times n$ ASM with a 1 in position r on the top row (counted from left to right) and k on the bottom row (counted from right to left).)

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n. Then the cardinality of $\mathscr{B}_{nk}$ is equal to the cardinality of the set of the monotone triangles with all entries m_{ij} in the first $n-1-k$ columns equal to their minimum values $j-i+1$.

TSSCPP and monotone triangles

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Example

$n = 3$, $k = 0$: The first 2 columns are equal to the maximum values 3.

$b \in \mathscr{B}_{3,0}$	$\begin{array}{cc} 3 & 3 \\ & 3 \end{array}$
$U_1(b)$	2
$U_2(b)$	2
$U_3(b)$	2

TSSCPP and monotone triangles

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For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_{3,0}} t^{U_k(b)} = t^2.$$

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \dots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

$n = 3$, $k = 1$: The first column equals the maximum values 3.

$b \in \mathscr{B}_{3,1}$	$\begin{smallmatrix} 3 & 3 \\ & 3 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 3 \\ & 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 3 \\ & 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 2 \\ & 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 2 \\ & 1 \end{smallmatrix}$
$U_1(b)$	2	1	0	2	1
$U_2(b)$	2	2	1	1	0
$U_3(b)$	2	2	1	1	0

TSSCPP and monotone triangles

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Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_{3,1}} t^{U_k(b)} = 1 + 2t + 2t^2.$$

TSSCPP and monotone triangles

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For $n \geq 2$ and $k = 0, \dots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

$n = 3$, $k = 2$: No restriction.

$b \in \mathscr{B}_{3,2}$	$\begin{array}{cc}3&3\\ &3\end{array}$	$\begin{array}{cc}3&3\\ &2\end{array}$	$\begin{array}{cc}3&3\\ &1\end{array}$	$\begin{array}{cc}3&2\\ &2\end{array}$	$\begin{array}{cc}3&2\\ &1\end{array}$	$\begin{array}{cc}2&2\\ &2\end{array}$	$\begin{array}{cc}2&2\\ &1\end{array}$
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

TSSCPP and monotone triangles

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For $n \geq 2$ and $k = 0, \dots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_{3,2}} t^{U_k(b)} = 2 + 3t + 2t^2.$$

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
  $$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
  $$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{O,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
  $$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
  $$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{0,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{0,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{O,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flips

Example

$n = 7$, Flip on the off-diagonal part $b_{2,4} = 5$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7, \quad 5 + b'_{2,4} = \min(7, 6) + \max(5, 4)$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7, \quad 5 + b'_{2,4} = 6 + 5$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7$, Change $b_{2,4} = 5$ to $b'_{2,4} = 6$.

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	6	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7$, Flip on the diagonal part $b_{2,1} = 6$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7, \quad 6 + b'_{2,1} = 7 + 6$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7$, Change $b_{2,1} = 6$ to $b'_{2,1} = 7$.

7	7	7	7	7	7	
7	7	7	7	7	7	6
	7	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

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Example $n = 7$, $k = 1$, Apply π_1 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n - 1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \leq i \leq n - k$.

Example $n = 7$, $k = 1$, Then we obtain the following $\pi_1(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	6	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					1

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \leq i \leq n-k$.

Example $n = 7$, $k = 2$, Apply π_2 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 2$, Then we obtain the following $\pi_2(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	7	6	5	5
		5	5	4	4
			4	4	4
				3	3
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \leq i \leq n-k$.

Example $n = 7$, $k = 3$, Apply π_3 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 3$, Then we obtain the following $\pi_3(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	5	5	5
		5	4	4	4
			4	4	3
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 4$, Apply π_4 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 4$, Then we obtain the following $\pi_4(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	6	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 5$, Apply π_5 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 5$, Then we obtain the following $\pi_5(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 6$, Apply π_6 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 6$, Then we obtain the following $\pi_6(b) \in \mathscr{B}_6$.

7	7	7	7	7	6
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

Conjecture 4

Definition

Define the involution $\rho : \mathscr{B}_n \to \mathscr{B}_n$ by

$$\rho = \pi_2 \pi_4 \pi_6 \cdots .$$

Conjecture (Conjecture 4 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, $0 \leq r \leq n$ be integers. Then the number of elements of $\mathscr{B}_n$ with $\rho(b) = b$ and $U_1(b) = r$ is the same as the number of n by n alternating sign matrices a invariant under the half turn in their own planes (that is $a_{ij} = a_{n+1-i,n+1-j}$ for $1 < i,j < n$) and satisfying $a_{1,r} = 1$.

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Let $n \geq 2$ and r, $0 \leq r \leq n$ be integers. Then the number of elements of $\mathscr{B}_n$ with $p(b) = b$ and $U_1(b) = r$ is the same as the number of n by n alternating sign matrices a invariant under the half turn in their own planes (that is $a_{ij} = a_{n+1-i,n+1-i}$ for $1 < i, j < n$) and satisfying $a_{1,r} = 1$.

Conjecture 6

Definition

Define the involution $\gamma : \mathscr{B}_n \to \mathscr{B}_n$ by

$$\gamma = \pi_1 \pi_3 \pi_5 \cdots .$$

Conjecture (Conjecture 6 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 3$ an odd integer and i, $0 \leq i \leq n-1$ be an integer. Then the number of b in $\mathscr{B}_n$ with $\gamma(b) = b$ and $U_2(b) = i$ is the same as the number of n by n alternating sign matrices with $a_{i1} = 1$ and which are invariant under the vertical flip (that is $a_{ij} = a_{i,n+1-j}$ for $1 \leq i, j \leq n$).

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Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1\le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

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Example

$\mathscr{P}_1$ consists of the single PP $\emptyset$.

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Example

$\mathscr{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \boxed{1}$$

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Example

$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Restricted column-strict plane partitions

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Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \leq i,j}$ subject to the constraints that

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Example

$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

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Let n be a positive integer.
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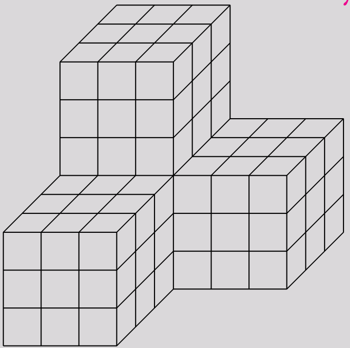
Example

π_1 $\qquad n = 3$



$\longrightarrow \quad \emptyset$

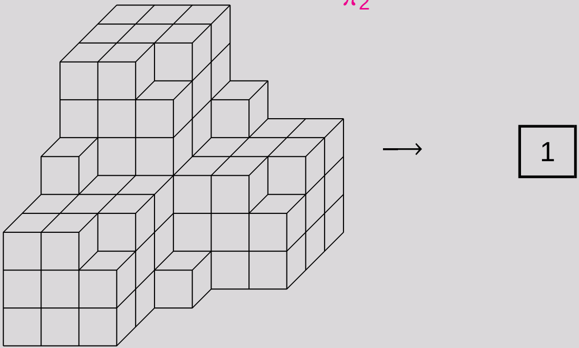
Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_2 $\qquad$ $n = 3$



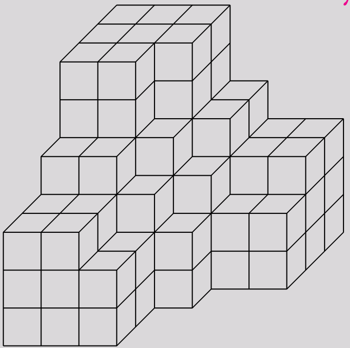
Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_3 $\qquad n = 3$



$\longrightarrow$

1	1

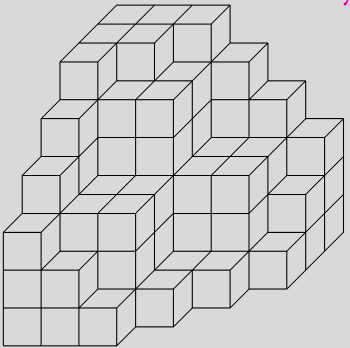
Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_4 $\qquad n = 3$



$\longrightarrow$ $\boxed{2}$

Another bijection

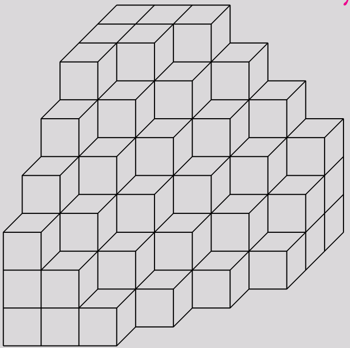
Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

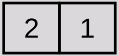
Example

π_5

$n = 3$



$\longrightarrow$



Another bijection

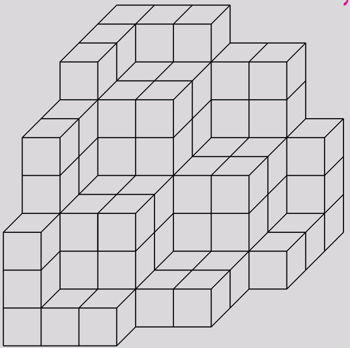
Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_6

$n = 3$



$\longrightarrow$

2
1

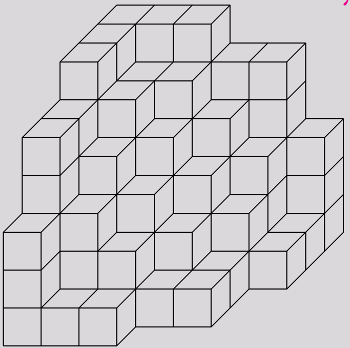
Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_7 $\qquad n = 3$



$\longrightarrow$

2	1
1	

Composition of the bijectons

Corollary

Let n be a positive integer.
Then there is a bijection φ_n from $\mathscr{B}_n$ to $\mathscr{P}_n$.

Example

The case of $n = 3$

$b \in \mathscr{B}_3$	$\begin{array}{cc} 3 & 3 \\ & 3 \end{array}$	$\begin{array}{cc} 3 & 3 \\ & 2 \end{array}$	$\begin{array}{cc} 3 & 3 \\ & 1 \end{array}$	$\begin{array}{cc} 3 & 2 \\ & 2 \end{array}$	$\begin{array}{cc} 3 & 2 \\ & 1 \end{array}$	$\begin{array}{cc} 2 & 2 \\ & 2 \end{array}$	$\begin{array}{cc} 2 & 2 \\ & 1 \end{array}$
$c \in \mathscr{P}_3$	$\emptyset$	1	$\begin{array}{cc} 1 & 1 \end{array}$	2	$\begin{array}{cc} 2 & 1 \end{array}$	$\begin{array}{c} 2 \\ 1 \end{array}$	$\begin{array}{cc} 2 & 1 \\ 1 & \end{array}$

Composition of the bijectons

Corollary

Let n be a positive integer.
Then there is a bijection φ_n from $\mathscr{B}_n$ to $\mathscr{P}_n$.

Example

The case of $n = 3$

$b \in \mathscr{B}_3$	$\begin{array}{cc}3&3\\&3\end{array}$	$\begin{array}{cc}3&3\\&2\end{array}$	$\begin{array}{cc}3&3\\&1\end{array}$	$\begin{array}{cc}3&2\\&2\end{array}$	$\begin{array}{cc}3&2\\&1\end{array}$	$\begin{array}{cc}2&2\\&2\end{array}$	$\begin{array}{cc}2&2\\&1\end{array}$
$c \in \mathscr{P}_3$	$\emptyset$	$\begin{array}{c}1\end{array}$	$\begin{array}{cc}1&1\end{array}$	$\begin{array}{c}2\end{array}$	$\begin{array}{cc}2&1\end{array}$	$\begin{array}{c}2\\1\end{array}$	$\begin{array}{cc}2&1\\1&\end{array}$

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,

Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

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4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, Saturated parts

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \leq i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 1$, $\overline{U}_1(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

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Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
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Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 2$, $\overline{U}_2(c) = 5$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 3$, $\overline{U}_3(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 4$, $\overline{U}_4(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \leq i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 5$, $\overline{U}_5(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 6$, $\overline{U}_6(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 7$, $\overline{U}_7(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Relation between $U_k(b)$ and $\overline{U}_k(c)$

Theorem

For $n \geq 1$ and $k = 1, \ldots, n$, assume that the bijection φ_n maps $b \in \mathscr{B}_n$ to $c = \varphi(b) \in \mathscr{P}_n$. Then

$$\overline{U}_k(c) = n - 1 - U_k(b).$$

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$$\overline{U}_k(c) = n - 1 - U_k(b).$$

Example

$n = 3$, $b \in \mathscr{B}_3$

b	$\begin{smallmatrix}3&3\\ &3\end{smallmatrix}$	$\begin{smallmatrix}3&3\\ &2\end{smallmatrix}$	$\begin{smallmatrix}3&3\\ &1\end{smallmatrix}$	$\begin{smallmatrix}3&2\\ &2\end{smallmatrix}$	$\begin{smallmatrix}3&2\\ &1\end{smallmatrix}$	$\begin{smallmatrix}2&2\\ &2\end{smallmatrix}$	$\begin{smallmatrix}2&2\\ &1\end{smallmatrix}$
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

Relation between $U_k(b)$ and $\overline{U}_k(c)$

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For $n \geq 1$ and $k = 1, \ldots, n$, assume that the bijection φ_n maps $b \in \mathscr{B}_n$ to $c = \varphi(b) \in \mathscr{P}_n$. Then

$$\overline{U}_k(c) = n - 1 - U_k(b).$$

Example

$n = 3$, $c \in \mathscr{P}_3$

c	$\emptyset$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$
$\overline{U}_1(c)$	0	1	2	0	1	1	2
$\overline{U}_2(c)$	0	0	1	1	2	1	2
$\overline{U}_3(c)$	0	0	1	1	2	1	2

From RCSPPs to lattce paths

Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \le y \le x\}$ *be the vertex set, and direct an edge from u to v whenever* $v - u = (1, -1)$ *or* $(0, -1)$*.*
Let $u_j = (n - j, n - j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \ldots, n$*, and let* $\boldsymbol{u} = (u_1, \ldots, u_n)$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n)$*. We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with n-tuples of nonintersecting D-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$*.*

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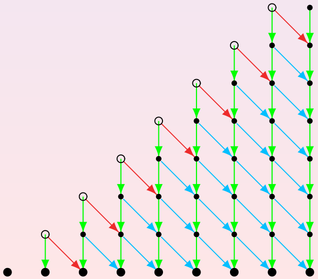


Example of lattice paths

Example

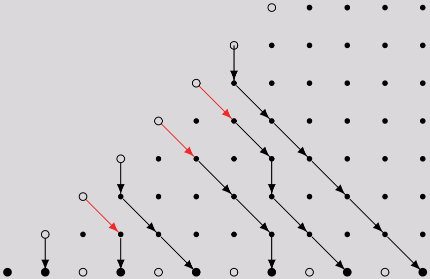
$n = 7$, $c \in \mathscr{P}_7$: RCSPP

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Example of lattice paths

Example

Lattice paths



Weight of each edge

Definition

Let $u \to v$ be an edge in from u to v.

1. We assign the weight

$$\begin{cases} \prod_{k=j}^{n} t_k \cdot x_j & \text{if } j = i, \\ t_j x_j & \text{if } j < i, \end{cases}$$

to the horizontal edge from $u = (i, j)$ to $v = (i + 1, j - 1)$.

2. We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j - 1)$.

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Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_n \\ \mathrm{sh}\, c = \lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, T_{n-i} x_{n-i})\right)_{1 \leq i,j \leq n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	$t_1 x_1$	$t_1^2 t_2 t_3 x_1^2$	$t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

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$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

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where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	$\boxed{1}$	$\boxed{1}\boxed{1}$	$\boxed{2}$	$\boxed{2}\boxed{1}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{l}\boxed{2}\boxed{1}\\\boxed{1}\end{array}$
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

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where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	1	1 1	2	2 1	2 1	2 1 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n and N, let $B_n^N(t) = (b_{ij}(t))_{0 \le i \le n-1,\ 0 \le j \le n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1} t & \text{otherwise.} \end{cases}$$

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Example

If $n = 3$ and $N = 2$, then

$$B_3^2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & 1 & 1+t & t \end{pmatrix}$$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $J_n = (\delta_{i,n+1-j})_{1\le i,j\le n}$ be the $n \times n$ anti-diagonal matrix.

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $J_n = (\delta_{i,n+1-j})_{1\leq i,j\leq n}$ be the $n \times n$ anti-diagonal matrix.

Example

If $n = 4$, then

$$J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $\overline{S}_n = (\overline{s}_{i,j})_{1\leq i,j\leq n}$ be the $n \times n$ skew-symmetricl matrix whose (i, j)th entry is

$$\overline{s}_{i,j} = \begin{cases} (-1)^{j-i-1} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ (-1)^{j-i} & \text{if } i > j. \end{cases}$$

A Pfaffian expression for the refined TSSCPP conj.

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For positive integers n, let $\overline{S}_n = (\overline{s}_{i,j})_{1 \le i,j \le n}$ be the $n \times n$ skew-symmetricl matrix whose (i,j)th entry is

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Example

If $n = 4$, then

$$\overline{S}_4 = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

A Pfaffian expression for the refined TSSCPP conj.

Theorem

Let n be a positive integer and let N be an even integer such that $N \geq n-1$. If k is an integer such that $1 \leq k \leq n$, then

$$\sum_{c \in \mathscr{P}_n} t^{\overline{U}_k(c)} = \mathrm{Pf}\begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \overline{S}_{n+N} \end{pmatrix}.$$

A Pfaffian expression for the refined TSSCPP conj.

Example

If $n = 3$ and $N = 2$ then

$$
\mathrm{Pf}\left(\begin{array}{ccc|ccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1+t & t \\
0 & 0 & 0 & 0 & 1 & t & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\
0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\
-1 & -t & 0 & 1 & -1 & 0 & 1 & -1 \\
-1-t & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\
-t & 0 & 0 & 1 & -1 & 1 & -1 & 0
\end{array}\right).
$$

A constant term identity for the refined TSSCPP conj.

Theorem

Let n be a positive integer. If k is an integer such that $1 \le k \le n$, then $\sum_{c \in \mathscr{P}_n} t^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right) \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \le i < j \le n} \frac{1}{1 - x_i x_j}.$$

Example

If $n = 3$, then the constant term of

$$\left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_1}{x_3}\right)\left(1 - \frac{x_2}{x_3}\right)\left(1 + \frac{t}{x_2}\right)\left(1 + \frac{1}{x_3}\right)\left(1 + \frac{t}{x_3}\right) \times \frac{1}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_1 x_2)(1 - x_1 x_3)(1 - x_2 x_3)}$$

is equal to $2 + 3t + 2t^2$.

A constant term identity for the refined TSSCPP conj.

Theorem

Let n be a positive integer. If k is an integer such that $1 \leq k \leq n$, then $\sum_{c \in \mathscr{P}_n} t^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right) \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}.$$

Example

If $n = 3$, then the constant term of

$$\left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_1}{x_3}\right)\left(1 - \frac{x_2}{x_3}\right)\left(1 + \frac{t}{x_2}\right)\left(1 + \frac{1}{x_3}\right)\left(1 + \frac{t}{x_3}\right)$$
$$\times \frac{1}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_1 x_2)(1 - x_1 x_3)(1 - x_2 x_3)}$$

is equal to $2 + 3t + 2t^2$.

A Pfaffian expression for the doubly refined TSSCPP enumeration

Definition

For positive integers n and N, let $B_n^N(t,u) = (b_{ij}(t,u))_{0\leq i\leq n-1,\ 0\leq j\leq n+N-1}$ be the $n\times(n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t,u) = \begin{cases} \delta_{0,j} & \text{if } i=0, \\ \delta_{0,j-i} + \delta_{0,j-i-1}tu & \text{if } i=1, \\ \binom{i-2}{j-i} + \binom{i-2}{j-i-1}(t+u) + \binom{i-2}{j-i-2}tu & \text{otherwise.} \end{cases}$$

A Pfaffian expression for the doubly refined TSSCPP enumeration

Example

If $n = 3$ and $N = 2$, then

$$B_3^2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & tu & 0 & 0 \\ 0 & 0 & 1 & t+u & tu \end{pmatrix}$$

A Pfaffian expression for the doubly refined TSSCPP enumeration

Theorem

Let n be a positive integer and let N be an even integer such that $N \geq n-1$. If k is an integer such that $2 \leq k \leq n$, then

$$\sum_{c \in \mathscr{P}_n} t^{\overline{U}_1(c)} u^{\overline{U}_k(c)} = \operatorname{Pf} \begin{pmatrix} O_n & J_n B_n^N(t,u) \\ -{}^t B_n^N(t,u) J_n & \overline{S}_{n+N} \end{pmatrix}.$$

A Pfaffian expression for the doubly refined TSSCPP enumeration

Example

If $n = 3$ and $N = 2$ then

$$\mathrm{Pf}\left(\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & t+u & tu \\ 0 & 0 & 0 & 0 & 1 & tu & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\ -1 & -tu & 0 & 1 & -1 & 0 & 1 & -1 \\ -t-u & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ -tu & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right).$$

A constant term identity for the doubly refined TSSCPP enumeration

Definition

Let $h_i(t, u; x)$ denote the function defined by

$$h_i(t,u;x) = \begin{cases} 1 & \text{if } i = 0, \\ 1 + tux & \text{if } i = 1, \\ (1+x)^{i-2}(1+tx)(1+ux) & \text{if } i \geq 2. \end{cases}$$

Theorem

Let n be a positive integer. If k is an integer such that $2 \leq k \leq n$, then $\sum_{c \in \mathscr{P}_n} t^{\overline{U}_1(c)} u^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\mathbf{x}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=1}^{n} h_{i-1}\left(t, u; x_i^{-1}\right) \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}.$$

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A constant term identity for the doubly refined TSSCPP enumeration

Example

If $n = 3$, then the constant term of

$$\left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_1}{x_3}\right)\left(1 - \frac{x_2}{x_3}\right)\left(1 + \frac{tu}{x_2}\right)\left(1 + \frac{t}{x_3}\right)\left(1 + \frac{u}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $1 + t + tu + t^2u + tu^2 + ut^2u^2$.

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

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Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

If $n = 3$ and $k = 0$, $\mathscr{P}_{3,0}$ consists of the single PP:

$$\emptyset.$$

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

If $n = 3$ and $k = 1$, $\mathscr{P}_{3,1}$ consists of the following 5 PPs:

$\emptyset$ | 1 | 1 1 | 2 | 2 1

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

If $n = 3$ and $k = 2$, $\mathscr{B}_{3,2}$ consists of the followng 7 PPs

$\emptyset$ $\quad$ $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$

A constant term identity

Theorem

Let n be a positive integer. The restriction of φ_n to $\mathscr{B}_{nk}$ gives a bijection from $\mathscr{B}_{nk}$ to $\mathscr{P}_{nk}$.

Theorem

Let n be a positive integer. If $0 \leq k \leq n-1$ and $1 \leq r \leq n$, then $\sum_{c\in\mathscr{P}_{nk}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\leq i<j\leq n} \left(1-\frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1+\frac{1}{x_i}\right)^{i-2} \left(1+\frac{t}{x_i}\right) \times \frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\leq i,j\leq n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\leq i<j\leq n}(x_j-x_i)(1-x_ix_j)}.$$

A constant term identity

Theorem

Let n be a positive integer. The restriction of φ_n to $\mathscr{B}_{nk}$ gives a bijection from $\mathscr{B}_{nk}$ to $\mathscr{P}_{nk}$.

Theorem

Let n be a positive integer. If $0 \le k \le n-1$ and $1 \le r \le n$, then $\sum_{c\in\mathscr{P}_{nk}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=2}^{n}\left(1+\frac{1}{x_i}\right)^{i-2}\left(1+\frac{t}{x_i}\right)$$

$$\times\frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\le i,j\le n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}.$$

Example of $n = 3$

Example

If $n = 3$ and $k = 0$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times \frac{\det\begin{pmatrix} 1-x_1^5 & x_1-x_1^4 & x_1^2-x_1^3 \\ 1-x_2^5 & x_2-x_1^4 & x_2^2-x_2^3 \\ 1-x_3^5 & x_3-x_1^4 & x_3^2-x_3^3 \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to 1.

Example of $n = 3$

Example

If $n = 3$ and $k = 1$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times \frac{\det\begin{pmatrix} 1-x_1^{6} & x_1-x_1^{5} & x_1^2-x_1^{5} \\ 1-x_2^{6} & x_2-x_1^{5} & x_2^2-x_2^{5} \\ 1-x_3^{6} & x_3-x_1^{5} & x_3^2-x_3^{5} \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2+2t+t^2$.

Example of $n = 3$

Example

If $n = 3$ and $k = 2$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times \frac{\det\begin{pmatrix} 1-x_1^{7} & x_1-x_1^{6} & x_1^2-x_1^{5} \\ 1-x_2^{7} & x_2-x_1^{6} & x_2^2-x_2^{5} \\ 1-x_3^{7} & x_3-x_1^{6} & x_3^2-x_3^{5} \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2+3t+2t^2$.

Twisted Bender-Knuth involution

The Bender-Knuth involution s_k on tableaux which swaps the number of k's and $(k-1)$'s, for each i.

Example

$n = 7, c \in \mathscr{P}_3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$, $c \in \mathscr{P}_3$

5	5	4	2	2
4	4	3	1	
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2	1			
1				

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Example

$n = 7$ Apply $\widetilde{\pi}_2$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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1				

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If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_2(c) \in \mathscr{P}_3$.

5	5	4	2	1
4	4	3	1	
3	2	1		
2	1			
1				

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If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_3$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_3(c) \in \mathscr{P}_3$.

5	5	4	3	2
4	4	3	1	
3	3	2		
2	1			
1				

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Example

$n = 7$ Apply $\widetilde{\pi}_4$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_4(c) \in \mathscr{P}_3$.

5	5	4	2	2
4	3	3	1	
3	2	2		
2	1			
1				

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Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_5$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_5(c) \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_6$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_6(c) \in \mathscr{P}_3$.

6	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

Let $c \in \mathscr{P}_n$. Set λ_i to be the number of parts ≥ 2 in the ith row of c. We set $\lambda_0 = n - 1$ by convention. Let k_i denote the number of 1's in the ith row. Let $\widetilde{\pi}_1$ be the involution on $\mathscr{P}_n$ that changes the number of 1's in the ith row from k_i to $\lambda_{i-1} - \lambda_i - k_i$.

Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_1(c) \in \mathscr{P}_3$.

5	5	4	2	2	1
4	4	3	1		
3	2	2			
2	1				

Flips in words of RCSPP

Theorem

Let n be a positive integer and let $k = 1, \dots, n-1$. If $b \in \mathscr{B}_n$, then we have

$$\widetilde{\pi}_k\left(\varphi_n\left(b\right)\right) = \varphi_n\left(\pi_k\left(b\right)\right).$$

Definition

We define involutions on $\mathscr{P}_n$

$$\widetilde{\rho} = \widetilde{\pi}_2\widetilde{\pi}_4\widetilde{\pi}_6\cdots,$$
$$\widetilde{\gamma} = \widetilde{\pi}_1\widetilde{\pi}_3\widetilde{\pi}_5\cdots,$$

and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Flips in words of RCSPP

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Let n be a positive integer and let $k = 1, \dots, n-1$. If $b \in \mathscr{B}_n$, then we have

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and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_1^{\widetilde{\rho}} = \{\emptyset\}$

Invariants under $\widetilde{\rho}$

Example

$$\mathscr{P}_2^{\widetilde{\rho}} = \left\{\emptyset, \boxed{1}\right\}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_3^{\widetilde{\rho}}$ is composed of the following 3 RCSPPs:

$$\emptyset \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_4^{\widetilde{\rho}}$ is composed of the following 10 elements:

$\emptyset$ $\quad$ $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline 2 & 1 & \color{red}{1} \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 2 & \color{red}{2} \\ \hline 1 & 1 \\ \hline\end{array}$

$\begin{array}{|c|c|c|}\hline 2 & \color{red}{2} & \color{red}{1} \\ \hline 1 & 1 \\ \cline{1-2}\end{array}$ $\quad$ $\begin{array}{|c|}\hline \color{red}{3} \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|}\hline \color{red}{3} \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline \color{red}{3} & \color{red}{2} \\ \hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline \color{red}{3} & \color{red}{2} & \color{red}{1} \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_5^{\widetilde{\rho}}$ has 25 elements, and $\mathscr{P}_6^{\widetilde{\rho}}$ has 140 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

1	1

3	2	1
1		

3	3	1
2	2	
1		

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

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Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

The following $c \in \mathscr{P}_{11}$ is invariant under $\widetilde{\gamma}$:

$c =$

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Remove all 1's from $c \in \mathscr{P}_{11}^{\widetilde{\gamma}}$.

$c =$

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Then we obtain a PP in which each row has even length.

$c =$

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Identify 3 and 2, 5 and 4, 7 and 6.

$c =$

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

Invariants under $\widetilde{\gamma}$

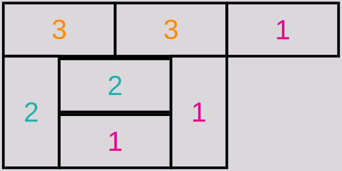
Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Repace 3 and 2 by dominos containing 1, 5 and 4 by dominos containing 2, 7 and 6 by dominos containing 3.

$d =$



Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

Domino plane partitions

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1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

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Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

$\mathscr{D}_1^{\mathrm{R}} = \mathscr{D}_2^{\mathrm{R}} = \{\emptyset\}$.

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Example

$\mathscr{D}_3^{\mathrm{R}}$ is composed of the following 3 elements:

$$\emptyset, \qquad \boxed{\;1\;}, \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}.$$

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Example

$\mathscr{D}_4^{\mathrm{R}}$ is composed of the following 4 elements:

$\emptyset$, [1], [1][1], [2][1].

$\mathscr{D}_5^{\mathrm{R}}$ has 26 elements, $\mathscr{D}_6^{\mathrm{R}}$ has 50 elements, and $\mathscr{D}_7^{\mathrm{R}}$ has 646 elements.

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{\mathrm{R}}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^0_n(t) = (R^0_{ij})_{0\leq i,j\leq n-1}$ be the $n \times n$ matrix where

$$R^0_{ij} = \binom{i+j-1}{2i-j} + \left(\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right)t + \binom{i+j-1}{2i-j}t^2$$

with the convention that $R^0_{0,0} = R^0_{0,1} = 1$. Then we obtain

$$\sum_{c\in\mathscr{P}_{2n+1}} t^{\overline{U}_2(c)} = \det R^0_n(t).$$

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Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{R}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

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The determinants

Example

if $n = 2$, then $\sum_{c \in \mathscr{P}_5^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 \\ 0 & 1 + t + t^2 \end{pmatrix}$$

which is equal to $1 + t + t^2$.

The determinants

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if $n = 3$, then $\sum_{c \in \mathscr{P}_7^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+t+t^2 & 1+2t+t^2 \\ 0 & t & 3+4t+3t^2 \end{pmatrix}$$

which is equal to $3 + 6t + 8t^2 + 6t^3 + 3t^4$.

The determinants

Example

if $n = 4$, then $\sum_{c \in \mathscr{P}_7^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1+t+t^2 & 1+2t+t^2 & t \\ 0 & t & 3+4t+3t^2 & 4+7t+4t^2 \\ 0 & 0 & 1+4t+t^2 & 10+15t+10t^2 \end{pmatrix}$$

which is equal to $26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$.

Determinant evaluation

Theorem (Andrews-Burge)

Let

$$M_n(x,y) = \det\left(\binom{i+j+x}{2i-j} + \binom{i+j+y}{2i-j}\right)_{0\le i,j\le n-1}.$$

Then

$$M_n(x,y) = \prod_{k=0}^{n-1} \Delta_{2k}(x+y),$$

where $\Delta_0(u) = 2$ and for $j > 0$

$$\Delta_{2j}(u) = \frac{(u+2j+2)_j(\frac{1}{2}u+2j+\frac{3}{2})_{j-1}}{(j)_j(\frac{1}{2}u+j+\frac{3}{2})_{j-1}}.$$

A weak version of Conjecture 6

Theorem

Let n be a positive integer. Then

$$\det R_n^{o}(1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

This proves tha the number of $b \in \mathscr{B}_{2n+1}$ invariant under γ is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$.

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The end

Thank you!