An Identity for Compound Determinant and its Application

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Introduction

Abstract

In this talk we give a formula for a compound determinant and use it to derive a Schur function identity. This compound determinant is a variant of Sylvester's determinant whose row is parametrized by n-element subsets of $\{1, 2, \ldots, s + n - 1\}$ and column is parametrized by compositions of n with at most s parts. We introduce a partial order on the set of compositions of n with at most s parts, and use this partial order to compute the determinant. This determinant identity has an application to compute a determinant whose entries are certain Schur functions, and this result generalize a Schur function identity obtained in the paper "A determinant formula for a holonomic q-difference system associated with Jackson integrals of type BC_n" by K. Aomoto and M. Ito.

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Preliminaries

Definition

Let r and s be integers.

- $[r, s] = \{r, r+1, \dots, s\}$
- $[r] = [1, r]$
- If S is a finite set, let $\binom{S}{r}$ denote the set of all r-element subsets of S.

Example

$[4] = \{1, 2, 3, 4\}$

$\binom{[4]}{2} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$

$\#\binom{[4]}{2} = \binom{4}{2} = 6$

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For $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$ in $\binom{[N]}{r}$, we write $I < J$ if there is an index k such that

$$i_1 = j_1, \quad \ldots, \quad i_{k-1} = j_{k-1}, \quad i_k < j_k :$$

This gives a total ordering on $\binom{[N]}{r}$, which is called the *lexicographic order*.

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$\{1, 3, 4, 7\} < \{1, 3, 5, 6\}$

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Minors

Definition

Let A be any $M \times N$ matrix, and let $I = \{i_1, \dots, i_r\} \subseteq [M]$ (resp. $J = \{j_1, \dots, j_r\} \subseteq [N]$) be a row (resp. column) index set.

Let $A^I_J = A^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ denote the matrix obtained from A by choosing rows indexed by I and columns indexed by J.

If $r = M$ and $I = [M]$ (i.e. we choose all rows), then we write $A_J = A_{j_1,\dots,j_r}$ for $A^{[M]}_J$.

Similary we write $A^I = A^{i_1,\dots,i_r}$ for $A^I_{[N]}$ when $r = N$ and $J = [N]$.

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If $I = \{1, 3\}$ and $J = \{2, 4\}$, then

$$A^I_J = A^{1,3}_{2,4} = \begin{pmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{pmatrix}.$$

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Sylvester's Determinant

Theorem

Let n and m be positive integers such that $m \leq n$.
Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix.
Then we have

$$\det\left(\det A^I_J\right)_{I,J \in \binom{[n]}{m}} = (\det A)^{\binom{n-1}{m-1}},$$

where the rows and columns of the matrix on the left hand side are arranged in increasing order with respect to $<$.

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Compound Determinant

Definition

Let s and n be positive integers.

Let $\mathscr{Z}_{s,n}$ denote the set of compositions of n which has at most s parts, i.e.

$$\mathscr{Z}_{s,n} = \{\mu = (\mu_1, \dots, \mu_s) \,|\, \mu_1 \geq 0, \dots, \mu_s \geq 0,\ \mu_1 + \cdots + \mu_s = n\},$$

and let $\mathscr{Z}^0_{s,n}$ denote the set of compositions of n which has exactly s parts, i.e.

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Compound Determinant

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$s = 3$, $n = 2$

$$\mathscr{Z}_{3,2} = \{(2,0,0),(1,1,0),(1,0,1),(0,2,0),(0,1,1),(0,0,2)\}$$

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Proposition

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$$\sharp\mathscr{Z}_{s,n} = \binom{s+n-1}{n},$$

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$s = 3$, $n = 2$, $\mathscr{Z}_{3,2}$

$$(2,0,0) < (1,1,0) < (1,0,1) < (0,2,0) < (0,1,1) < (0,0,2)$$

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Compound Determinant

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To each $\mu \in \mathscr{Z}_{s,n}$, we associate an subset $\iota(\mu) \in \binom{[sn]}{n}$ defined by

$$\iota(\mu) = \bigsqcup_{i=1}^{s} [(i-1)n+1, (i-1)n+\mu_i] .$$

This gives an injection $\iota : \mathscr{Z}_{s,n} \to \binom{[sn]}{n}$, and one readily sees that, for $\lambda, \mu \in \mathscr{Z}_{s,n}$, $\iota(\lambda) < \iota(\mu)$ if and only if $\lambda < \mu$.

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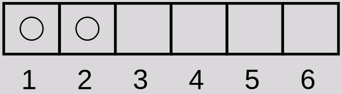
$s = 3$, $n = 2$, $\iota(2,0,0) = \{1, 2\}$



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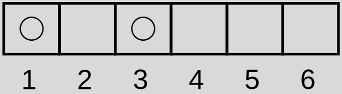
Example

$s = 3$, $n = 2$, $\iota(1,1,0) = \{1,3\}$



Compound Determinant

Definition

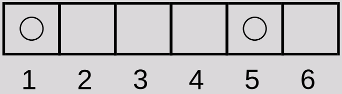
To each $\mu \in \mathscr{Z}_{s,n}$, we associate an subset $\iota(\mu) \in \binom{[sn]}{n}$ defined by

$$\iota(\mu) = \bigsqcup_{i=1}^{s} [(i-1)n + 1, (i-1)n + \mu_i] \,.$$

This gives an injection $\iota : \mathscr{Z}_{s,n} \to \binom{[sn]}{n}$, and one readily sees that, for $\lambda, \mu \in \mathscr{Z}_{s,n}$, $\iota(\lambda) < \iota(\mu)$ if and only if $\lambda < \mu$.

Example

$s = 3$, $n = 2$, $\iota(1,0,1) = \{1,5\}$



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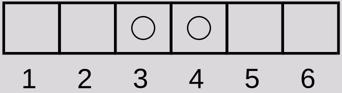
Example

$s = 3$, $n = 2$, $\iota(0, 2, 0) = \{3, 4\}$



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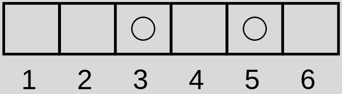
Example

$s = 3$, $n = 2$, $\iota(0, 1, 1) = \{3, 5\}$



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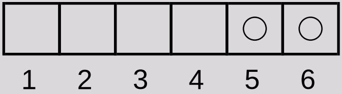
This gives an injection $\iota : \mathscr{Z}_{s,n} \to \binom{[sn]}{n}$, and one readily sees that, for $\lambda, \mu \in \mathscr{Z}_{s,n}$, $\iota(\lambda) < \iota(\mu)$ if and only if $\lambda < \mu$.

Example

$s = 3$, $n = 2$, $\iota(0,0,2) = \{5,6\}$



Compound Determinant

theorem

Let s and n be positive integers and $A = (a_{ij})_{1 \leq i \leq s+n-1,\ 1 \leq j \leq sn}$ be an $(s+n-1) \times sn$ matrix. We put

$$\mathscr{R} = \binom{[s+n-1]}{n}, \quad \mathscr{C} = \{\iota(\mu) : \mu \in \mathscr{Z}_{s,n}\}.$$

Then we have

$$\det\left(\det A^I_J\right)_{I \in \mathscr{R}, J \in \mathscr{C}} = \prod_{\nu \in \mathscr{Z}^0_{s,s+n-1}} \det A_{\iota(\nu)},$$

where the rows and columns of the matrix on the left hand side are arranged in increasing order with respect to $<$.

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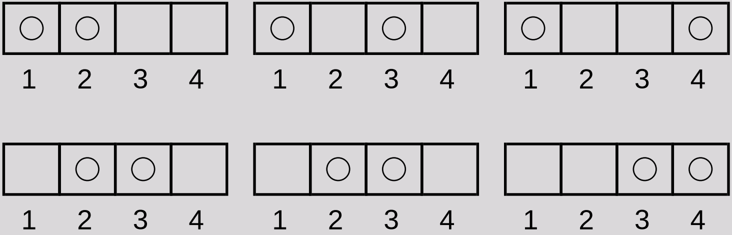
Compound Determinant

Example

If $s = 3$ and $n = 2$, then
$\mathscr{R} = \binom{[3+2-1]}{2} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$



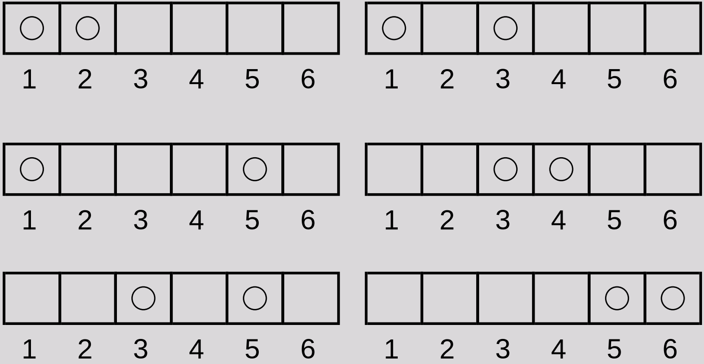
Compound Determinant

Example

If $s = 3$ and $n = 2$, then

$\mathscr{Z}_{3,2} = \{(2,0,0),(1,1,0),(1,0,1),(0,2,0),(0,1,1),(0,0,2)\}$

$\mathscr{C} = \{\iota(\mu) : \mu \in \mathscr{Z}_{3,2}\} = \{\{1,2\},\{1,3\},\{1,5\},\{3,4\},\{3,5\},\{5,6\}\}$



Compound Determinant

Example

If $s = 3$ and $n = 2$, then the left-hand side determinant $\det\left(\det A^I_J\right)_{I\in\mathscr{R},J\in\mathscr{C}}$ equals

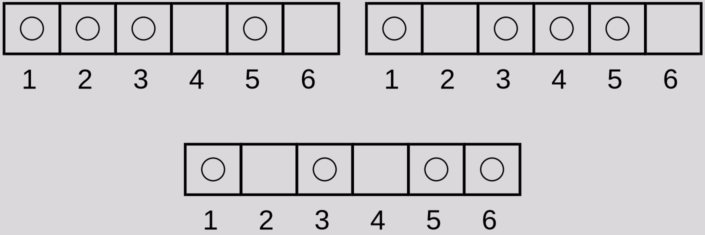
$$\det\begin{pmatrix}
\det A^{12}_{12} & \det A^{12}_{13} & \det A^{12}_{15} & \det A^{12}_{34} & \det A^{12}_{35} & \det A^{12}_{56} \\
\det A^{13}_{12} & \det A^{13}_{13} & \det A^{13}_{15} & \det A^{13}_{56} & \det A^{13}_{35} & \det A^{13}_{56} \\
\det A^{14}_{12} & \det A^{14}_{13} & \det A^{14}_{15} & \det A^{14}_{56} & \det A^{14}_{35} & \det A^{14}_{56} \\
\det A^{23}_{12} & \det A^{23}_{13} & \det A^{23}_{15} & \det A^{23}_{56} & \det A^{23}_{35} & \det A^{23}_{56} \\
\det A^{24}_{12} & \det A^{24}_{13} & \det A^{24}_{15} & \det A^{24}_{56} & \det A^{24}_{35} & \det A^{24}_{56} \\
\det A^{34}_{12} & \det A^{34}_{13} & \det A^{34}_{15} & \det A^{34}_{56} & \det A^{34}_{35} & \det A^{34}_{56}
\end{pmatrix}.$$

Compound Determinant

Example

If $s = 3$ and $n = 2$, then $\mathscr{Z}^0_{3,3+2-1} = \{(2,1,1),(1,2,1),(1,1,2)\}$

$\{\iota(\mu) : \mu \in \mathscr{Z}^0_{3,4}\} = \{\{1,2,3,5\},\{1,3,4,5\},\{1,3,5,6\}\}$



Compound Determinant

Example

If $s = 3$ and $n = 2$, then the right-hand side $\det A_{1235} \det A_{1345} \det A_{1356}$ equals

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{45} \end{pmatrix} \det \begin{pmatrix} a_{11} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{43} & a_{44} & a_{45} \end{pmatrix}$$

$$\times \det \begin{pmatrix} a_{11} & a_{13} & a_{15} & a_{16} \\ a_{21} & a_{23} & a_{25} & a_{26} \\ a_{31} & a_{33} & a_{35} & a_{36} \\ a_{41} & a_{43} & a_{45} & a_{46} \end{pmatrix}.$$

An application to a Schur function determinant

Definition

A *partition* is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$$

of non-negative integers in decreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \ldots$ and containing only finitely many non-zero terms.

Example

$\lambda = (5441)$ is a partition of 14 with length 4.

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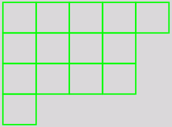

An application to a Schur function determinant

Definition (Tableaux)

Given a partition λ, A *tableaux* T of shape λ is a filling of the diagram with numbers whereas the numbers must strictly increase down each column and weakly from left to right along each row. Let $x = (x_1, x_2, \dots)$ be variables. The *weight* of tableaux T is

$$\mathrm{w}(T) = x_1^{\sharp 1's} x_2^{\sharp 2's} \cdots$$

Example

A Tableau T of shape (5441).

5	5	4	4	2
3	3	3	2	
2	2	1	1	
1				

The weight of T is $\mathrm{w}(T) = x^2 x^4 x^3 x^2 x^2$

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An application to a Schur function determinant

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The *Schur function $s_\lambda(x)$* is, by definition,

$$s_\lambda(x) = \sum_T \mathrm{w}(T),$$

where the sum runs over all tableaux of shape λ.

Example

When $\lambda = (22)$,

1	1
2	2

1	1
2	3

1	1
3	3

1	2
2	3

1	2
3	3

2	2
3	3

$$s_\lambda(x) = x_1^2 x_2^2 + x_1^2 x_3^2 + \dots$$

An application to a Schur function determinant

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$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$$

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An application to a Schur function determinan

Definition (Schur functions)

For $X = (x_1, \ldots, x_n)$ and a partition λ such that $\ell(\lambda) \leq n$, let

$$s_\lambda(X) = \frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}}.$$

$s_\lambda(X)$ is called *the Schur function* corresponding to λ.

Example

If $n = 2$ and $\lambda = (2^2)$, then

$$s_{(2^2)}(x_1, x_2) = \frac{1}{x_1 - x_2} \det \begin{pmatrix} x_1^3 & x_1^2 \\ x_2^3 & x_2^2 \end{pmatrix}$$

An application to a Schur function determinan

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An application to a Schur function determinant

If we apply the main theorem to the matrix $A = (a_{ij})$ given by

$$a_{i,(k-1)n+j} = \left(x_j^{(k)}\right)^{i-1} \quad (1 \le i \le s+n-1, 1 \le k \le s, 1 \le j \le n),$$

then we obtain the following formula.

Corollary

To $\mu \in Z_{s,n}$, we associate

$$X_\mu = (x_1^{(1)}, \cdots, x_{\mu_1}^{(1)}, x_1^{(2)}, \cdots, x_{\mu_2}^{(2)}, \cdots, x_1^{(s)}, \cdots, x_{\mu_s}^{(s)}).$$

Then we have

$$\det\left(s_\lambda(X_\mu)\right)_{\lambda \subset ((s-1)^n), \mu \in Z_{s,n}} = \pm \prod_{1 \le k < l \le s} \prod_{i,j=1}^{n} \left(x_i^{(k)} - x_j^{(l)}\right)^{\binom{s+n-i-j-1}{s-2}},$$

where the rows are indexed by partitions $\lambda = (\lambda_1, \cdots, \lambda_n)$ such that $\lambda_1 \le s-1$.

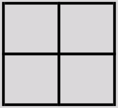
An application to a Schur function determinant

Example

$s = 3$, $n = 2$,

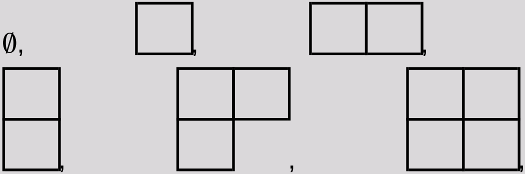
The partitions contained the rectangle

$$\lambda \in (3-1)^2 =$$



are as follows:

$\emptyset$,



An application to a Schur function determinant

The left-hand side determinant equals

$$\begin{vmatrix} s_\emptyset(X_{2,0,0}) & s_\emptyset(X_{1,1,0}) & s_\emptyset(X_{1,0,1}) & s_\emptyset(X_{0,2,0}) & s_\emptyset(X_{0,1,1}) & s_\emptyset(X_{0,0,2}) \\ s_1(X_{2,0,0}) & s_1(X_{1,1,0}) & s_1(X_{1,0,1}) & s_1(X_{0,2,0}) & s_1(X_{0,1,1}) & s_1(X_{0,0,2}) \\ s_2(X_{2,0,0}) & s_2(X_{1,1,0}) & s_2(X_{1,0,1}) & s_2(X_{0,2,0}) & s_2(X_{0,1,1}) & s_2(X_{0,0,2}) \\ s_{11}(X_{2,0,0}) & s_{11}(X_{1,1,0}) & s_{11}(X_{1,0,1}) & s_{11}(X_{0,2,0}) & s_{11}(X_{0,1,1}) & s_{11}(X_{0,0,2}) \\ s_{21}(X_{2,0,0}) & s_{21}(X_{1,1,0}) & s_{21}(X_{1,0,1}) & s_{21}(X_{0,2,0}) & s_{21}(X_{0,1,1}) & s_{21}(X_{0,0,2}) \\ s_{22}(X_{2,0,0}) & s_{22}(X_{1,1,0}) & s_{22}(X_{1,0,1}) & s_{22}(X_{0,2,0}) & s_{22}(X_{0,1,1}) & s_{22}(X_{0,0,2}) \end{vmatrix}$$

where $X_{2,0,0} = (x_1^{(1)}, x_2^{(1)})$, $X_{1,1,0} = (x_1^{(1)}, x_1^{(2)})$, $X_{1,0,1} = (x_1^{(1)}, x_1^{(3)})$, $X_{0,2,0} = (x_1^{(2)}, x_2^{(2)})$, $X_{0,1,1} = (x_1^{(2)}, x_1^{(3)})$, and $X_{0,0,2} = (x_1^{(3)}, x_2^{(3)})$.

This determinant equals

$\pm(x_1^{(1)} - x_1^{(2)})^2(x_1^{(1)} - x_2^{(2)})(x_2^{(1)} - x_1^{(2)})(x_1^{(1)} - x_1^{(3)})^2(x_1^{(1)} - x_2^{(3)})(x_2^{(1)} - x_1^{(3)})(x_1^{(2)} - x_1^{(3)})^2(x_1^{(2)} - x_2^{(3)})(x_2^{(2)} - x_1^{(3)})$.

Proof of the Compound Determinant

How to prove the Compound determinant?

We introduce an order on $\mathscr{L}_{s,n}$ and prove by induction.

Definition

Let s and n be positive integers, and let A be an $(s+n-1)\times sn$ matrix. For $J \in \binom{[sn]}{n}$, we write

$$\mathscr{V}_J(A) = \left(\det A^I_J\right)_{I\in\binom{[s+n-1]}{n}},$$

and, for $K \in \binom{[sn]}{s-1}$, we write

$$\overline{\mathscr{V}}_K(A) = \left((-1)^{|I|-\frac{n(n+1)}{2}} \det A^{\overline{I}}_K\right)_{I\in\binom{[s+n-1]}{n}},$$

which are both $\binom{s+n-1}{n}$-dimensional column vectors where the entries are arranged in the lexicographic order of indices.

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Proof of the Compound Determinant

Example

If $s = 3$, $n = 2$, $J = \{1, 3\} \in \binom{[6]}{2}$ and $K = \{4, 6\} \in \binom{[6]}{2}$, then $\mathscr{V}_J(A)$ equals

$$ {}^t\left(\det A^{12}_{13} \quad \det A^{13}_{13} \quad \det A^{14}_{13} \quad \det A^{23}_{13} \quad \det A^{24}_{13} \quad \det A^{34}_{13}\right), $$

and $\overline{\mathscr{V}}_K(A)$ equals

$$ {}^t\left(\det A^{34}_{46} \quad -\det A^{24}_{46} \quad \det A^{23}_{46} \quad \det A^{14}_{46} \quad -\det A^{13}_{46} \quad \det A^{12}_{46}\right). $$

Proof of the Compound Determinant

Proposition

By the Laplace expansion formula, we have

$$\left\langle \mathscr{V}_J(A), \overline{\mathscr{V}}_K(A) \right\rangle = {}^t\mathscr{V}_J(A)\overline{\mathscr{V}}_K(A) = (-1)^{s(J,K)} \det A_{J\sqcup K}.$$

Example

If $s = 3$, $n = 2$, $J = \{1, 3\} \in \binom{[6]}{2}$ and $K = \{4, 6\} \in \binom{[6]}{2}$, then we have

$$\begin{aligned} \left\langle \mathscr{V}_J(A), \overline{\mathscr{V}}_K(A) \right\rangle &= \det A^{12}_{13} \det A^{34}_{46} - \det A^{13}_{13} \det A^{24}_{46} \\ &+ \det A^{14}_{13} \det A^{23}_{46} + \det A^{23}_{13} \det A^{14}_{46} - \det A^{24}_{13} \det A^{13}_{46} \\ &+ \det A^{34}_{13} \det A^{12}_{46} = \det A_{1346} \end{aligned}$$

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Proof of the Compound Determinant

Definition

Let s and n be positive integers, and fix a positive integer k such that $1 \leq k \leq s$. We introduce a partial order $\preceq_k$ on the set $\mathscr{Z}_{s,n}$ of compostions as follows. For λ and μ in $\mathscr{Z}_{s,n}$, we define $\lambda \preceq_k \mu$ if $\lambda_i \leq \mu_i$ for all $i \neq k$.

Example

$\lambda = (2, 0, 1, 3), \mu = (2, 1, 2, 1) \in \mathscr{Z}_{4,6}$, and we have $\lambda \preceq_4 \mu$ since $\lambda_i \leq \mu_i$ for $i \neq 4$.

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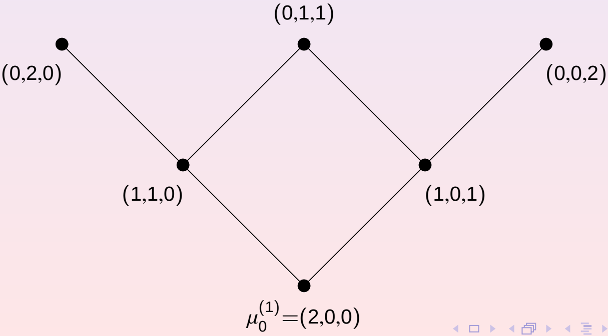
Proof of the Compound Determinant

$(\mathscr{Z}_{s,n}, \preceq_k)$ is a graded poset with the rank function $\rho^{(k)}(\mu) = n - \mu_k = \sum_{i \neq k} \mu_i$ for $\mu = (\mu_1, \dots, \mu_s) \in \mathscr{Z}_{s,n}$.
Let $\mu_0^{(k)}$ denote the minimum element of $(\mathscr{Z}_{s,n}, \preceq_k)$.
Let $P_i^{(k)} = \left\{ \mu \in \mathscr{Z}_{s,n} \,|\, \rho^{(k)}(\mu) = i \right\}$.

$(\mathscr{Z}_{3,2}, \preceq_1)$



Proof of the Compound Determinant

Definition

Let $P_i^{(k)} = \left\{ \mu \in \mathscr{Z}_{s,n} \,|\, \rho^{(k)}(\mu) = i \right\}$ be the subset of rank i elements. $\mathscr{Z}_{s,n} = P_0^{(k)} \sqcup \cdots \sqcup P_n^{(k)}$ is a disjoint union. Let $P^{(k)} = \bigsqcup_{i=0}^{n-1} P_i^{(k)} = \mathscr{Z}_{s,n} \setminus P_n^{(k)}$.

Example

$s = 3$, $n = 2$.

$$P_0^{(1)} = \{(2,0,0)\},$$
$$P_1^{(1)} = \{(1,1,0),(1,0,1)\},$$
$$P_2^{(1)} = \{(0,2,0),(0,1,1),(0,0,2)\}.$$

$$P^{(1)} = P_0^{(1)} \sqcup P_1^{(1)} = \{(2,0,0),(1,1,0),(1,0,1)\}$$

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Proof of the Compound Determinant

Definition

To each $\mu \in P^{(k)}$, define $\phi^{(k)}(\mu) \in \binom{[sn]}{s-1}$ by

$$\phi^{(k)}(\mu) = \bigsqcup_{\substack{i=1 \\ i \neq k}}^{s} \{(i-1)n + \mu_i + 1\}.$$

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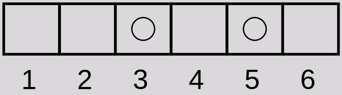
Example

If $s = 3$, $n = 2$ and $k = 1$,
$\phi^{(1)}(2,0,0) = \{3,5\}$



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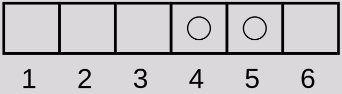
Example

If $s = 3$, $n = 2$ and $k = 1$,
$\phi^{(1)}(1,1,0) = \{4,5\}$



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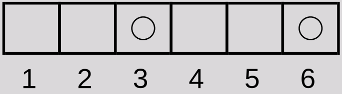
Example

If $s = 3$, $n = 2$ and $k = 1$,
$\phi^{(1)}(1,0,1) = \{3,6\}$



Proof of the Compound Determinant

Proposition

Let s and n be positive integers, and let A be an $(s+n-1)\times sn$ matrix. Fix a color $k\in\mathscr{C}$. Let $\lambda\in\mathscr{L}_{s,n}$ and $\mu\in P^{(k)}$. Then we have

$$\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\phi^{(k)}(\mu)}(A)\right\rangle = 0,$$

unless $\lambda\preceq_k\mu$.

Proof

Assume $\lambda\not\preceq_k\mu$. Then there exists $l\neq k$ such that $\lambda_l>\mu_l$. Since this implies $(l-1)n+\mu_l+1\in\phi^{(k)}(\mu)\cap\iota(\lambda)$, we obtain $\phi^{(k)}(\mu)\cap\iota(\lambda)\neq\emptyset$. Thus we have $\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\phi^{(k)}(\mu)}(A)\right\rangle = 0$.

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Proof of the Compound Determinant

Definition

Let

$$\mathscr{M}(A) = \left(\mathscr{V}_{\iota(\mu)}(A)\right)_{\mu \in \mathscr{Z}_{s,n}} = \left(\det A^{I}_{\iota(\mu)}\right)_{I \in \binom{[s+n-1]}{n},\, \mu \in \mathscr{Z}_{s,n}},$$

where the indices μ are arranged in the increasing order with respect to $<$.

Lamma

Let s and n be positive integers. Let A be an $(s+n-1) \times sn$ matrix. Then there exist non-negative integers m_ν, $\nu \in \mathscr{Z}^0_{s,n}$, and a constant $c \in \mathbb{Q}$ such that

$$\det \mathscr{M}(A) = c \prod_{\nu \in \mathscr{Z}^0_{s,s+n-1}} (\det A_\nu)^{m_\nu}.$$

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Proof of the Compound Determinant

Definition

Assume we are given by a map

$$\Phi : \mathscr{Z}_{s,n} \to \binom{[sn]}{s-1}.$$

Set $\widehat{\mathscr{M}}(\Phi, A)$ to be the $\binom{s+n-1}{n} \times \binom{s+n-1}{n}$ matrix defined by

$$\widehat{\mathscr{M}}(\Phi, A) = \left((-1)^{|I| - \frac{n(n+1)}{2}} \det A^{\bar{I}}_{\Phi(\mu)} \right)_{I \in \binom{[s+n-1]}{n},\, \mu \in \mathscr{Z}_{s,n}}.$$

Proposition

$$\det \mathscr{M}(A) \cdot \det \widehat{\mathscr{M}}(\Phi, A) = \det \left(\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\Phi(\mu)}(A) \right\rangle \right)_{\lambda, \mu \in \mathscr{Z}_{s,n}}$$

$$= \det \left((-1)^{s(\iota(\lambda), \Phi(\mu))} \det A_{\iota(\lambda) \sqcup \Phi(\mu)} \right)_{\lambda, \mu \in \mathscr{Z}_{s,n}}.$$

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Proof of the Compound Determinant

Sketch of Proof

We define a map $\pi : \mathscr{Z}_{s,n} \mapsto \mathscr{C}$ determined by the following condition:

> *For each* $\mu = (\mu_1, \dots, \mu_s) \in \mathscr{Z}_{s,n}$*, let* k *be the least index such that* $\mu_k = \max\{\mu_l : l = 1, \dots, s\}$*. We let* $\pi(\mu) = k$*.*

Define the map Φ by $\Phi(\mu) = \phi^{(\pi(\mu))}(\mu)$. Then we claim that

$$\det\left(\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\Phi(\mu)}(A) \right\rangle\right)_{\lambda,\mu\in\mathscr{Z}_{s,n}} = \pm \prod_{\mu\in\mathscr{Z}_{s,n}} \det A_{\iota(\mu)\sqcup\Phi(\mu)}.$$

Note that $\iota(\mu) \sqcup \phi^{(k)}(\mu) \in \mathscr{Z}^{0}_{s,s+n-1}$ for any k, and $\det A_J$, $J \in \binom{[sn]}{s+n-1}$, are irreducible polynomials in the unique factorization domain $\mathbb{Q}[a_{ij}]$.

Proof of the Compound Determinant

Example

If $s = 3$ and $n = 2$, then the map $\pi : \mathscr{Z}_{3,2} \to \mathscr{C}$ determined by the condition is given by $\pi(2,0,0) = 1$,$\pi(1,1,0) = 1$, $\pi(1,0,1) = 1$, $\pi(0,2,0) = 2$, $\pi(0,1,1) = 2$, $\pi(0,0,2) = 3$.
Thus, if we take $\Phi = \phi^{(\pi(\mu))}$, then the above 6×6 matrix $\widehat{\mathscr{M}}(\Phi, A)$ equals

$$\begin{pmatrix} \det A^{34}_{35} & \det A^{34}_{45} & \det A^{34}_{36} & \det A^{34}_{15} & \det A^{34}_{16} & \det A^{34}_{13} \\ -\det A^{24}_{35} & -\det A^{24}_{45} & -\det A^{24}_{36} & -\det A^{24}_{15} & -\det A^{24}_{16} & -\det A^{24}_{13} \\ \det A^{23}_{35} & \det A^{23}_{45} & \det A^{23}_{36} & \det A^{23}_{15} & \det A^{23}_{16} & \det A^{23}_{13} \\ \det A^{14}_{35} & \det A^{14}_{45} & \det A^{14}_{36} & \det A^{14}_{15} & \det A^{14}_{16} & \det A^{14}_{13} \\ -\det A^{13}_{35} & -\det A^{13}_{45} & -\det A^{13}_{36} & -\det A^{13}_{15} & -\det A^{13}_{16} & -\det A^{13}_{13} \\ \det A^{12}_{35} & \det A^{12}_{45} & \det A^{12}_{36} & \det A^{12}_{15} & \det A^{12}_{16} & \det A^{12}_{13} \end{pmatrix}.$$

Proof of the Compound Determinant

Example

Then we obtain that $\det\left(\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\Phi(\mu)}(A)\right\rangle\right)_{\lambda,\mu\in\mathscr{Z}_{s,n}}$ equals

$$\begin{pmatrix} \det A_{1235} & \det A_{1245} & \det A_{1236} & 0 & 0 & 0 \\ 0 & \det A_{1345} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\det A_{1356} & 0 & 0 & 0 \\ 0 & 0 & 0 & \det A_{1345} & \det A_{1346} & 0 \\ 0 & 0 & 0 & 0 & \det A_{1356} & 0 \\ 0 & 0 & 0 & 0 & 0 & \det A_{1356} \end{pmatrix},$$

whose determinant is $-\det A_{1235}\left(\det A_{1345}\right)^2\left(\det A_{1356}\right)^3$.

Proof of the Compound Determinant

Lamma

Let s and n be positive integers. Let A be an $(s+n-1)\times sn$ matrix. Then there exist a constant $c \in \mathbb{Q}$ such that

$$\det \mathscr{M}(A) = c \prod_{\nu \in \mathscr{Z}^0_{s,s+n-1}} \det A_\nu.$$

Sketch of Proof

We choose another map Φ such that the factors $\det A_\nu$, $\nu \in \mathscr{Z}^0_{s,s+n-1}$, appears just once in $\det\left(\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\Phi(\mu)}(A)\right\rangle\right)_{\lambda,\mu\in\mathscr{Z}_{s,n}}$, which implies $m_\nu = 0$ or 1. Comparing the degrees of the both sides, we conclude that $m_\nu = 1$.

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Proof of the Compound Determinant

Example

If $s = 3$, $n = 2$, then we take an appropriate map $\Phi : \mathscr{Z}_{3,2} \to \binom{[6]}{2}$, and define the 6×6 matrix $\widehat{\mathscr{M}}(\Phi, A)$

$$\begin{pmatrix} \det A^{34}_{35} & \det A^{34}_{45} & \det A^{34}_{36} & \det A^{34}_{25} & \det A^{34}_{46} & \det A^{34}_{23} \\ -\det A^{24}_{35} & -\det A^{24}_{45} & -\det A^{24}_{36} & -\det A^{24}_{25} & -\det A^{24}_{46} & -\det A^{24}_{23} \\ \det A^{23}_{35} & \det A^{23}_{45} & \det A^{23}_{36} & \det A^{23}_{25} & \det A^{23}_{46} & \det A^{23}_{23} \\ \det A^{14}_{35} & \det A^{14}_{45} & \det A^{14}_{36} & \det A^{14}_{25} & \det A^{14}_{46} & \det A^{14}_{23} \\ -\det A^{13}_{35} & -\det A^{13}_{45} & -\det A^{13}_{36} & -\det A^{13}_{25} & -\det A^{13}_{46} & -\det A^{13}_{23} \\ \det A^{12}_{35} & \det A^{12}_{45} & \det A^{12}_{36} & \det A^{12}_{25} & \det A^{12}_{46} & \det A^{12}_{23} \end{pmatrix}.$$

Proof of the Compound Determinant

Then we obtain that
${}^t\mathscr{M}(A)\widehat{\mathscr{M}}(\Phi,A) = \det\left(\left\langle \mathscr{V}_{\iota(\lambda)}(A), \overline{\mathscr{V}}_{\Phi(\mu)}(A)\right\rangle\right)_{\lambda,\mu\in\mathscr{Z}_{s,n}}$ equals

$$\begin{pmatrix} \det A_{1235} & \det A_{1245} & \det A_{1236} & 0 & \det A_{1246} & 0 \\ 0 & \det A_{1345} & 0 & -\det A_{1235} & \det A_{1346} & 0 \\ 0 & 0 & -\det A_{1356} & 0 & -\det A_{1456} & \det A_{1235} \\ 0 & 0 & 0 & \det A_{2345} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\det A_{3456} & 0 \\ 0 & 0 & 0 & 0 & 0 & \det A_{2356} \end{pmatrix},$$

whose determinant is
$\det A_{2345} \det A_{3456} \det A_{2356} \prod_{\nu\in\mathscr{Z}^0_{3,5}} \det A_{\iota(\nu)}$.

Proof of the Compound Determinant

The last step of the proof

To show that $c = 1$, we substitute $a_{i,j} = x_j^{s+n-i}$ into the both sides, and compare the leading coefficients.

Open problems

Definition (Hall-Little function)

Let X denote the set of variables $x_1, \dots, x_n$, and for any subset E of X, let $p(E)$ denote the product of the elements of E. Suppose a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is of the form $(\mu_1^{r_1}, \dots, \mu_k^{r_k})$, where $\mu_1 > \cdots > \mu_k \geq 0$, and the r_i are positive integers whose sum is n. Then the Hall-Little function is defined to be

$$P_\lambda(x_1, \dots, x_n; t) = \sum_f p\left(f^{-1}(1)\right)^{\mu_1} \cdots p\left(f^{-1}(k)\right)^{\mu_k} \prod_{f(x_i)<f(x_j)} \frac{x_i - tx_j}{x_i - x_j}$$

summed over all surjective mappings $f : X \to \{1, 2, \dots, k\}$ such that $\left|f^{-1}(i)\right| = r_i$ for $1 \leq i \leq k$.

Open problems

Example

If $n = 2$ and $\lambda = (2) = (2^1, 0^1)$, then there are two surjections:
$f_1 : x_1 \mapsto 1, x_2 \mapsto 2,$
$f_2 : x_1 \mapsto 2, x_2 \mapsto 1$

$$P_{(2)}(x_1, x_2; t) = x_1^2 \frac{x_1 - tx_2}{x_1 - x_2} + x_2^2 \frac{x_2 - tx_1}{x_2 - x_1}$$
$$= x_1^2 + x_1x_2 + x_2^2 - tx_1x_2$$

Open problems

Conjecture

To $\mu \in Z_{s,n}$, we associate

$$X_\mu = (x_1^{(1)}, \cdots, x_{\mu_1}^{(1)}, x_1^{(2)}, \cdots, x_{\mu_2}^{(2)}, \cdots, x_1^{(s)}, \cdots, x_{\mu_s}^{(s)}).$$

Then we have

$$\det\left(P_\lambda(X_\mu)\right)_{\lambda\subset((s-1)^n),\mu\in Z_{s,n}} = \pm \prod_{1\le k<l\le s} \prod_{i,j=1}^{n} \left(x_i^{(k)} - x_j^{(l)}\right)^{\binom{s+n-i-j-1}{s-2}},$$

where the rows are indexed by partitions $\lambda = (\lambda_1, \cdots, \lambda_n)$ such that $\lambda_1 \le s-1$.

Problem

What happens if we replace the Schur functions with the Macdonald polynomials?

Open problems

Conjecture

To $\mu \in Z_{s,n}$, we associate

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What happens if we replace the Schur functions with the Macdonald polynomials?

The end

Thank you!