Andrews-Stanley の分割関数と直交多項式

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Contents of this talk

References

- M. Ishikawa, "Minor summation formula and a proof of Stanley's open problem", `arXiv:math.CO/0408204`, to appear in Ramanujan J.
- Masao Ishikawa, Hiroyuki Tagawa, Soichi Okada and Jiang Zeng, "Generalizations of Cuachy's determinant and Schur's Pfaffian", `arXiv:math.CO/0411280`, to appear Adv. in Appl. Math.
- M. Ishikawa and Jiang Zeng, "The Andrews-Stanley partition function and Al-Salam-Chihara polynomials", `arXiv:math.CO/0506128`.

Preliminaries

A q-shifted factorial is, by definition,

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)\cdots(1-aq^{n-1}) & n = 1, 2, \ldots. \end{cases}$$

We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Notation

Since products of q-shifted factorials occur so often, to simplify them we shall use the more compact notations

$$(a_1, \dots, a_m; q)_n = (a_1, q)_n \cdots (a_n, q)_n$$

$$(a_1, \dots, a_m; q)_\infty = (a_1, q)_\infty \cdots (a_n, q)_\infty$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Basic hypergeometric series

We shall define an ${}_r\phi_s$ basic hypergeometric series by

$$
{}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right]
= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n
$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s+1$.

Partitions

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition, and n is called the weight of the partition, denoted by $|\lambda|$. Many times the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be denoted by λ, and we shall write $\lambda \vdash n$ to denote "λ is a partition of n". The number of (non-zero) parts is the length, denoted by $\ell(\lambda)$.

Example

The empty sequence ∅ forms the only partition of zero.

$n = 1$: (1);

$n = 2$: (2), (1^2);

$n = 3$: (3), (21), (1^3);

$n = 4$: (4), (31), (2^2), (21^2), (1^4);

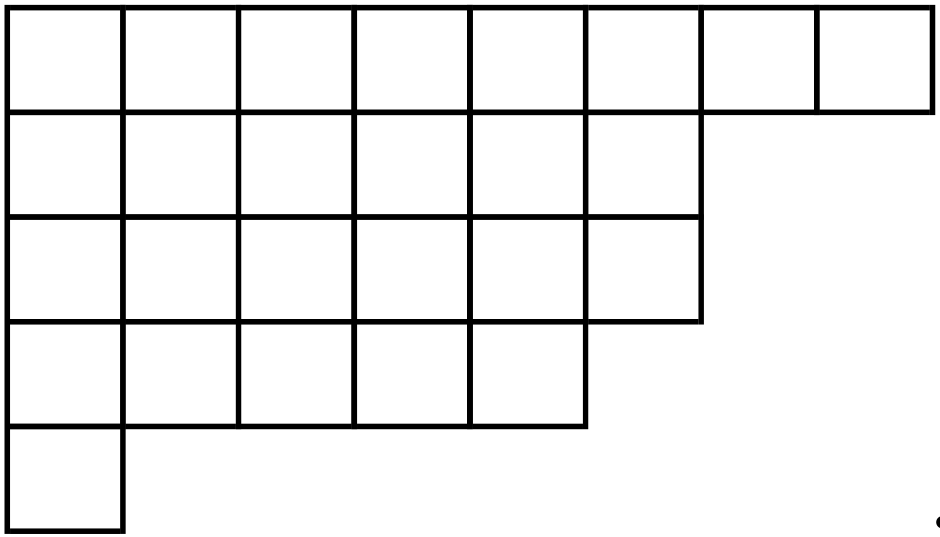
$n = 5$: (5), (41), (32), (31^2), (2^21), (21^3), (1^5);

Young Diagram

To each partition $\boldsymbol{\lambda}$ is associated its graphical representation (Young diagram) $\mathcal{D}_{\lambda}$, which formally is the set of points with integral coordinates (i, j) in the plane such that if $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_r)$, then $(i, j) \in \mathcal{D}_{\lambda}$ if and only if $1 \leq j \leq \boldsymbol{\lambda}_i$. We sometimes identify the Ferres graph $\mathcal{D}_{\lambda}$ with the partition $\boldsymbol{\lambda}$ and use the same symbol $\boldsymbol{\lambda}$ to express its Young diagram.

Example

The Young diagram of the partition $(8, 6, 6, 5, 1)$ is



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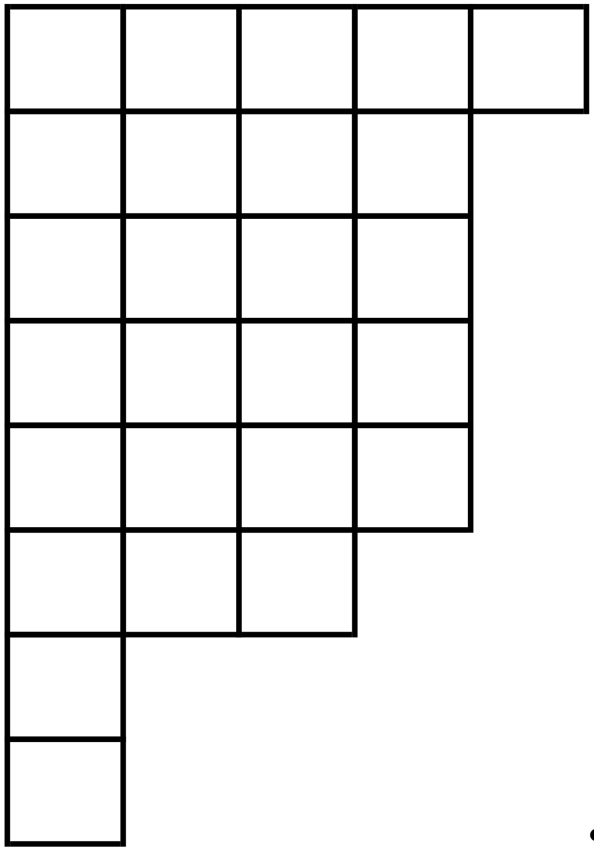
Conjugate

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition, we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_r)$ by choosing λ'_i as the number of parts of λ that are $\geq i$. The partition λ' is called the conjugate of λ.

Example

The conjugate of the partition $(86^2 51)$ is $(54^4 31^2)$



The generating function

Theorem (Euler)

For $|q| < 1$,

$$\sum_{\lambda} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where the sum runs over all partitions λ.

More generally,

$$\sum_{\lambda} z^{\ell(\lambda)} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-zq^n}$$

where the sum runs over all partitions λ.

Proof

$$\prod_{n=1}^{\infty} \frac{1}{1 - zq^n} = (1 + zq + z^2q^2 + z^3q^3 + \cdots)$$

$$\times (1 + zq^2 + z^2q^4 + z^3q^6 + \cdots)$$

$$\times (1 + zq^3 + z^2q^6 + z^3q^9 + \cdots)$$

$$\times (1 + zq^4 + z^2q^8 + z^3q^{12} + \cdots)$$

$$\cdots$$

$$= \sum_{\lambda} z^{\ell(\lambda)} q^{|\lambda|}$$

The generating function 2

A similar argument is vald to prove the following theorem:

Theorem

For $|q| < 1$,

$$\sum_{\lambda} q^{|\lambda|} = \prod_{n=1}^{N} \frac{1}{1-q^n}$$

where the sum runs over all partitions λ where each part of λ is $\leq N$.

More generally,

$$\sum_{\lambda} z^{\ell(\lambda)} q^{|\lambda|} = \prod_{n=1}^{N} \frac{1}{1-zq^n}$$

where the sum runs over all partitions λ where each part of λ is $\leq N$.

Number of odd parts

Definition

Let λ be a partition of λ of some integer. Let $\mathcal{O}(\lambda)$ denote the number of odd parts of λ.

Example

If $\lambda = (86^2 51)$, then $\lambda' = (54^4 31^2)$, $\mathcal{O}(\lambda) = 2$ and $\mathcal{O}(\lambda) = 4$.

Andrews' Theorem

Theorem (G.E.Andrews)

$$\sum_{\lambda} z^{\mathcal{O}(\lambda)} y^{\mathcal{O}(\lambda')} q^{|\lambda|}$$

$$= \frac{\sum_{j=1}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^4;q^4)_N}$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N$.

$$\sum_{\lambda} z^{\mathcal{O}(\lambda)} y^{\mathcal{O}(\lambda')} q^{|\lambda|}$$

$$= \frac{\sum_{j=1}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^4;q^4)_{N+1}}$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N+1$.

(G.E.Andrews, "On a partition function of Richard Stanley", Electron. J. Combin. 11(2) (2004) #1.)

The four parameter weight

Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ, which is equal to $a^5 b^4 c^3 d^2$.

a	b	a	b	a
c	d	c	d	
a	b	a	b	
c				

Boulet's Theorem

Theorem (Boulet)

Let $q = abcd$. If $|a|, |b|, |c|, |d| < 1$, then

$$\sum_{\lambda} \omega(\lambda) = \frac{(-a;q)_\infty(-abc;q)_\infty}{(q;q)_\infty(ab;q)_\infty(ac;q)_\infty}$$

where the sum runs over all partitions λ.

(C.Boulet, "A four parameter partition identity", arXiv:math.CO/0308012, to appear in Ramanujan J.)

Generalization

Theorem

Let $q = abcd$. Then

$$\sum_{\lambda} \omega(\lambda) = \frac{(-a;q)_N}{(q;q)_N(ac;q)_N} \, {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N+1} \end{matrix}; q, -bq\right),$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N$.

$$\sum_{\lambda} \omega(\lambda)$$

$$= \frac{(-a;q)_{N+1}}{(q;q)_N (ac;q)_{N+1}} \, {}_2\phi_1 \left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N} \end{matrix} ; q, -b \right),$$

where the sum runs over all partitions λ where each part of λ is $\leq 2N+1$.

Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomial $Q_n(x) = Q_n(x;\alpha,\beta|q)$ is, by definition,

$$Q_n(x;\alpha,\beta|q)$$

$$= (\alpha u;q)_n u^{-n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1}q^{-n+1}u^{-1} \end{matrix}; q, \alpha^{-1}qu\right),$$

where $x = \frac{u+u^{-1}}{2}$ (R. Koelof and R.F.Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue Delft University of Technology, Report no. 98-17 (1998), p.80).

Al-Salam-Chihara Recurrence relation

The Al-salam polynomials satisfy the three-term recurrence relation

$$2xQ_n(x) = Q_{n+1}(x) + (\alpha + \beta)q^n Q_n(x) + (1 - q^n)(1 - \alpha\beta q^{n-1})Q_{n-1}(x),$$

with $Q_{-1}(x) = 0$, $Q_0(x) = 1$.

Associated Al-Salam-Chihara Recurrence relation

We also consider a more general recurrence relation:

$$2x\widetilde{Q}_n(x) = \widetilde{Q}_{n+1}(x) + t(\alpha+\beta)q^n\widetilde{Q}_n(x) + (1-tq^n)(1-t\alpha\beta q^{n-1})\widetilde{Q}_{n-1}(x),$$

which we call the associated Al-Salam-Chihara recurrence relation.

Solutions of AASC Recurrence relation

Let

$$\widetilde{Q}_n^{(1)}(x) = u^{-n}\,(t\alpha u;q)_n \times {}_2\phi_1\left(\begin{matrix} t^{-1}q^{-n},\beta u^{-1} \\ t^{-1}\alpha^{-1}q^{-n+1}u^{-1}\end{matrix};q,\alpha^{-1}qu\right),$$

$$\widetilde{Q}_n^{(2)}(x) = u^{n}\,\frac{(tq;q)_n(t\alpha\beta;q)_n}{(t\beta uq;q)_n} \times {}_2\phi_1\left(\begin{matrix} tq^{n+1},\alpha^{-1}qu \\ t\beta q^{n+1}u\end{matrix};q,\alpha u\right).$$

Then $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are linearly independent solutions of the AASC recurrence relation.

Associated Askey-Wilson polynomials

M.E.H. Ismail and M. Rahman “The associated Askey-Wilson polynomials”, Trans. Amer. Math. Soc. **328** (1991), 201 – 237.

Generating Function (ordinary partitions)

Let us consider

$$\Phi_N = \Phi_N(a, b, c, d; z) = \sum_{\substack{\lambda \\ \lambda_1 \leq N}} \omega(\lambda) z^{\ell(\lambda)},$$

where the sum runs over all partitions λ such that each part of λ is less than or equal to N.

Example

For example, the first few terms can be computed directly as follows:

$$\Phi_0 = 1,$$

$$\Phi_1 = \frac{1+az}{1-acz^2},$$

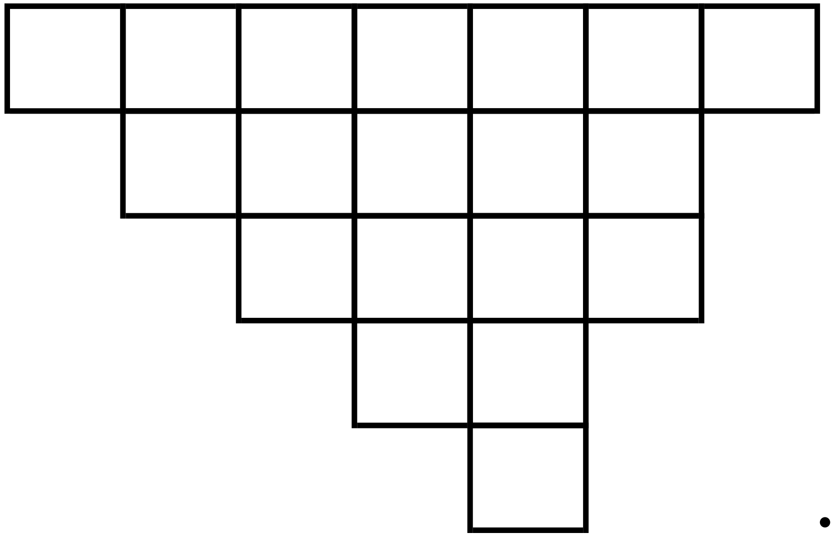
$$\Phi_2 = \frac{1+a(1+b)z+abcz^2}{(1-acz^2)(1-qz^2)},$$

$$\Phi_3 = \frac{1+a(1+b+ab)z+abc(1+a+ad)z^2+a^3bcd}{(1-z^2ac)(1-z^2q)(1-z^2acq)}$$

Strict Partitions

A partition μ all of whose parts are distinct (have multiplicity 1) is called a strict partition. For a strict partition $\mu = (\mu_1, > \mu_2 > \cdots > \mu_r)$, the shifted diagram $\mathcal{S}_\mu$ is obtained from the Young diagram of μ by moving the ith row $(i-1)$ squares to the right, for each $i > 1$.

If $\mu = (7, 5, 4, 2, 1)$ then $\mathcal{S}_\mu$ is



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Generating Function (strict partitions)

Let

$$\Psi_N = \Psi_N(a,b,c,d;z) = \sum \omega(\mu) z^{\ell(\mu)},$$

where the sum is over all strict partitions μ such that each part of μ is less than or equal to N.

Example

For example, we have

$$\begin{aligned}
\Psi_0 &= 1, \\
\Psi_1 &= 1 + az, \\
\Psi_2 &= 1 + a(1+b)z + abcz^2, \\
\Psi_3 &= 1 + a(1+b+ab)z \\
&\qquad + abc(1+a+ad)z^2 + a^3bcdz^3.
\end{aligned}$$

Example

$N = 3$

$\ell(\mu) = 0$

$$\emptyset,$$

$\ell(\mu) = 1$

$$\begin{array}{|c|}\hline a \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline a & b & a \\ \hline\end{array}\,,$$

$\ell(\mu) = 2$

$$\begin{array}{cc} \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} \\ \hline & \multicolumn{1}{|c|}{c} \\ \cline{2-2} \end{array} \qquad \begin{array}{ccc} \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{a} \\ \hline & \multicolumn{1}{|c|}{c} & \\ \cline{2-2} \end{array} \qquad \begin{array}{ccc} \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{a} \\ \hline & \multicolumn{1}{|c|}{c} & \multicolumn{1}{c|}{d} \\ \cline{2-3} \end{array}\,,$$

$\ell(\mu) = 3$

$$\begin{array}{ccc} \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{a} \\ \hline & \multicolumn{1}{|c|}{c} & \multicolumn{1}{c|}{d} \\ \cline{2-3} & & \multicolumn{1}{|c|}{a} \\ \cline{3-3} \end{array}\,.$$

Relation between Φ_N and Ψ_N

Theorem

$$\Phi_N(a,b,c,d;z) = \frac{\Psi_N(a,b,c,d;z)}{(z^2q;q)_{\lfloor N/2 \rfloor}(z^2ac;q)_{\lceil N/2 \rceil}}.$$

Proof

The idea is the bijection in Boulet's paper.

Recurrence equation

Theorem

Set $q = abcd$ and put $X_N = \Psi_{2N}$ and $Y_N = \Psi_{2N+1}$. Then X_N and Y_N satisfy

$$\begin{aligned} X_{N+1} &= \left\{1 + ab + a(1+bc)z^2q^N\right\} X_N \\ &\qquad - ab(1-z^2q^N)(1-acz^2q^{N-1})X_{N-1}, \\ Y_{N+1} &= \left\{1 + ab + abc(1+ad)z^2q^N\right\} Y_N \\ &\qquad - ab(1-z^2q^N)(1-acz^2q^N)Y_{N-1}, \end{aligned}$$

where $X_0 = 1$, $Y_0 = 1 + az$, $X_1 = 1 + a(1+b)z + abcz^2$ and

$$Y_1 = 1 + a(1+b+ab)z + abc(1+a+ad)z^2 + a^3bcdz^3.$$

Reduction to AASC

Corollary

Especially, if we put $X'_N = (ab)^{-\frac{N}{2}} X_N$ and $Y'_N = (ab)^{-\frac{N}{2}} Y_N$, then X'_N and Y'_N satisfy

$$\left\{(ab)^{\frac{1}{2}} + (ab)^{-\frac{1}{2}}\right\} X'_N = X'_{N+1} - a^{\frac{1}{2}} b^{-\frac{1}{2}} (1+bc) z^2 q^N X'_N + (1 - z^2 q^N)(1 - acz^2 q^{N-1}) X'_{N-1},$$

$$\left\{(ab)^{\frac{1}{2}} + (ab)^{-\frac{1}{2}}\right\} Y'_N = Y'_{N+1} - a^{\frac{1}{2}} b^{\frac{1}{2}} c(1+ad) z^2 q^N Y'_N + (1 - z^2 q^N)(1 - a^2 bc^2 d z^2 q^{N-1}) Y'_{N-1}.$$

Solution (even)

$$
\begin{aligned}
X_N = \frac{(-az^2q, -abc; q)_\infty}{(-a, -abcz^2; q)_\infty} \Bigg\{ & (s_0^X X_1 - s_1^X X_0) \\
& \times (-abcz^2; q)_N \, {}_2\phi_1 \left(\begin{matrix} q^{-N}z^{-2}, -b^{-1} \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix} ; q, -c^{-1}q \right) \\
& + (r_1^X X_0 - r_0^X X_1) \\
& \times (ab)^N \frac{(qz^2, acz^2; q)_N}{(-aqz^2; q)_N} \, {}_2\phi_1 \left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -aq^{N+1}z^2 \end{matrix} ; q, -abc \right) \Bigg\},
\end{aligned}
$$

where

$$r_0^X = {}_2\phi_1\left(\begin{matrix} z^{-2}, -b^{-1} \\ -(abc)^{-1}z^{-2}q \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^X = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -az^2q \end{matrix}; q, -abc\right),$$

$$r_1^X = (1+abcz^2)\,{}_2\phi_1\left(\begin{matrix} z^{-2}q^{-1}, -b^{-1} \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_1^X = \frac{ab(1-z^2q)(1-acz^2)}{1+az^2q}\,{}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -az^2q^2 \end{matrix}; q, -abc\right).$$

Solution (odd)

$$
\begin{aligned}
Y_N = {} & \frac{(-a^2bcdz^2q, -abc; q)_\infty}{(-a^2bcd, -abcz^2; q)_\infty} \Bigg\{ (s_0^Y Y_1 - s_1^Y Y_0) \\
& \times (-abcz^2; q)_N \, {}_2\phi_1 \left(\begin{matrix} q^{-N} z^{-2}, -acd \\ -(abc)^{-1} q^{-N+1} z^{-2} \end{matrix} ; q, -c^{-1} q \right) \\
& + (r_1^Y Y_0 - r_0^Y Y_1) \\
& \times (ab)^N \frac{(qz^2, a^2bc^2dz^2; q)_N}{(-a^2bcdqz^2; q)_N} \, {}_2\phi_1 \left(\begin{matrix} q^{N+1} z^2, -c^{-1} q \\ -a^2bcdq^{N+1} z^2 \end{matrix} ; q, -abc \right) \Bigg\},
\end{aligned}
$$

where

$$r_0^Y = {}_2\phi_1\left(\begin{matrix} z^{-2}, -acd \\ (-abc)^{-1}qz^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$r_1^Y = (1+abcz^2)\,{}_2\phi_1\left(\begin{matrix} q^{-1}z^{-2}, -ac \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^Y = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -a^2bcdz^2q \end{matrix}; q, -abc\right),$$

$$s_1^Y = \frac{ab(1-z^2q)(1-a^2bc^2dz^2)}{1+a^2bcdz^2q}\,{}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -a^2bcdz^2q^2 \end{matrix}; q, -abc\right).$$

Limit

Set $q = abcd$. Let s_i^X, s_i^Y, X_i, Y_i $(i = 0, 1)$ be as in the above theorem. Then we have

$$\sum_{\mu} \omega(\mu) z^{|\mu|} = \frac{(-abc, -az^2q; q)_\infty}{(ab; q)_\infty} (s_0^X X_1 - s_1^X X_0)$$
$$= \frac{(-abc, -a^2bcdz^2q; q)_\infty}{(ab; q)_\infty} (s_0^Y Y_1 - s_1^Y Y_0),$$

where the sum runs over all strict partitions μ.

The ring of symmetric functions

The ring Λ of symmetric functions in countably many variables x_1, x_2, $\ldots$ is defined by the inverse limit. (See Macdonald's book "Symmetric functions and Hall polynomials, 2nd Edition", Oxford University Press, I, 2.)

Here we use the convention that $f(x)$ stands for a symmetric function in countably many variables $x = (x_1, x_2, \ldots)$, whereas $f(X)$ stands for a symmetric function in finitely many variables $X = (x_1, \ldots, x_n)$.

The Schur functions

For $X = (x_1, \ldots, x_n)$ and a partition λ such that $\ell(\lambda) \leq n$, let

$$s_\lambda(X) = \frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}}.$$

$s_\lambda(X)$ is called the Schur function corresponding to λ.

Let $\delta = (n-1, n-2, \ldots, 1, 0)$.

Example

When $\lambda = (2, 2)$ and $X = (x_1, x_2, x_3, x_4)$,

$$s_\lambda(x) = \frac{1}{\Delta(x)} \det \begin{pmatrix} x_1^5 & x_1^4 & x_1 & 1 \\ x_2^5 & x_2^4 & x_2 & 1 \\ x_3^5 & x_3^4 & x_3 & 1 \\ x_4^5 & x_4^4 & x_4 & 1 \end{pmatrix}$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$.

Note that $(2, 2, 0, 0) + (3, 2, 1, 0) = (5, 4, 1, 0)$.

Schur functions

Let $X = (x_1, \ldots, x_{2n})$ and let $T = (x_i^{j-1})_{1 \le i \le 2n, j \ge 1}$, i.e.

$$T = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \ldots \\ 1 & x_2 & x_2^2 & x_2^3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x_{2n} & x_{2n}^2 & x_{2n}^3 & \ldots \end{bmatrix}$$

Then we have

$$s_\lambda(X) = \frac{\det\left(\Delta_{I(\lambda)}(T)\right)_{1 \le i,j \le 2n}}{\det(x_i^{j-1})_{1 \le i,j \le 2n}}.$$

Example

If $n = 3$. and $\lambda = (5, 4, 4, 1, 0, 0)$, then

$$I(\lambda) = \{0, 1, 3, 7, 8, 10\},$$

and

$$s_\lambda(X) = \frac{\begin{vmatrix} 1 & x_1 & x_1^3 & x_1^7 & x_1^8 & x_1^{10} \\ 1 & x_2 & x_2^3 & x_2^7 & x_2^8 & x_2^{10} \\ 1 & x_3 & x_3^3 & x_3^7 & x_3^8 & x_3^{10} \\ 1 & x_4 & x_4^3 & x_4^7 & x_4^8 & x_4^{10} \\ 1 & x_5 & x_5^3 & x_5^7 & x_5^8 & x_5^{10} \\ 1 & x_6 & x_6^3 & x_6^7 & x_6^8 & x_6^{10} \end{vmatrix}}{\prod_{1 \le i < j \le 6} (x_j - x_i)}$$

Tableaux

Given a partition λ, A tableaux T of shape λ is a filling of the diagram with numbers whereas the numbers must strictly increase down each column and weakly from left to right along each row.

Schur functions

The Schur function $s_\lambda(x)$ is

$$s_\lambda(X) = \sum_T X^T,$$

where the sum runs over all tableaux of shape λ. Here $X^T = x_1^{\sharp 1\text{s in } T} x_2^{\sharp 2\text{s in } T} \cdots$

Example

A Tableau T of shape (5441).

1	1	1	2	2
2	2	3	4	
3	3	4	5	
5				

The weight of T is $x_1^3 x_2^4 x_3^3 x_4^2 x_5^2$.

Example

When $\lambda = (2,2)$ and $X = (x_1, x_2, x_3, x_4)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 4 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 4 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 4 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 4 & 4 \\ \hline\end{array}$$

$$\begin{aligned}
s_\lambda(X) = {} & x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2 + 2x_1x_2x_3x_4 \\
& + x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_2^2x_1x_3 + x_2^2x_1x_4 + x_2^2x_3x_4 \\
& + x_3^2x_1x_2 + x_3^2x_1x_4 + x_3^2x_2x_4 + x_4^2x_1x_2 + x_4^2x_1x_3 + x_4^2x_2x_3
\end{aligned}$$

Schur's P-functions

Let A_n denote the skew-symmetric matrix

$$\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1 \le i,j \le n}$$

and for each strict partition $\mu = (\mu_1, \dots, \mu_l)$ of length $l \le n$, let Γ_μ denote the $n \times l$ matrix $\left(x_j^{\mu_i}\right)$. Let

$$A_\mu(x_1, \dots, x_n) = \begin{pmatrix} A_n & \Gamma_\mu J_l \\ -J_l {}^t\Gamma_\mu & O_l \end{pmatrix}$$

which is a skew-symmetric matrix of $(n + l)$ rows and columns. Define $\mathrm{Pf}_\mu(x_1, \dots, x_n)$ to be $\mathrm{Pf}\, A_\mu(x_1, \dots, x_n)$ if $n + l$ is even, and to be $\mathrm{Pf}\, A_\mu(x_1, \dots, x_n, 0)$ if $n + l$ is odd.

Schur's P-functions

Schur's P-function $P_\mu(x_1, \dots, x_n)$ is defined to be

$$\frac{\mathrm{Pf}_\mu(x_1, \dots, x_n)}{\mathrm{Pf}_\emptyset(x_1, \dots, x_n)},$$

where it is well-known that

$$\mathrm{Pf}_\emptyset(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} \frac{x_i - x_j}{x_i + x_j}.$$

Example

When $X = (x_1, x_2, x_3, x_4)$ and $\mu = (4, 2, 1)$, $P_\mu(X)$ is given by

$$\mathrm{Pf} \begin{pmatrix} 0 & \frac{x_1-x_2}{x_1+x_2} & \frac{x_1-x_3}{x_1+x_3} & \frac{x_1-x_4}{x_1+x_4} & 1 & x_1 & x_1^2 & x_1^4 \\ -\frac{x_1-x_2}{x_1+x_2} & 0 & \frac{x_2-x_3}{x_2+x_3} & \frac{x_2-x_4}{x_2+x_4} & 1 & x_2 & x_2^2 & x_2^4 \\ -\frac{x_1-x_3}{x_1+x_3} & -\frac{x_2-x_3}{x_2+x_3} & 0 & \frac{x_3-x_4}{x_3+x_4} & 1 & x_3 & x_3^2 & x_3^4 \\ -\frac{x_1-x_4}{x_1+x_4} & -\frac{x_2-x_4}{x_2+x_4} & -\frac{x_3-x_4}{x_3+x_4} & 0 & 1 & x_4 & x_4^2 & x_4^4 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -x_1 & -x_2 & -x_3 & -x_4 & 0 & 0 & 0 & 0 \\ -x_1^2 & -x_2^2 & -x_3^2 & -x_4^2 & 0 & 0 & 0 & 0 \\ -x_1^4 & -x_2^4 & -x_3^4 & -x_4^4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

divided by $\prod_{i<j} \frac{x_i-x_j}{x_i+x_j}$.

Schur's Q-functions

Schur's Q-function $Q_\mu(x_1, \ldots, x_n)$ is defined to be

$$2^{\ell(\lambda)} P_\mu(x_1, \ldots, x_n).$$

Combinatorial definition of Schur's Q-functions

Let $\mathbb{P}'$ denote the ordered alphabet $\{1' < 1 < 2' < 2 < \cdots\}$. The symbol $1'$, $2'$,... are said to be marked, and we shall denote $|a|$ the unmarked version of any $a \in \mathbb{P}'$. Let μ be a strict partition. A marked shifted tableaux T of shape μ is a labeling of squares of $\mathcal{S}_\mu$ with symbols $\mathbb{P}'$ such that:

1. The labels increase (in the weak sense) along each row and down each column.
2. Each column contains at most one k, for each $k \geq 1$.
3. Each row contains at most one k', for each $k \geq 1$.

Let us define

$$x^{|T|} = \prod_k x_k^{\#k + \#k'}.$$

Schur's Q-functions

Schur's Q-function $Q_\mu(x_1, \ldots, x_n)$ is defined to be

$$\sum_T \boldsymbol{x}^{|T|}$$

summed over marked shifted tableaux of shape μ.

Example

If $\mu = (7, 5, 4, 2, 1)$ then

$1'$	1	2	2	$4'$	4	4
	2	3	3	$4'$	4	
		4	4	$6'$	7	
			5	$6'$		
				7		

is a marked shifted tableaux of shape μ.

Power Sum Symmetric Functions

Let r denote a positive integer.

$$p_r(X) = x_1^r + x_2^r + \cdots + x_n^r$$

is called the rth power sum symmetric function.

$$p_1(X) = x_1 + x_2 + \cdots + x_n$$

$$p_2(X) = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$p_3(X) = x_1^3 + x_2^3 + \cdots + x_n^3$$

An open problem by Richard Stanley

In FPSAC'03 R.P. Stanley gave the following conjecture in the open problem session:

Theorem

Let

$$z = \sum_{\lambda} \omega(\lambda) s_\lambda(x),$$

where the sum runs over all partitions λ.

Then we have

$$\log z - \sum_{n\ge 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \ge 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2$$
$$\in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

A simple version

Let

$$y = \sum_{\substack{\lambda \\ \lambda,\lambda' \text{ even}}} s_\lambda(x).$$

Here the sum runs over all partitions λ such that λ and λ' are even partitions (i.e. with all parts even).

Then we have

$$\log y - \sum_{n\geq 1} \frac{1}{4n} p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

Strategy of the proof

1. Step1. Express $\omega(\lambda)$ and z by a single Pfaffian.

   Use the minor summation formula of Pfaffians.

2. Step2. Express z by a single determinant.

   Use the homogenious version of Okada's gereralization of Schur's Pfaffian.

3. Step3. Show that

$$\log z - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

   Use Stembridge's criterion.

The goal of the proof

Put

$$w = \log z - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2$$

and use the following Stembridge's criterion to w.

Proposition (Stembridge)

Let $f(x_1, x_2, \dots)$ be a symmetric function with infinite variables. Then

$$f \in \mathbb{Q}[p_1, p_3, p_5, \dots]$$

if and only if

$$f(t, -t, x_1, x_2, \dots) = f(x_1, x_2, \dots).$$

Pfaffians

Assume we are given a $2n$ by $2n$ skew-symmetric matrix

$$A = (a_{ij})_{1 \le i,j \le 2n},$$

(i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring.

The Pfaffian of A is, by definition,

$$\mathrm{Pf}(A) = \frac{1}{n!} \sum \epsilon(\sigma_1, \sigma_2, \dots, \sigma_{2n-1}, \sigma_{2n}) a_{\sigma_1 \sigma_2} \cdots a_{\sigma_{2n-1} \sigma_{2n}}.$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-elements blocks, and where $\epsilon(\sigma_1, \sigma_2, \dots, \sigma_{2n-1}, \sigma_{2n})$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n} \end{pmatrix}.$$

Perfect matching

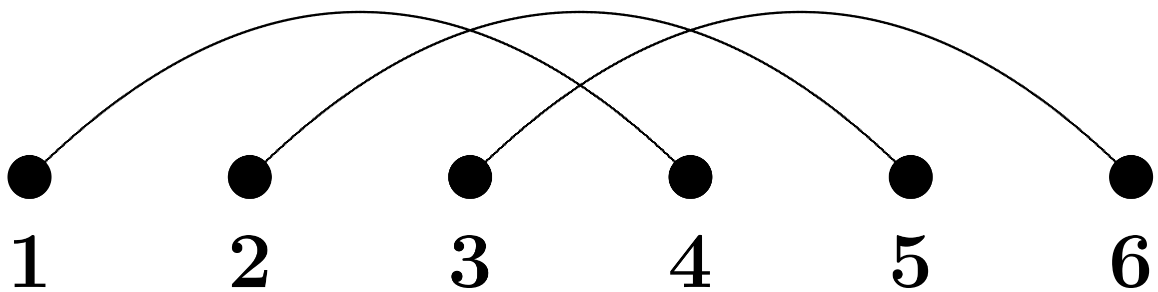

Figure 1: A perfect matching

Example

When $n = 2$,

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{21} & 0 & a_{23} & a_{24} \\ -a_{31} & -a_{32} & 0 & a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Notation

Fix a positive integer n.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition such that $\ell(\lambda) \leq n$,

then we put

$$l = (l_1, \ldots, l_n) = \lambda + \delta_n = (\lambda_1 + n - 1, \ldots, \lambda_n),$$

where $\delta_n = (n-1, n-2, \ldots, 1, 0)$,

and we write

$$I_n(\lambda) = \{l_n, l_{n-1}, \ldots, l_1\}.$$

We regard this set as a set of row/column indices.

Theorem

Define a skew-symmetric array $A = (\alpha_{ij})_{0 \le i,j}$ by

$$\alpha_{ij} = a^{\lceil (j-1)/2 \rceil} b^{\lfloor (j-1)/2 \rfloor} c^{\lceil i/2 \rceil} d^{\lfloor i/2 \rfloor}$$

for $i < j$.

Then we have

$$\operatorname{Pf}\left[A^{I_{2n}(\lambda)}_{I_{2n}(\lambda)}\right] = (abcd)^{\binom{n}{2}} \omega(\lambda).$$

Theorem

Let $\mu = (\mu_1, \dots, \mu_{2n})$ be a strict partition such that $\mu_1 > \cdots > \mu_n \geq 0$. Let $K(\mu) = \{\mu_{2n}, \dots, \mu_1\}$. Define a skew-symmetric matrix $B = (\beta_{ij})_{i,j \geq 0}$ by

$$\beta_{ij} = \begin{cases} a^{\lceil j/2 \rceil} b^{\lfloor j/2 \rfloor} z & \text{if } i = 0, \\ a^{\lceil j/2 \rceil} b^{\lfloor j/2 \rfloor} c^{\lceil i/2 \rceil} d^{\lfloor i/2 \rfloor} z^2, & \text{if } i > 0, \end{cases}$$

for $0 \leq i < j$. Then we have

$$\operatorname{Pf}\left[\Delta_{K(\mu)}^{K(\mu)}(B)\right] = \omega(\mu) z^{\ell(\mu)}.$$

Lemma

Let x_i and y_j be indeterminates, and let n is a non-negative integer.

Then

$$\operatorname{Pf}\left[x_i y_j\right]_{1\le i<j\le 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}.$$

Example

$A = (\alpha_{ij})_{0 \le i,j}$:

$$\begin{bmatrix} 0 & 1 & a & ab & a^2b & a^2b^2 & \dots \\ -1 & 0 & ac & abc & a^2bc & a^2b^2c & \dots \\ -a & -ac & 0 & abcd & a^2bcd & a^2b^2cd & \dots \\ -ab & -abc & -abcd & 0 & a^2bc^2d & a^2b^2c^2d & \dots \\ -a^2b & -a^2bc & -a^2bcd & -a^2bc^2d & 0 & a^2b^2c^2d^2 & \dots \\ -a^2b^2 & -a^2b^2c & -a^2b^2cd & -a^2b^2c^2d & -a^2b^2c^2d^2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

If $n = 3$. and $\lambda = (5, 4, 4, 1, 0, 0)$, then

$$I(\lambda) = \{0, 1, 3, 7, 8, 10\}.$$

$A_{I(\lambda)}^{I(\lambda)}$:

$$\begin{bmatrix} 0 & 1 & ab & a^3b^3 & a^4b^3 & a^5b^4 \\ -1 & 0 & abc & a^3b^3c & a^4b^3c & a^5b^4c \\ -ab & -abc & 0 & a^3b^3c^2d & a^4b^3c^2d & a^5b^4c^2d \\ -a^3b^3 & -a^3b^3c & -a^3b^3c^2d & 0 & a^4b^3c^4d^3 & a^5b^4c^4d^3 \\ -a^4b^3 & -a^4b^3c & -a^4b^3c^2d & -a^4b^3c^4d^3 & 0 & a^5b^4c^4d^4 \\ -a^5b^4 & -a^5b^4c & -a^5b^4c^2d & -a^5b^4c^4d^3 & -a^5b^4c^4d^4 & 0 \end{bmatrix}$$

$$\mathrm{Pf}\left(A_{I(\lambda)}^{I(\lambda)}\right) = a^8b^7c^6d^5 = (abcd)^3\omega(\lambda)$$

Finite Sum

We consider a weighted sum of Schur's P-functions and Q-functions

$$\xi_N(a,b,c,d;X_n) = \sum_{\substack{\mu \\ \mu_1 \le N}} \omega(\mu) P_\mu(x_1,\dots,x_n),$$

$$\eta_N(a,b,c,d;X_n) = \sum_{\substack{\mu \\ \mu_1 \le N}} \omega(\mu) Q_\mu(x_1,\dots,x_n),$$

where the sums run over all strict partitions μ such that each part of μ is less than or equal to N. More generally, we can unify these problems to finding the following sum:

$$\zeta_N(a,b,c,d;z;X_n) = \sum_{\substack{\mu \\ \mu_1 \le N}} \omega(\mu) z^{\ell(\mu)} P_\mu(x_1,\dots,x_n),$$

where the sum runs over all strict partitions μ such that each part of μ is less than or equal to N.

Infinite Sum

Further, let us put

$$\zeta(a,b,c,d;z;X_n) = \lim_{N\to\infty} \zeta_N(a,b,c,d;z;X_n)$$
$$= \sum_{\mu} \omega(\mu) z^{\ell(\mu)} P_\mu(X_n),$$

where the sum runs over all strict partitions μ. We also write

$$\xi(a,b,c,d;X_n) = \zeta(a,b,c,d;1;X_n) = \sum_{\mu} \omega(\mu) P_\mu(X_n),$$

where the sum runs over all strict partitions μ.

Theorem

Let n be a positive integer. Then

$$\zeta(a,b,c,d;z;X_n) = \begin{cases} \mathrm{Pf}\,(\gamma_{ij})_{1\le i<j\le n}/\mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even,} \\ \mathrm{Pf}\,(\gamma_{ij})_{0\le i<j\le n}/\mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$\gamma_{ij} = \frac{x_i - x_j}{x_i + x_j} + u_{ij}z + v_{ij}z^2$$

with

$$u_{ij} = \frac{a \det \begin{pmatrix} x_i + bx_i^2 & 1 - abx_i^2 \\ x_j + bx_j^2 & 1 - abx_j^2 \end{pmatrix}}{(1 - abx_i^2)(1 - abx_j^2)},$$

$$v_{ij} = \frac{abcx_ix_j \det \begin{pmatrix} x_i + ax_i^2 & 1 - a(b+d)x_i^2 - abdx_i^3 \\ x_j + ax_j^2 & 1 - a(b+d)x_j^2 - abdx_j^3 \end{pmatrix}}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2x_j^2)},$$

if $1 \le i, j \le n$, and

$$\gamma_{0j} = 1 + \frac{ax_j(1 + bx_j)}{1 - abx_j^2} z$$

if $1 \le j \le n$.

Especially, when $z = 1$, we have

$$\xi(a,b,c,d;X_n) = \begin{cases} \mathrm{Pf}\left(\widetilde{\gamma}_{ij}\right)_{1 \le i < j \le n} / \mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even,} \\ \mathrm{Pf}\left(\widetilde{\gamma}_{ij}\right)_{0 \le i < j \le n} / \mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$\widetilde{\gamma}_{ij} = \begin{cases} \frac{1+ax_j}{1-abx_j^2} & \text{if } i = 0, \\ \frac{x_i - x_j}{x_i + x_j} + \widetilde{v}_{ij} & \text{if } 1 \le i < j \le n, \end{cases} \quad with$$

$$\widetilde{v}_{ij} = \frac{a \det \begin{pmatrix} x_i + bx_i^2 & 1 - b(a+c)x_i^2 - abcx_i^3 \\ x_j + bx_j^2 & 1 - b(a+c)x_j^2 - abcx_j^3 \end{pmatrix}}{(1-abx_i^2)(1-abx_j^2)(1-abcdx_i^2x_j^2)}.$$

The key idea to prove theorems

We write the four parameter weight $\omega(\lambda)$ by a Pfaffian, and use the minor summation formula.

Notation

Let m, n and r be integers such that $r \leq m, n$. Let A be an m by n matrix. For any index sets

$$I = \{i_1, \ldots, i_r\}_< \subseteq [m],$$
$$J = \{j_1, \ldots, j_r\}_< \subseteq [n],$$

let $\Delta^I_J(A)$ denote the submatrix obtained by selecting the rows indexed by I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write $\Delta_J(A)$ for $\Delta^{[m]}_J(A)$.

Example

If $n = 6$ and $\boldsymbol{\lambda} = (5, 4, 4, 1, 0, 0)$, then

$$l = \boldsymbol{\lambda} + \delta_6 = (10, 8, 7, 3, 1, 0),$$

and

$$I_6(\boldsymbol{\lambda}) = \{0, 1, 3, 7, 8, 10\}.$$

Fact:

$$s_\lambda(X) = \frac{\det\left(\Delta_{I_n(\lambda)}(T)\right)_{1\le i,j\le n}}{\det(x_i^{j-1})_{1\le i,j\le n}}.$$

Theorem

Let n be a positive integer. Let

$$z_n = \sum_{\ell(\lambda)\le 2n} \omega(\lambda) s_\lambda(X_{2n})$$

be the sum restricted to $2n$ variables. Then we have

$$z_n = \frac{1}{\prod_{1\le i<j\le 2n}(x_i - x_j)} (abcd)^{-\binom{n}{2}} \operatorname{Pf}\left(p_{ij}\right)_{1\le i<j\le 2n},$$

where

$$p_{ij} = \frac{\det\begin{pmatrix} x_i + a x_i^2 & 1 - a(b+c)x_i - abcx_i^3 \\ x_j + a x_j^2 & 1 - a(b+c)x_j - abcx_j^3 \end{pmatrix}}{(1-abx_i^2)(1-abx_j^2)(1-abcdx_i^2x_j^2)}.$$

The idea of the proof of Theorem

- Write the Schur function $s_\lambda(X_{2n})$ by the quotient of determinants. (The denominator is the Vandermonde determinant.)
- Write the weight $\omega(\lambda)$ by the Pfaffian.
- Take the product of the Pfaffian and the determinant, and sum up over all columns.

Theorem (Minor summation formula)

Let n and N be non-negative integers such that $2n \le N$. Let $T = (t_{ij})_{1\le i\le 2n, 1\le j\le N}$ be a $2n$ by N rectangular matrix, and let $A = (a_{ij})_{1\le i,j\le N}$ be a skew-symmetric matrix of size N. Then

$$\sum_{I\in\binom{[N]}{2n}} \mathrm{Pf}\left(\Delta_I^I(A)\right)\det\left(\Delta_I(T)\right) = \mathrm{Pf}\left(TA\,{}^tT\right).$$

If we put $Q = (Q_{ij})_{1\le i,j\le 2n} = TA\,{}^tT$, then its entries are given by

$$Q_{ij} = \sum_{1\le k<l\le N} a_{kl}\det\left(\Delta_{kl}^{ij}(T)\right),$$

$(1 \le i, j \le 2n)$. Here we write $\Delta_{kl}^{ij}(T)$ for

$$\Delta_{\{kl\}}^{\{ij\}}(T) = \begin{vmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl} \end{vmatrix}.$$

Theorem (Minor summation formula 2)

Let $A = (a_{ij})_{1 \le i,j \le n}$ and $B = (b_{ij})_{1 \le i,j \le n}$ be skew symmetric matrices of size n. Then

$$\sum_{t=0}^{\lfloor n/2 \rfloor} z^t \sum_{I \in \binom{[n]}{2t}} \gamma^{|I|} \mathrm{Pf}\left(\Delta_I^I(A)\right) \mathrm{Pf}\left(\Delta_I^I(B)\right) = \mathrm{Pf} \begin{bmatrix} J_n \, {}^t A J_n & J_n \\ -J_n & C \end{bmatrix},$$

where $|I| = \sum_{i \in I} i$ and $C = (C_{ij})_{1 \le i,j \le n}$ is given by

$$C_{ij} = \gamma^{i+j} b_{ij} z.$$

Theorem (Minor summation formula 2')

Let n and N be nonnegative integers. Let $A = (a_{ij})$ and $B = (b_{ij})$ be skew symmetric matrices of size $(n+N)$. We divide the set of row/column indices into two subsets, i.e. the first n indices $I_0 = [n]$ and the last N indices $I_1 = [n+1, n+N]$. Then

$$\sum_{\substack{t \geq 0 \\ n+t \text{ even}}} z^{(n+t)/2} \sum_{I \in \binom{I_1}{t}} \gamma^{|I_0 \uplus I|} \mathrm{Pf}\left(\Delta^{I_0 \uplus I}_{I_0 \uplus I}(A)\right) \mathrm{Pf}\left(\Delta^{I_0 \uplus I}_{I_0 \uplus I}(B)\right)$$

$$= \mathrm{Pf}\begin{pmatrix} J_{n+N}\,{}^t A J_{n+N} & K_{n,N} \\ -{}^t K_{n,N} & C \end{pmatrix},$$

where $C = (C_{ij})_{1 \leq i,j, \leq n+N}$ is given by $C_{ij} = \gamma^{i+j} b_{ij} z$ and $K_{n,N} = J_{n+N} \widetilde{E}_{n,N}$ with

$$\widetilde{E}_{n,N} = \begin{pmatrix} O_n & O_{n,N} \\ O_{N,n} & E_N \end{pmatrix}.$$

The sum of $\omega(\mu)$

Let S_n denote the $n \times n$ skew-symmetric matrix whose (i, j)th entry is 1 for $0 \leq i < j \leq n$.

Theorem

Let N be a nonnegative integer.

$$\Psi_N(a, b, c, d; z) = \mathrm{Pf} \begin{bmatrix} S_{N+1} & J_{N+1} \\ -J_{N+1} & B \end{bmatrix},$$

where $B = (\beta_{ij})_{0 \leq i < j \leq N}$ is the $N \times N$ skew-symmetric matrix whose (i, j)th entry β_{ij} is defined above.

Example

For example, if $N = 3$, then the Pfaffian in the right-hand side looks like

$$\mathrm{Pf}\left[\begin{array}{cccc|cccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & az & abz & a^2bz \\
0 & 0 & -1 & 0 & -az & 0 & abcz^2 & a^2bcz^2 \\
0 & -1 & 0 & 0 & -abz & -abcz^2 & 0 & a^2bcdz^2 \\
-1 & 0 & 0 & 0 & -a^2bz & -a^2bcz^2 & -a^2bcdz^2 & 0
\end{array}\right],$$

and this is equal to $1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3$.

Recurrence Equation

Theorem

Let $\Psi_N = \Psi_N(a, b, c, d; z)$ be as above. Then we have

$$\Psi_{2N} = (1 + b)\Psi_{2N-1} + (a^N b^N c^N d^{N-1} z^2 - b)\Psi_{2N-2},$$

$$\Psi_{2N+1} = (1 + a)\Psi_{2N} + (a^{N+1} b^N c^N d^N z^2 - a)\Psi_{2N-1},$$

for any positive integer N.

Recurrence equation

Theorem

Set $q = abcd$ and put $X_N = \Psi_{2N}$ and $Y_N = \Psi_{2N+1}$. Then X_N and Y_N satisfy

$$\begin{aligned} X_{N+1} &= \left\{1 + ab + a(1+bc)z^2q^N\right\} X_N \\ &\qquad - ab(1 - z^2q^N)(1 - acz^2q^{N-1})X_{N-1}, \\ Y_{N+1} &= \left\{1 + ab + abc(1+ad)z^2q^N\right\} Y_N \\ &\qquad - ab(1 - z^2q^N)(1 - acz^2q^N)Y_{N-1}, \end{aligned}$$

where $X_0 = 1$, $Y_0 = 1 + az$, $X_1 = 1 + a(1+b)z + abcz^2$ and

$$Y_1 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3.$$

Reduction to AASC

Corollary

Especially, if we put $X'_N = (ab)^{-\frac{N}{2}} X_N$ and $Y'_N = (ab)^{-\frac{N}{2}} Y_N$, then X'_N and Y'_N satisfy

$$\left\{(ab)^{\frac{1}{2}} + (ab)^{-\frac{1}{2}}\right\} X'_N = X'_{N+1} - a^{\frac{1}{2}} b^{-\frac{1}{2}} (1+bc) z^2 q^N X'_N + (1-z^2q^N)(1-acz^2q^{N-1}) X'_{N-1},$$

$$\left\{(ab)^{\frac{1}{2}} + (ab)^{-\frac{1}{2}}\right\} Y'_N = Y'_{N+1} - a^{\frac{1}{2}} b^{\frac{1}{2}} c(1+ad) z^2 q^N Y'_N + (1-z^2q^N)(1-a^2bc^2dz^2q^{N-1}) Y'_{N-1}.$$

Assiciated Al-Salam-Chihara Recurrence equation

$$2x\widetilde{Q}_n(x) = \widetilde{Q}_{n+1}(x) + (\alpha + \beta)tq^n\widetilde{Q}_n(x) + (1 - tq^n)(1 - t\alpha\beta q^{n-1})\widetilde{Q}_{n-1}(x).$$

Two linearly independent solutions

$$\widetilde{Q}_n^{(1)}(x) = u^{-n}\,(t\alpha u;q)_n \times {}_2\phi_1\left(\begin{matrix} t^{-1}q^{-n}, \beta u^{-1} \\ t^{-1}\alpha^{-1}q^{-n+1}u^{-1} \end{matrix}; q, \alpha^{-1}qu\right),$$

$$\widetilde{Q}_n^{(2)}(x) = u^{n}\,\frac{(tq;q)_n(t\alpha\beta;q)_n}{(t\beta uq;q)_n} \times {}_2\phi_1\left(\begin{matrix} tq^{n+1}, \alpha^{-1}qu \\ t\beta q^{n+1}u \end{matrix}; q, \alpha u\right),$$

where $x = \frac{u+u^{-1}}{2}$.

Ismail and Rahman have presented two linearly independent solutions of the associated Askey-Wilson recurrence equation.

Casorati determinant

Let

$$W_n = \widetilde{Q}_n^{(1)}(x)\widetilde{Q}_{n-1}^{(2)}(x) - \widetilde{Q}_{n-1}^{(1)}(x)\widetilde{Q}_n^{(2)}(x)$$

denote the Casorati determinant of the AASC equation.

Since $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ both satisfy the recurrence equation, it is easy to see that W_n satisfies the recurrence equation

$$W_{n+1} = (1 - tq^n)(1 - t\alpha\beta q^{n-1})W_n.$$

$$W_1 = \frac{\lim_{n\to\infty} W_{n+1}}{(tq, t\alpha\beta; q)_\infty} = \frac{u^{-1}(t\alpha u, \beta u; q)_\infty}{(\alpha u, t\beta uq; q)_\infty}.$$

Initial condition problem

we need to find a polynomial solution of the AASC recurrence equation which satisfies a given initial condition, say $\widetilde{Q}_0(x) = \widetilde{Q}_0$ and $\widetilde{Q}_1(x) = \widetilde{Q}_1$. Since $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are linearly independent solutions of AASC recurrence equation, this $\widetilde{Q}_n(x)$ can be written as a linear combination of these functions, say

$$\widetilde{Q}_n(x) = C_1\,\widetilde{Q}_n^{(1)}(x) + C_2\,\widetilde{Q}_n^{(2)}(x).$$

Solution

If we substitute the initial condition $\widetilde{Q}_0(x) = \widetilde{Q}_0$ and $\widetilde{Q}_1(x) = \widetilde{Q}_1$ into this equation and solve the linear equation, then we obtain

$$C_1 = \frac{1}{W_1}\left\{\widetilde{Q}_1\widetilde{Q}_0^{(2)}(x) - \widetilde{Q}_0\widetilde{Q}_1^{(2)}(x)\right\},$$

$$C_2 = \frac{1}{W_1}\left\{\widetilde{Q}_0\widetilde{Q}_1^{(1)}(x) - \widetilde{Q}_1\widetilde{Q}_0^{(1)}(x)\right\}.$$

Solution (even)

$$\begin{aligned}
X_N = {} & \frac{(-az^2q, -abc; q)_\infty}{(-a, -abcz^2; q)_\infty} \Bigg\{ (s_0^X X_1 - s_1^X X_0) \\
& \times (-abcz^2; q)_N \; {}_2\phi_1 \left(\begin{matrix} q^{-N}z^{-2}, -b^{-1} \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix} ; q, -c^{-1}q \right) \\
& + (r_1^X X_0 - r_0^X X_1) \\
& \times (ab)^N \frac{(qz^2, acz^2; q)_N}{(-aqz^2; q)_N} \; {}_2\phi_1 \left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -aq^{N+1}z^2 \end{matrix} ; q, -abc \right) \Bigg\},
\end{aligned}$$

where

$$r_0^X = {}_2\phi_1\left(\begin{matrix} z^{-2}, -b^{-1} \\ -(abc)^{-1}z^{-2}q \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^X = {}_2\phi_1\left(\begin{matrix} z^{2}q, -c^{-1}q \\ -az^{2}q \end{matrix}; q, -abc\right),$$

$$r_1^X = (1+abcz^2)\,{}_2\phi_1\left(\begin{matrix} z^{-2}q^{-1}, -b^{-1} \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_1^X = \frac{ab(1-z^2q)(1-acz^2)}{1+az^2q}\,{}_2\phi_1\left(\begin{matrix} z^{2}q^{2}, -c^{-1}q \\ -az^{2}q^{2} \end{matrix}; q, -abc\right).$$

Solution (odd)

$$
\begin{aligned}
Y_N = \frac{(-a^2bcdz^2q, -abc; q)_\infty}{(-a^2bcd, -abcz^2; q)_\infty} \Bigg\{ & (s_0^Y Y_1 - s_1^Y Y_0) \\
& \times (-abcz^2; q)_N \, {}_2\phi_1 \left(\begin{matrix} q^{-N}z^{-2}, -acd \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix} ; q, -c^{-1}q \right) \\
& + (r_1^Y Y_0 - r_0^Y Y_1) \\
& \times (ab)^N \frac{(qz^2, a^2bc^2dz^2; q)_N}{(-a^2bcdqz^2; q)_N} \, {}_2\phi_1 \left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -a^2bcdq^{N+1}z^2 \end{matrix} ; q, -abc \right) \Bigg\},
\end{aligned}
$$

where

$$r_0^Y = {}_2\phi_1\left(\begin{matrix} z^{-2}, -acd \\ (-abc)^{-1}qz^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$r_1^Y = (1 + abcz^2)\, {}_2\phi_1\left(\begin{matrix} q^{-1}z^{-2}, -ac \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^Y = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -a^2bcdz^2q \end{matrix}; q, -abc\right),$$

$$s_1^Y = \frac{ab(1 - z^2q)(1 - a^2bc^2dz^2)}{1 + a^2bcdz^2q}\, {}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -a^2bcdz^2q^2 \end{matrix}; q, -abc\right).$$

Limit

Set $q = abcd$. Let s_i^X, s_i^Y, X_i, Y_i $(i = 0, 1)$ be as in the above theorem. Then we have

$$\sum_{\mu} \omega(\mu) z^{|\mu|} = \frac{(-abc, -az^2q; q)_\infty}{(ab; q)_\infty} (s_0^X X_1 - s_1^X X_0)$$

$$= \frac{(-abc, -a^2bcdz^2q; q)_\infty}{(ab; q)_\infty} (s_0^Y Y_1 - s_1^Y Y_0),$$

where the sum runs over all strict partitions μ.

Cauchy's determinant

$$\det\left[\frac{1}{x_i+y_j}\right]_{1\le i,j\le n}=\frac{\Delta_n(X)\Delta_n(Y)}{\prod_{1\le i,j\le n}(x_i+y_j)}.$$

Schur's Pfaffian

$$\operatorname{Pf}\left[\frac{x_i-x_j}{x_i+x_j}\right]_{1\le i,j\le 2n}=\prod_{1\le i<j\le 2n}\frac{x_i-x_j}{x_i+x_j}.$$

(I. Schur, "Über die Darstellung der symmetrischen und der alternirenden Gruppe durch gebrochene lineare Substitutionen", J. Reine Angew. Math. **139** (1911), 155–250.)

Here $\Delta_n(X)=\prod_{1\le i<j\le n}(x_i-x_j)$.

A generalization

M. Ishikawa, S. Okada, H. Tagawa and J. Zeng “Generalizations of Cauchy’s determinant and Schur’s Pfaffian”, `arXiv:math.CO/0411280`.

We gathered more generalizations of Cauchy’s determinant and Schur’s Pfaffian and their applications.

Theorem (The Desnanot–Jacobi formulae)

(1) If A is a square matrix, then we have

$$\det A_1^1 \cdot \det A_2^2 - \det A_2^1 \cdot \det A_1^2 = \det A \cdot \det A_{1,2}^{1,2}.$$

(2) If A is a skew-symmetric matrix, then we have

$$\mathrm{Pf}\, A_{1,2}^{1,2} \cdot \mathrm{Pf}\, A_{3,4}^{3,4} - \mathrm{Pf}\, A_{1,3}^{1,3} \cdot \mathrm{Pf}\, A_{2,4}^{2,4} + \mathrm{Pf}\, A_{1,4}^{1,4} \cdot \mathrm{Pf}\, A_{2,3}^{2,3} = \mathrm{Pf}\, A \cdot \mathrm{Pf}\, A_{1,2,3,4}^{1,2,3,4}$$

Thank you!