Euler-Mahonian Statistics of Ordered Partitions

Transfer matrix method and determinant evaluation

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\(^a\)joint work with Anisse Kasraoui and Jiang Zeng
An ordered partition of a set $S$ into $k$ blocks is a sequence $B_1 - B_2 - \cdots - B_k$ such that:

$\spadesuit B_i \neq \emptyset, \quad 1 \leq i \leq k$;
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Ordered Partitions
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Set $[n] := \{1, \ldots, n\}$.

$$\pi = \{2, 9\} - \{3\} - \{1, 4, 8\} - \{5, 6\} - \{7\}$$

is an ordered partition of $[9]$ with 5 blocks.
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Definition

$\mathcal{OP}_{n}^{k} := \{\text{ordered partitions of } \lfloor n \rfloor \text{ with } k \text{ blocks}\}$. 

The Stirling number $S(n, k)$ of the second kind satisfy:

$$S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k).$$

The Stirling number $S(n, k)$ of the 2nd kind counts the number of (unordered) partitions of $[n]$ into $k$ blocks.

$$\text{cardinal}(\mathcal{OP}_n^k) = k! \cdot S(n, k).$$
q-Stirling numbers

q-integers and q-factorials

\[ [n]_q = 1 + q + q^2 + \cdots + q^{n-1}, \]
$q$-Stirling numbers

$q$-integers and $q$-factorials

\[ [n]_q = 1 + q + q^2 + \cdots + q^{n-1}, \]
\[ [n]_q! = [n]_q[n - 1]_q \cdots [1]_q. \]
The *q*-Stirling number $S_q(n, k)$ of the second kind satisfy:

$$S_q(n, k) = q^{k-1}S_q(n - 1, k - 1) + [k]_qS_q(n - 1, k).$$

where $S_q(n, k) = \delta_{n,k}$ if $n = 0$ or $k = 0$. (Carlitz)
The first few values of the $q$-Stirling numbers $S_q(n, k)$ read
### Table

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<table>
<thead>
<tr>
<th>$n \setminus k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$q$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$2q + 2q^2$</td>
<td>$q^3$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$3q + 5q^2 + 3q^3$</td>
<td>$3q^3 + 5q^4 + 3q^5$</td>
<td>$q^6$</td>
</tr>
</tbody>
</table>
Definition 1 (Steingrímsson) A statistic $STAT$ on ordered partitions is said Euler-Mahonian if
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Steingrímsson:

Find Euler-Mahonian statistics on ordered partitions.
Steingrímsson defines a system of statistics:

\texttt{ros, rob, rcs, rcb, lob, los, lcs, lcb, lsb, rsb, bInv, inv, cinv.}
Steingrímsson’s Conjecture

Steingrímsson defines a system of statistics:

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Conjecture 2 (Steingrímsson, 1997)

The following combinations of \( \text{SYSTEM} \)

\[ \text{mak} + \text{bInv}, \quad \text{lmak}' + \text{bInv}, \quad \text{cinvLSB}, \]
\[ \text{mak}' + \text{bInv}, \quad \text{lmak} + \text{bInv}, \]

are Euler-mahonian on \( \mathcal{OP} \).
Given an ordered partition $\pi$ in $\mathcal{OP}_n^k$, each entry of $\pi$ is divided into four classes:

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Singleton, Opener, Closer, Transient

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Given an ordered partition $\pi$ in $\mathcal{OP}_n^k$, each entry of $\pi$ is divided into four classes:

- ★ singleton: an entry of a singleton block;
- ★ opener: the smallest entry of a non-singleton block;
- ★ closer: the largest entry of a non-singleton block;
- ★ transient: none of the above.
Given an ordered partition $\pi$ in $\mathcal{OP}_n^k$, each entry of $\pi$ is divided into four classes:

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- **transient**: none of the above.

The above sets are denoted by $\mathcal{O}(\pi)$, $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$ and $\mathcal{T}(\pi)$, respectively.
Example

We can classify each entry of an ordered partition into four categories. If
\[ \pi = \{3\ 5\} - \{2\ 4\ 6\} - \{1\} - \{7\ 8\}, \]

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Example

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★ singletons: 1.

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- **transients**: 4.
We can classify each entry of an ordered partition into four categories.
if $\pi = \{3 5\} - \{2 4 6\} - \{1\} - \{7 8\}$,

- **singletons:** $1$.
- **openers:** $2, 3, 7$.
- **closers:** $5, 6, 8$.
- **transients:** $4$.

$S(\pi) = \{1\}$, $O(\pi) = \{2, 3, 7\}$, $C(\pi) = \{5, 6, 8\}$, $T(\pi) = \{4\}$. 
Let $w_i$ denote the block index containing $i$, namely the integer $j$ such that $i \in B_j$. 
Let \( w_i \) denote the block index containing \( i \), namely the integer \( j \) such that \( i \in B_j \).

\[
\text{ros}_i(\pi) = \#\{ j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid i > j, w_j > w_i \},
\]

where \( (\mathcal{O} \cup \mathcal{S})(\pi) = \mathcal{O}(\pi) \cup \mathcal{S}(\pi) \).
Let $w_i$ denote the block index containing $i$, namely the integer $j$ such that $i \in B_j$. Then

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where $(O \cup S)(\pi) = O(\pi) \cup S(\pi)$. 

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

$$\text{ros}_i : \quad / \quad / \quad / \quad / \quad / \quad /$$
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\[
\pi = 6 \quad 8 \quad 5 \quad 1 \quad 4 \quad 7 \quad 3 \quad 9 \quad 2
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\[
\text{ros}_i : \quad 4 \quad / \quad / \quad / \quad / \quad / \quad / \quad / \quad / \quad / \quad /
\]
Let $w_i$ denote the block index containing $i$, namely the integer $j$ such that $i \in B_j$.

Let $\pi$ be a permutation defined as follows:

$$
\pi = 6 \ 8 \ -5 \ -1 \ 4 \ 7 \ -3 \ 9 \ -2
$$

The rosi statistic is given by:

$$
ros_i(\pi) = \#\{j \in (O \cup S)(\pi) \mid i > j, w_j > w_i\},
$$

where $(O \cup S)(\pi) = O(\pi) \cup S(\pi)$. 

For $\pi = 6 \ 8 \ -5 \ -1 \ 4 \ 7 \ -3 \ 9 \ -2$, we have $ros_i : 4 \ 4$.
Let $w_i$ denote the block index containing $i$, namely the integer $j$ such that $i \in B_j$. Then,
\[
\text{ros}(i) = \# \{ j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid i > j, w_j > w_i \},
\]
where $(\mathcal{O} \cup \mathcal{S})(\pi) = \mathcal{O}(\pi) \cup \mathcal{S}(\pi)$.

\[
\pi = 6 \quad 8 \quad - \quad 5 \quad - \quad 1 \quad 4 \quad 7 \quad - \quad 3 \quad 9 \quad - \quad 2
\]

\[
\text{ros}_i : 4 \quad 4 \quad / \quad 3 \quad / \quad / \quad / \quad /
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$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

$$\text{ros}_i : 4 \ 4 \ / \ 3 \ / \ 0 \ / \ /$$
Let $w_i$ denote the block index containing $i$, namely the integer $j$ such that $i \in B_j$.

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$$\text{ros}_i: \ 4 \ 4 / 3 / 0 \ 2 / \ / \ /$$
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\text{ros}_i : \ 4 4 / 3 / 0 2 2 / 1 1 / \]
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\[
\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2 \\
\mathrm{ros}_i: \quad 4 \ 4 \ / \ 3 \ / \ 0 \ 2 \ 2 \ / \ 1 \ 1 \ / \ 0
\]
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\]

where \((O \cup S)(\pi) = O(\pi) \cup S(\pi)\).

\[
\pi = 6 \quad 8 \quad - \quad 5 \quad - \quad 1 \quad 4 \quad 7 \quad - \quad 3 \quad 9 \quad - \quad 2
\]

\[
\text{ros}_3 : \begin{array}{c}
4 \quad 4 \\
/ \quad 3 \\
/ \quad 0 \quad 2 \quad 2 \\
/ \quad 1 \quad 1 \\
/ \quad 0
\end{array}
\]

\[
\text{ros}(\pi) = 17
\]
rob (right-opener-big)

\[ \text{rob}_i(\pi) = \# \{ j \in (O \cup S)(\pi) \mid i < j, w_j > w_i \}, \]

where \((O \cup S)(\pi) = O(\pi) \cup S(\pi)\).

\[ \pi = 68 - 5 - 147 - 39 - 2 \]

\[ \text{rob}_i : \quad 00 / 0 / 200 / 00 / 0 \]

\[ \text{rob}(\pi) = 2 \]
rcs (right-closer-small)

$$rcs_i(\pi) = \# \{ j \in (C \cup S)(\pi) | i > j, w_j > w_i \},$$

where $$(C \cup S)(\pi) = C(\pi) \cup S(\pi).$$

$$\pi = \begin{array}{cccccc}
6 & 8 & - & 5 & - & 1 4 7 & - & 3 9 & - & 2 \\
\end{array}$$

$$rcs_i: \begin{array}{cccccc}
2 & 3 & / & 1 & / & 0 1 1 & / & 1 1 & / & 0 \\
\end{array}$$

$$rcs(\pi) = 10$$
rcb (right-closer-big)

\[
rcb_i(\pi) = \# \{ j \in (C \cup S)(\pi) \mid i < j, w_j > w_i \},
\]

where \((C \cup S)(\pi) = C(\pi) \cup S(\pi)\).

\[
\pi = 6 8 \quad 5 \quad 1 4 7 \quad 3 9 \quad 2 \\
rcb_i : \quad 2 1 \quad / \quad 2 \quad / \quad 2 1 1 \quad / \quad 0 0 \quad / \quad 0 \\
rcb(\pi) = 9
\]
los (left-opener-small)

\[ \text{los}_i(\pi) = \# \{ j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid i > j, w_j < w_i \}, \]

where \((\mathcal{O} \cup \mathcal{S})(\pi) = \mathcal{O}(\pi) \cup \mathcal{S}(\pi)\).

\[ \pi = \begin{array}{ccccccc}
6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2 \\
\end{array} \\
\text{los}_i : \begin{array}{ccccccc}
0 & 0 & / & 0 & / & 0 & 0 & 2 & / & 1 & 3 & / & 1 \\
\end{array} \\
\text{los}(\pi) = 7 \]
Other Statistics

lob (left-opener-big)

\[
\text{lob}_i(\pi) = \#\{ j \in (\mathcal{O} \cup S)(\pi) \mid i < j, w_j < w_i \},
\]

where \((\mathcal{O} \cup S)(\pi) = \mathcal{O}(\pi) \cup S(\pi)\).

\[
\pi = \begin{array}{ccccccc}
6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2 \\
\end{array}
\]

\[
\text{lob}_i : \begin{array}{ccccccc}
0 & 0 & / & 1 & / & 2 & 2 & 0 & / & 2 & 0 & / & 3 \\
\end{array}
\]

\[
\text{lob}(\pi) = 10
\]
Other Statistics

lcs (left-closer-small)

\[ \text{lcs}_i(\pi) = \# \{ j \in (C \cup S)(\pi) | i > j, w_j < w_i \}, \]

where \((C \cup S)(\pi) = C(\pi) \cup S(\pi)\).

\[ \pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2 \]

\[ \text{lcs}_i : \ 0 \ 0 \ / \ 0 \ / \ 0 \ 0 \ 1 \ / \ 0 \ 3 \ / \ 0 \]

\[ \text{lcs}(\pi) = 4 \]
lcb (left-closer-big)

\[ lcb_i(\pi) = \# \{ j \in (C \cup S)(\pi) \mid i < j, w_j < w_i \}, \]

where \((C \cup S)(\pi) = C(\pi) \cup S(\pi)\).

\[ \pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2 \]

\[ lcb_i : \quad 0 \ 0 \ / \ 1 \ / \ 2 \ 2 \ 1 \ / \ 3 \ 0 \ / \ 4 \]

\[ lcb(\pi) = 13 \]
\( \text{rsb}_i(\pi) \) is the number of blocks \( B \) in \( \pi \) to the right of the block containing \( i \) such that the opener of \( B \) is smaller than \( i \) and the closer of \( B \) is greater than \( i \).

\[
\pi = 6 \quad 8 \quad - \quad 5 \quad - \quad 147 \quad - \quad 39 \quad - \quad 2
\]

\( \text{rsb}_i : \quad / \quad / \quad / \quad / \quad / \quad / \)
rsb (right-small-big)

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\[
\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2
\]

\[
\text{rsb}_i : 2 \ / \ / \ / \ / \ / \ /
\]
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\( rsb_i : \quad 2 \ 1 \ / \ / \ / \ / \ / \ / \ / \)
\( \text{rsb\, (right-small-big)} \)

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\[ \pi = 6 \; 8 \; - \; 5 \; - \; 1 \; 4 \; 7 \; - \; 3 \; 9 \; - \; 2 \]

\[ \text{rsb}_i \,: \; 2 \; 1 \; / \; 2 \; / \; / \; / \; / \]
rsb (right-small-big)

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\[
\pi = 68 - 5 - 147 - 39 - 2
\]

rsb\(_i\): 2 1 / 2 / 0 / / /
rsb (right-small-big)

\[ rsb_i(\pi) \text{ is the number of blocks } B \text{ in } \pi \text{ to the right of the block containing } i \text{ such that the opener of } B \text{ is smaller than } i \text{ and the closer of } B \text{ is greater than } i. \]

\[ \pi = 6 \quad 8 \quad - \quad 5 \quad - \quad 1 \quad 4 \quad 7 \quad - \quad 3 \quad 9 \quad - \quad 2 \]

\[ rsb_i : \quad 2 \quad 1 \quad / \quad 2 \quad / \quad 0 \quad 1 \quad / \quad / \quad / \]
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\[
\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2
\]

\[
\text{rsb}_i: \quad 2 \ 1 \ / \ 2 \ / \ 0 \ 1 \ 1 \ / \ /
\]
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\[
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\]

\[
\text{rsb}_i : \ 2 \ 1 \ / \ 2 \ / \ 0 \ 1 \ / \ 0 \ / \ 0
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\[
\pi = 6 8 \quad \text{–} \quad 5 \quad \text{–} \quad 1 4 7 \quad \text{–} \quad 3 9 \quad \text{–} \quad 2 \\
\text{rsb}_i : \quad 2 1 \quad / \quad 2 \quad / \quad 0 1 \quad / \quad 0 0 \quad / 
\]
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\[
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\[ \pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2 \]
\[ \text{rsb}_i : \quad 2 \ 1 \ / \ 2 \ / \ 0 \ 1 \ / \ 0 \ 0 \ / \ 0 \]
\[ \text{rsb}(\pi) = 7 \]
**lsb (left-small-big)**

$$\text{lsb}_i(\pi)$$ is the number of blocks $$B$$ in $$\pi$$ to the left of the block containing $$i$$ such that the opener of $$B$$ is smaller than $$i$$ and the closer of $$B$$ is greater than $$i$$.
lsb (left-small-big)

\( lsb_i(\pi) \) is the number of blocks \( B \) in \( \pi \) to the left of the block containing \( i \) such that the opener of \( B \) is smaller than \( i \) and the closer of \( B \) is greater than \( i \).

\[
\pi = 68 - 5 - 147 - 39 - 2
\]

\( lsb_i : 00 / 0 / 001 / 10 / 1 \)
\( \text{lsb}_i(\pi) \) is the number of blocks \( B \) in \( \pi \) to the left of the block containing \( i \) such that the opener of \( B \) is smaller than \( i \) and the closer of \( B \) is greater than \( i \).

\[
\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2
\]

\[
\text{lsb}_i : \ 0 \ 0 \ / \ 0 \ / \ 0 \ 0 \ 1 \ / \ 1 \ 0 \ / \ 1
\]

\[
\text{lsb}(\pi) = 3
\]
If \( \pi \in \mathcal{OP}_k^n \), there is a unique permutation \( \sigma \) in \( S_k \) such that

\[
\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},
\]

where \( B_1 - B_2 - \cdots - B_k \) is a partition.
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

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$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

where $B_1 - B_2 - \cdots - B_k$ is a partition.

$$\pi = \begin{array}{cccccccc}
6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2 \\
1 & 4 & 7 & - & 2 & - & 3 & 9 & - & 5 & - & 6 & 8
\end{array}$$
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

where $B_1 - B_2 - \cdots - B_k$ is a partition.

$$\pi = 6 8 - 5 - 1 4 7 - 3 9 - 2$$
$$1 4 7 - 2 - 3 9 - 5 - 6 8$$

$\text{perm}(\pi) = 54132$
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

where $B_1 - B_2 - \cdots - B_k$ is a partition.

We set

$$\text{perm}(\pi) = \sigma,$$
If \( \pi \in \mathcal{OP}_k^n \), there is a unique permutation \( \sigma \) in \( S_k \) such that

\[
\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},
\]

where \( B_1 - B_2 - \cdots - B_k \) is a partition.

We set

\[
\text{perm}(\pi) = \sigma,
\]

\[
\text{inv} \, \pi = \text{inv} \, \sigma,
\]
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

where $B_1 - B_2 - \cdots - B_k$ is a partition.

We set

$$\text{perm}(\pi) = \sigma,$$

$$\text{inv} \pi = \text{inv} \sigma,$$

$$\text{cinv} \sigma = \binom{n}{2} - \text{inv} \sigma.$$
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

where $B_1 - B_2 - \cdots - B_k$ is a partition.

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

\[
\begin{align*}
\text{perm}(\pi) &= 54132 \\
\text{inv}(\pi) &= 8
\end{align*}
\]
If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation $\sigma$ in $S_k$ such that

$$\pi = \sigma_1 - \sigma_2 - \cdots - \sigma_k,$$

where $B_1 - B_2 - \cdots - B_k$ is a partition.

$$\pi = 6\ 8 - 5 - 1\ 4\ 7 - 3\ 9 - 2$$

$\text{perm}(\pi) = 54132$

$\text{inv}(\pi) = 8$

$\text{cinv}(\pi) = \binom{5}{2} - 8 = 2$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$. 
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

A partial order on blocks:

$B_i > B_j$ if all the letters of $B_i$ are greater than those of $B_j$; in other words, if the opener of $B_i$ is greater than the closer of $B_j$. 
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

A partial order on blocks:

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$$\pi = 68 - 5 - 147 - 39 - 2$$
Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}_n^k \).

A partial order on blocks:

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\[ \pi = 6\ 8\ -\ 5\ -\ 1\ 4\ 7\ -\ 3\ 9\ -\ 2 \]

\( \{6, 8\} > \{5\} \).
Block Operations

Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}_n^k \).

A partial order on blocks:

\[ B_i > B_j \] if all the letters of \( B_i \) are greater than those of \( B_j \); in other words, if the opener of \( B_i \) is greater than the closer of \( B_j \).

\[ \pi = 6 8 - 5 - 1 4 7 - 3 9 - 2 \]

\( \{6, 8\} > \{2\} \).
Block Operations

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$\pi = 6 8 - 5 - 1 4 7 - 3 9 - 2$

$\{5\} > \{2\}$.
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

A partial order on blocks:

$B_i > B_j$ if all the letters of $B_i$ are greater than those of $B_j$; in other words, if the opener of $B_i$ is greater than the closer of $B_j$.

$$\pi = 68 - 5 - 147 - 39 - 2$$

$$\{3, 9\} > \{2\}.$$
Block Operations

Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}_n^k \).

**Block inversion:**

A block inversion in \( \pi \) is a pair \((i, j)\) such that \( i < j \) and \( B_i > B_j \). We denote by \( b\text{Inv}\pi \) the number of block inversions in \( \pi \). We also set \( c\text{bInv} = \binom{k}{2} - b\text{Inv} \).
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

**Block inversion:**

A block inversion in $\pi$ is a pair $(i, j)$ such that $i < j$ and $B_i > B_j$. We denote by $b\text{Inv}\,\pi$ the number of block inversions in $\pi$. We also set $c\text{bInv} = \left(\begin{array}{c}k \\ 2\end{array}\right) - b\text{Inv}$.

\[ \pi = 6\quad 8\quad -\quad 5\quad -\quad 1\quad 4\quad 7\quad -\quad 3\quad 9\quad -\quad 2 \]
Block Operations

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

**Block inversion:**

A block inversion in $\pi$ is a pair $(i, j)$ such that $i < j$ and $B_i > B_j$. We denote by $bInv(\pi)$ the number of block inversions in $\pi$. We also set $cbInv = \binom{k}{2} - bInv$.

$\pi = 6 8 - 5 - 1 4 7 - 3 9 - 2$,

$bInv(\pi) = 4$, $cbInv(\pi) = \binom{5}{2} - 4 = 6$. 
Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}_n^k \).

**Block descent:**

A block descent in \( \pi \) is a block \( B_i \) such that \( i \) and \( B_i > B_{i+1} \).
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

**Block descent:**

A block descent in $\pi$ is a block $B_i$ such that $i$ and $B_i > B_{i+1}$.

\[
\pi = 6 8 - 5 - 1 4 7 - 3 9 - 2
\]

\[
\{6 8\} > \{5\}, \{3 9\} > \{2\}.
\]
Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}_n^k \).

**Block descent:**

The block block major index, denote by \( b\text{Maj} \pi \), is *the sum of indices of block descents in* \( \pi \). We also set \( c_b\text{Maj} = \binom{k}{2} - b\text{Maj} \).
Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}_n^k \).

**Block descent:**

The block block major index, denote by \( \text{bMaj} \pi \), is the sum of indices of block descents in \( \pi \). We also set \( \text{cbMaj} = \binom{k}{2} - \text{bMaj} \).

\[
\pi = 6 \overline{8} - 5 - 1 \overline{4} 7 - 3 \overline{9} - 2
\]

\[
1 \quad 2 \quad 3 \quad 4 \quad 5
\]

\[\text{bMaj} \pi = 1 + 4 = 5, \quad \text{cbMaj} \pi = \binom{5}{2} - 5 = 5.\]
mak and lmak

Definition [Steingrímsson (Foata & Zeilberger)]

mak = ros + lcs,
mak and lmak

Definition [Steingrímsson (Foata & Zeilberger)]

\[
\begin{align*}
\text{mak} &= \text{ros} + \text{lcs}, \\
\text{lmak} &= n(k - 1) - [\text{los} + \text{rcs}],
\end{align*}
\]
mak and lmak

Definition [Steingrímsson (Foata & Zeilberger)]

\[ \text{mak} = \text{ros} + \text{lcs}, \]
\[ \text{lmak} = n(k - 1) - [\text{los} + \text{rcs}], \]
\[ \text{mak}' = \text{lob} + \text{rcb}, \]
mak and lmak

Definition [Steingrímsson (Foata & Zeilberger)]

\[ \text{mak} = \text{ros} + \text{lcs}, \]
\[ \text{lmak} = n(k - 1) - [\text{los} + \text{rcs}], \]
\[ \text{mak}' = \text{lob} + \text{rcb}, \]
\[ \text{lmak}' = n(k - 1) - [\text{lcb} + \text{rob}]. \]
mak and lmak

Definition [Steingrímsson (Foata & Zeilberger)]

\[
\begin{align*}
\text{mak} &= \text{ros} + \text{lcs}, \\
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\text{mak}' &= \text{lob} + \text{rcb}, \\
\text{lmak}' &= n(k - 1) - [\text{lcb} + \text{rob}].
\end{align*}
\]

Proposition 3 (Ksavrelol & Zeng)

\[
\begin{align*}
\text{mak} &= \text{lmak}', \quad \text{and} \quad \text{mak}' = \text{lmak}.
\end{align*}
\]
Definition
Let $\mathcal{OP}^k$ be the set of all ordered partitions with $k$ blocks.

$$cinv_{\text{LSB}} := \text{lsb} + \text{cbInv} + \binom{k}{2}$$
Definition
Let $\mathcal{OP}^k$ be the set of all ordered partitions with $k$ blocks.

\[
cinvLSB := lsb + cbInv + \binom{k}{2}
\]
\[
cmajLSB := lsb + cbMaj + \binom{k}{2}
\]
Definition

Let $\mathcal{OP}^k$ be the set of all ordered partitions with $k$ blocks.

\[
cinvLSB := \text{lsb} + \text{cbInv} + \binom{k}{2}
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\]

\[
\pi = 6 8 \quad - \quad 5 \quad - \quad 1 4 7 \quad - \quad 3 9 \quad - \quad 2
\]
Definition

Let $\mathcal{OP}^k$ be the set of all ordered partitions with $k$ blocks.

$cinv_{\text{LSB}} := \text{lsb} + \text{cbInv} + \binom{k}{2}$

$cmaj_{\text{LSB}} := \text{lsb} + \text{cbMaj} + \binom{k}{2}$

$\pi = 6 \quad 8 \quad - \quad 5 \quad - \quad 1 \quad 4 \quad 7 \quad - \quad 3 \quad 9 \quad - \quad 2$

$\text{lsb} \pi = 3$, $\text{cbInv} \pi = 6$, $\text{cbMaj} \pi = 5$. 
Definition
Let $\mathcal{OP}^k$ be the set of all ordered partitions with $k$ blocks.

\[
cinvLSB := \text{lsb} + \text{cbInv} + \binom{k}{2}
\]
\[
cmajLSB := \text{lsb} + \text{cbMaj} + \binom{k}{2}
\]

\[
\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2
\]
\[
cinvLSB \ \pi = 3 + 6 + \binom{5}{2} = 19.
\]
**cinvLSB, cmajLSB**

**Definition**

Let \( \mathcal{OP}^k \) be the set of all ordered partitions with \( k \) blocks.

\[
cinvLSB := \text{lsb} + \text{cbInv} + \binom{k}{2}
\]

\[
cmajLSB := \text{lsb} + \text{cbMaj} + \binom{k}{2}
\]

\[
\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2
\]

\[
cinvLSB \ \pi = 3 + 6 + \binom{5}{2} = 19,
\]

\[
cmajLSB \ \pi = 3 + 5 + \binom{5}{2} = 18.
\]
Consider the following generating functions of $OP^k$: 
Consider the following generating functions of $\mathcal{OP}^k$:

$$\varphi_k(a; x, y, t, u) = \sum_{\pi \in \mathcal{OP}^k} x^{(\text{mak}+\text{bInv})\pi} y^{\text{cinvLSB} \pi} t^{\text{inv} \pi} u^{\text{cinv} \pi} a^{\mid\pi\mid},$$

where $\mid\pi\mid = n$ if $\pi$ is an ordered partition of $[n]$. 
Consider the following generating functions of $\mathcal{OP}^k$:

$$
\psi_k(a; x, y, t, u) = \sum_{\pi \in \mathcal{OP}^k} x^{(l\text{mak} + b\text{Inv})\pi} y^{\text{cinvLSB}} \pi t^{\text{inv}} \pi u^{\text{cinv}} \pi a^{|\pi|},
$$

where $|\pi| = n$ if $\pi$ is an ordered partition of $[n]$. 
Main Result

Definition

\[ [n]_{p,q} = \frac{p^n - q^n}{p - q} : p, q \text{-integer} \]
**Main Result**

**Definition**

\[
[n]_{p,q} = \frac{p^n - q^n}{p-q} : p, q\text{-integer}
\]

\[
[n]_{p,q}! = [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q} : p, q\text{-factorial}
\]
Main Result

Definition

\[
[n]_{p,q} = \frac{p^n - q^n}{p-q} : p, q\text{-integer}
\]

\[
[n]_{p,q}! = [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q} : p, q\text{-factorial}
\]

\[
[k]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} : p, q\text{-binomial coefficient}
\]
One of the main results of our paper is the following theorem:
One of the main results of our paper is the following theorem:

**Theorem** We have

\[
\varphi_k(a; x, y, t, u) = \frac{a^k (xy)^\binom{k}{2} [k]_{tx,uy}!}{\prod_{i=1}^{k} (1 - a[i] x, y)}.
\]
One of the main results of our paper is the following theorem:

**Theorem** We have

$$
\varphi_k(a; x, y, t, u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx, uy}!}{\prod_{i=1}^{k} (1 - a[i]_{x,y})},
$$

$$
\psi_k(a; x, y, t, u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx, uy}!}{\prod_{i=1}^{k} (1 - a[i]_{x,y})}.
$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $OP_n^k$. 
Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( OP^k_n \).

The restriction \( B_j \cap [i] \) of a block \( B_j \) on \([i]\) is said to be \textbf{active} if \( B_j \neq [i] \) and \( B_j \cap [i] \neq \emptyset \).
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

The restriction $B_j \cap [i]$ of a block $B_j$ on $[i]$ is said to be **complete** if $B_j \subseteq [i]$. 
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.  

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the **trace** of the ordered partition $\pi$.  

---

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Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$. Then

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the **trace** of the ordered partition $\pi$.

**Example**

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$
Let \( \pi = B_1 - B_2 - \cdots - B_k \) be in \( \mathcal{OP}^k_n \).

\[
T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),
\]

where \( B_j(\leq i) = B_j \cap [i] \), while empty sets are omitted. The sequence \( (T_i(\pi))_{1 \leq i \leq n} \) is called the trace of the ordered partition \( \pi \).

\[\text{Example}\]

\[\pi = \begin{array}{ccccccc} 6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2 \\
T_1(\pi) = & 1 \end{array}\]
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Example**

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

$$T_2(\pi) = \begin{array}{c} 1 \\ \end{array} \ - \ 2$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Example**

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

$$T_3(\pi) = 1 \ - \ 3 \ - \ 2$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Example**

$$\pi = \begin{array}{cccccccc}
6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2 \\
\end{array}$$

$$T_4(\pi) = \begin{array}{cccccccc}
1 & 4 & - & 3 & - & 2 \\
\end{array}$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Example**

$$\pi = 68 - 5 - 147 - 39 - 2$$

$$T_5(\pi) = 5 - 14 - 3 - 2$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $OP_k^n$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

Example

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

$$T_6(\pi) = 6 \ - \ 5 \ - \ 1 \ 4 \ - \ 3 \ - \ 2$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$. Then

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Example**

$$\pi = 6 \ 8 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ 9 \ - \ 2$$

$$T_7(\pi) = 6 \ - \ 5 \ - \ 1 \ 4 \ 7 \ - \ 3 \ - \ 2$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.\[T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),\]
where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Example**

$\pi = 6 8 - 5 - 1 4 7 - 3 9 - 2$

$T_8(\pi) = 6 8 - 5 - 1 4 7 - 3 - 2$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the *trace* of the ordered partition $\pi$.

**Example**

$$\pi = \begin{array}{ccccccc}
6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2
\end{array}$$

$$T_9(\pi) = \begin{array}{ccccccc}
6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2
\end{array}$$
Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the trace of the ordered partition $\pi$.

**Definition**

$$x_i = \# \text{ complete blocks of } T_i(\pi) : \text{ abscissa}$$

$$y_i = \# \text{ active blocks of } T_i(\pi) : \text{ height}$$

Let us call $\{(x_i, y_i)\}_{1 \leq i \leq n}$ the form of $\pi$. 
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

- \( i \)-th trace of \( \pi \) surjection
- 1-th trace of \( \pi \)
- \( \{1, \ldots\} \)
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

- \(i\)-th trace of \(\pi\) \hspace{1cm} \text{surjection} \hspace{1cm} \text{form of } \pi

\[ 2\text{-th trace of } \pi \quad \{1, \cdots \} - \{2, \cdots \} \]

\(Euler-Mahonian\ Statistics\ of\ Ordered\ Partitions\)
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\(i\)-th trace of \(\pi\) \rightarrow \text{surjection} \rightarrow \text{form of } \pi

3-th trace of \(\pi\)
\(\{3, \cdots\} - \{1, \cdots\} - \{2, \cdots\} \)
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\[ \text{surjection} \quad \text{i-th trace of } \pi \quad \text{form of } \pi \]

4-th trace of \( \pi \)
\[ \{3, \cdots\} - \{1, 4\} - \{2, \cdots\} \]
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

- \(i\)-th trace of \(\pi\)
- 5-th trace of \(\pi\)

\{3, 5, \ldots\} - \{1, 4\} - \{2, \ldots\}
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

6-th trace of \(\pi\) \(\longrightarrow\) form of \(\pi\)

i-th trace of \(\pi\) \(\longrightarrow\) surjection

active blocks

complete blocks
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\(i\)-th trace of \(\pi\)  \hspace{1cm} \text{surjection} \hspace{1cm} \text{form of } \pi

7-th trace of \(\pi\)

\[ \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, \cdots \} \]
Path

\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\(i\)-th trace of \(\pi\)

form of \(\pi\)

\(8\)-th trace of \(\pi\)

\{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}

complete blocks

active blocks
\[
\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}
\]

Thus the following path correspond to the ordered partition \(\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \).
\[
\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}.
\]
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}. \]
\[ T_6(\pi) = \{6\} - \{3, 5, \ldots \} - \{1, 4\} - \{2, \ldots \}. \]
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}. \]

\[ T_6(\pi) = \{6\} - \{3, 5, \cdots \} - \{1, 4\} - \{2, \cdots \}. \]

Form of \( T_6(\pi) = (2, 2) \)
\( \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}. \)

\( T_6(\pi) = \{6\} - \{3, 5, \ldots\} - \{1, 4\} - \{2, \ldots\}. \)

Form of \( T_6(\pi) = (2, 2) \)

2 + 2 + 1 = 5 possibilities to open a new block or insert a singleton into \( T_6(\pi) \).
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}. \]
\[ T_6(\pi) = \{6\} - \{3, 5, \ldots\} - \{1, 4\} - \{2, \ldots\}. \]

Form of \( T_6(\pi) = (2, 2) \)

2 possibilities to close an active block or add a transient into \( T_6(\pi) \).

\[ \{6\} - \{3, 5, \ldots\} - \{1, 4\} - \{2, \ldots\} \]

↑ 1

↑ 2
Definition

A path diagram of depth $k$ and length $n$
Definition
A path diagram of depth $k$ and length $n$ is a pair $(\omega, \xi)$:

- $\omega$ is a path in $\mathbb{N}^2$ of length $n$ from $(0, 0)$ to $(k, 0)$, whose steps are North, East, South-East or Null.
Definition
A path diagram of depth $k$ and length $n$ is a pair $(\omega, \xi)$:

$$\star \xi = (\xi_i)_{1 \leq i \leq n}$$

is a sequence of integers.
Definition
A path diagram of depth \( k \) and length \( n \) is a pair \((\omega, \xi)\):

\[ \xi = (\xi_i)_{1 \leq i \leq n} \]

is a sequence of integers such that:

\[ 1 \leq \xi_i \leq q \]

if the \( i \)-th step is Null or South-East, of height \( q \),

Euler-Mahonian Statistics of Ordered Partitions – p.23/35
Definition
A path diagram of depth $k$ and length $n$ is a pair $(\omega, \xi)$:

$\star \xi = (\xi_i)_{1 \leq i \leq n}$ is a sequence of integers such that:

$\spadesuit 1 \leq \xi_i \leq q$ if the $i$-th step is Null or South-East, of height $q$,

$\spadesuit 1 \leq \xi_i \leq p + q + 1$ if the $i$-th step is North or East, of abscissa $p$ and height $q$. 
$$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$$

$i$-th trace of $\pi$

1-th trace of $\pi$

$\{1, \cdots \}$

$\xi_1 = 1$

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\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\[ i\text{-th trace of } \pi \quad \text{bijection} \quad \text{path diagram of } \pi \]

2-th trace of \( \pi \)
\[ \{1, \cdots\} - \{2, \cdots\} \]

\[ \xi_2 = 2 \]
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\[ i\text{-th trace of } \pi \quad \text{bijection} \quad \text{path diagram of } \pi \]

3-th trace of \( \pi \)
\[ \{3, \cdots \} - \{1, \cdots \} - \{2, \cdots \} \]

\[ \xi_3 = 1 \]
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

The \(i\)-th trace of \(\pi\): bijection

4-th trace of \(\pi\):
\[\{3, \cdots\} - \{1, 4\} - \{2, \cdots\}\]

\[\xi_4 = 2\]
\( \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \)

\( i \)-th trace of \( \pi \)  

\( \xi_5 = 1 \)

Euler-Mahonian Statistics of Ordered Partitions – p.24/35
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

The 6-th trace of \( \pi \) is:
\[ \xi_6 = 1 \]
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

**Path**

\[ \xi_7 = 1 \]

**i-th trace of \( \pi \) → bijection → path diagram of \( \pi \)**

7-th trace of \( \pi \)

\{6\} − \{3, 5, 7\} − \{1, 4\} − \{2, \cdots \}
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\( i \)-th trace of \( \pi \) 

path diagram of \( \pi \)

8-th trace of \( \pi \)
\[ \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

\( \xi_8 = 1 \)
\[ \pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\} \]

Thus we obtain
\[ \omega = (N, N, N, S-E, \text{Null}, E, S-E, S-E). \]
\[ \xi = (1, 2, 1, 2, 1, 1, 1, 1) \]
The digraph $D_k$

\[
\begin{align*}
\alpha^p \theta^q [p + q + 1]_{\varepsilon, \eta} & \quad \text{if } N \text{ or } E; \\
\beta^p [q]_{\gamma, \delta} & \quad \text{if } \text{Null or S-E.}
\end{align*}
\]
The digraph $D_k$

\[ n_k = 1 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2} \]

\[
\begin{cases}
\alpha^p \theta^q [p + q + 1]_{\varepsilon, \eta} \\
\beta^p [q]_{\gamma, \delta}
\end{cases}
\] if N or E;
if Null or S-E.
The digraph $D_k$

(a) if the $i$-th step of $\omega$ is North (resp. East), then $i \in O(\pi)$ (resp. $i \in S(\pi)$) and

$$(\text{lcs} + \text{rcs})_i(\pi) = p_{i-1}, \quad \text{los}_i(\pi) = \xi_i - 1,$$

$$(\text{lsb} + \text{rsb})_i(\pi) = q_{i-1}, \quad \text{ros}_i(\pi) = p_{i-1} + q_{i-1} + 1 - \xi_i;$$
The digraph $D_k$

(b) if the $i$-th step of $\omega$ is South-East (resp. Null), then $i \in C(\pi)$ (resp. $i \in T(\pi)$) and

\[
\begin{align*}
(lcs + rcs)_i(\pi) &= p_{i-1}, \\
(\text{lsb})_i(\pi) &= \xi_i - 1, \\
(lsb + rsb)_i(\pi) &= q_{i-1} - 1, \\
(\text{rsb})_i(\pi) &= q_{i-1} - \xi_i.
\end{align*}
\]
The digraph $D_k$

\[
Q_k(a; \alpha, \beta, \gamma, \delta, \varepsilon, \eta, \theta) := \sum_{\pi \in \mathcal{OP}^k} \alpha^{(lcs + rcs)(\mathcal{O} \cup \mathcal{S}) \pi} \beta^{(lcs + rcs)(\mathcal{T} \cup \mathcal{C}) \pi} \gamma^{rsb(\mathcal{T} \cup \mathcal{C}) \pi} \\
\times \delta^{\text{lsb}(\mathcal{T} \cup \mathcal{C}) \pi} \varepsilon^{\text{ros}(\mathcal{O} \cup \mathcal{S}) \pi} \eta^{\text{los}(\mathcal{O} \cup \mathcal{S}) \pi} \theta^{(\text{lsb} + \text{rsb})(\mathcal{O} \cup \mathcal{S}) \pi} a^{|\pi|}
\]

\[
= \sum_{w \in D_k : (0,0) \rightarrow (0,k)} \text{val}(w) a^{|w|}
\]
Transfer-Matrix Method

- $D = (V, E)$ a digraph.
Transfer-Matrix Method

- $D = (V, E)$ a digraph.
- $\text{val} : E \mapsto \mathbb{R}$ a valuation.
Transfer-Matrix Method

- \( D = (V, E) \) a digraph.
- \( val : E \mapsto \mathbb{R} \) a valuation.

Let \( A \) be the adjacency matrix of \( D \), i.e

\[
A_{ij} = val(v_i, v_j).
\]
Transfer-Matrix Method

- $D = (V, E)$ a digraph.
- $val : E \rightarrow \mathbb{R}$ a valuation.

Let $A$ be the adjacency matrix of $D$, i.e.

$$A_{ij} = val(v_i, v_j).$$

Example
Transfer-Matrix Method

- $D = (V, E)$ a digraph.
- $\text{val} : E \mapsto \mathbb{R}$ a valuation.

Let $A$ be the adjacency matrix of $D$, i.e

$$A_{ij} = \text{val}(v_i, v_j).$$

Example

$$A = \begin{pmatrix}
0 & 0 & s \\
st & t^2 & 0 \\
t & s^3 & 0
\end{pmatrix}$$
A walk of length $k$ is a sequence $w = v_{i_0} v_{i_1} \ldots v_{i_k}$ of points of $D$ such that $(v_{i_r}, v_{i_{r+1}}) \in E$. 
A walk of length $k$ is a sequence $w = v_{i_0} v_{i_1} \ldots v_{i_k}$ of points of $D$ such that $(v_{i_r}, v_{i_{r+1}}) \in E$.

**Theorem**

$$\sum_{w : v_{i} \rightarrow v_{j}} val(w) z^{|w|} = (-1)^{i+j} \frac{\det(I - zA; j, i)}{\det(I - zA)}.$$
A walk of length \( k \) is a sequence \( w = v_{i_0}v_{i_1} \ldots v_{i_k} \) of points of \( D \) such that \((v_{i_r}, v_{i_r+1}) \in E\).

Example

\[
A = \begin{pmatrix}
0 & 0 & s \\
st & t^2 & 0 \\
t & s^3 & 0
\end{pmatrix}
\]

\( w_0 = v_3v_2v_2v_1v_3v_1 \) walk of length \( |w_0| = 5 \) and \( \text{val}(w_0) = s^3 \times t^2 \times st \times s \times t = s^5t^4 \).
A walk of length \( k \) is a sequence \( w = v_{i_0} v_{i_1} \ldots v_{i_k} \) of points of \( D \) such that \( (v_{i_r}, v_{i_{r+1}}) \in E \).

Example

\[
\sum_{w:v_1 \to v_3} \text{val}(w)z^{|w|} = \frac{\det(I_2 - z A_2; 3, 1)}{\det(I_2 - z A_2)}
\]

\[
= \frac{z s(1 - zt^2)}{1 - zt^2 + z^3 s^5 t + z^2 ts - z^3 t^3 s}
\]
The determinant expression is given by:

\[ Q_k(a; t_1, t_2, t_3, t_4, t_5, t_6, t_7) = \sum_{w \in D_k:(0,0) \to (0,k)} \text{val}(w) a^{|w|} \]

**Transfer-matrix method**

\[ = (-1)^{1+n_k} \frac{\det(I - aA_k; n_k, 1)}{\det(I - aA_k)} \]
For instance, when $k = 2$, we have

\[
A_2 = \begin{pmatrix}
0 & 1 & 1 & | & 0 & 0 & 0 \\
0 & 1 & 1 & | & t_7 [2]_{t_5,t_6} & t_7 [2]_{t_5,t_6} & 0 \\
0 & 0 & 0 & | & 0 & t_1 [2]_{t_5,t_6} & t_1 [2]_{t_5,t_6} \\
0 & 0 & 0 & | & [2]_{t_3,t_4} & [2]_{t_3,t_4} & 0 \\
0 & 0 & 0 & | & 0 & t_2 & t_2 \\
0 & 0 & 0 & | & 0 & 0 & 0
\end{pmatrix}
\]
\[ Q_2(a; t) = -\frac{\det(I_2 - aA_2; 6, 1)}{\det(I_2 - aA_2)} \]

\[ = a^2[2]_{t_5,t_6} (at_2t_7 + t_1(1 - a[2]_{t_3,t_4})) \]

\[ = \frac{a^2[2]_{t_5,t_6} (at_2t_7 + t_1(1 - a[2]_{t_3,t_4}))}{(1 - a)(1 - a[2]_{t_3,t_4})(1 - at_2)}. \]
In order to prove Steingrímsson’s conjecture, it is sufficient to evaluate the following special cases of $Q_k(a; t)$:

\[
\begin{align*}
  f_k(a; x, y, t, u) &= Q_k(a; x, x, x, y, t, u, y), \\
  g_k(a; x, y, t, u) &= Q_k(a; 1, x, 1, xy, t, u, y).
\end{align*}
\]
The goal of our proof is the following identity:

\[ f_k(a; x, y, t, u) = \frac{a^k x^{(k)} [k]_{t, u}!}{\prod_{i=1}^{k} (1 - a[i]_{x, y})}, \]

\[ g_k(a; x, y, t, u) = \frac{a^k [k]_{t, u}!}{\prod_{i=1}^{k} (1 - ax^{k-i} [i]_{xy})}. \]
Let $A'_k$ and $A''_k$ be the matrices obtained from $A_k$ by making the substitutions. Let

$$M_k = I_k - aA'_k \quad \text{and} \quad N_k = I_k - aA''_k.$$ 

Then we derive from the above formula that

$$f_k(a; x, y, t, u) = \frac{(-1)^{1+n_k} \det(M_k; n_k, 1)}{\det M_k},$$

$$g_k(a; x, y, t, u) = \frac{(-1)^{1+n_k} \det(N_k; n_k, 1)}{\det N_k}.$$
Matrix $M_k$

Example

$k = 1$

$$M_1 = \begin{pmatrix} 1 & -a & -a \\ 0 & 1 - a & -a \\ 0 & 0 & 1 \end{pmatrix}$$
Matrix $M_k$

**Example** $k = 2$

$$M_2 = \begin{pmatrix}
1 & -a & -a & 0 & 0 & 0 \\
0 & 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\
0 & 0 & 1 & 0 & -ax(t+u) & -ax(t+u) \\
0 & 0 & 0 & 1-a(x+y) & -a(x+y) & 0 \\
0 & 0 & 0 & 0 & 1-ax & -ax \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. $$
The matrix $M_k$ is defined inductively as follows:

$$M_k = \begin{pmatrix} M_{k-1} & \overline{M}_{k-1} \\ O_{k+1,n_{k-1}} & \hat{M}_{k-1} \end{pmatrix}.$$ 

Here $\hat{M}_{k-1}$ is the $(k + 1) \times (k + 1)$ matrix

$$\hat{M}_{k-1} = (\delta_{ij} - ax^{i-1}[n + 1 - i]_{x,y}(\delta_{ij} + \delta_{i+1,j}))_{1 \leq i,j \leq k+1}.$$
The matrix $M_k$ is defined inductively as follows:

$$M_k = \left( \begin{array}{c|c} M_{k-1} & M_{k-1} \\ \hline O_{k+1,n_{k-1}} & \hat{M}_{k-1} \end{array} \right).$$

Here $\overline{M}_{k-1}$ is the $n_{k-1} \times (k + 1)$ matrix

$$\overline{M}_{k-1} = \left( \begin{array}{c} O_{n_{k-2},k+1} \\ \hline \hat{M}_{k-1} \end{array} \right).$$
The matrix $M_k$ is defined inductively as follows:

$$M_k = \begin{pmatrix} M_{k-1} & M_{k-1} \\ O_{k+1,n_k-1} & \tilde{M}_{k-1} \end{pmatrix}.$$

with the $k \times (k + 1)$ matrix

$$\tilde{M}_{k-1} = (-a x^{i-1} y^{k-i} [k]_{t,u}(\delta_{ij} + \delta_{i+1,j}))_{1 \leq i \leq k, 1 \leq j \leq k+1}.$$
The matrix $M_k$ is defined inductively as follows:

$$M_k = \begin{pmatrix}
M_{k-1} & \overline{M}_{k-1} \\
O_{k+1,n_k-1} & \widehat{M}_{k-1}
\end{pmatrix}.$$ 

**Theorem**

$$\det(M_k; n_k, 1) = (-1)^\binom{k}{2} a^k x^{\binom{k}{2}} [k]_{t,u}!$$

$$\times \prod_{m=1}^{k-1} \prod_{i=1}^{m} \left(1 - ax^i [m - i + 1]_{x,y}\right).$$
Matrix $M_k$

The matrix $M_k$ is defined inductively as follows:

$$M_k = \begin{pmatrix}
M_{k-1} & \overline{M}_{k-1} \\
O_{k+1,n_k-1} & \widehat{M}_{k-1}
\end{pmatrix}.$$

Proof

Use

$$\det \left( \begin{array}{c|c}
A & B \\
\hline C & D
\end{array} \right) = \det A \cdot \det \left( D - CA^{-1}B \right).$$
Example \( k = 2 \)

\[
N_2(\lambda, a) = \begin{pmatrix}
\lambda & -a & -a & 0 & 0 & 0 \\
0 & \lambda - a & -a & -ay[2]_{t,u} & -ay[2]_{t,u} & 0 \\
0 & 0 & \lambda & 0 & -a[2]_{t,u} & -a[2]_{t,u} \\
0 & 0 & 0 & \lambda - a(1 + xy) & -a(1 + xy) & 0 \\
0 & 0 & 0 & 0 & \lambda - ax & -ax \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]
The matrix $N_k$ is defined inductively as follows:

$$N_k(\lambda, a) = \left( \begin{array}{c|c} N_{k-1}(\lambda, a) & \overline{N}_{k-1}(\lambda, a) \\ \hline O_{k+1,n_{k-1}} & \widehat{N}_{k-1}(\lambda, a) \end{array} \right)$$

Here $\widehat{N}_{k-1}(\lambda, a)$ is the $(k + 1) \times (k + 1)$ matrix

$$\widehat{N}_{n-1}(\lambda, a) = \left( \lambda \delta_{ij} - ax^{i-1}[n + 1 - i]_{xy}(\delta_{ij} + \delta_{i+1,j}) \right)_{1 \leq i,j \leq n+1}$$
The matrix $N_k$ is defined inductively as follows:

$$N_k(\lambda, a) = \begin{pmatrix}
N_{k-1}(\lambda, a) & \overline{N}_{k-1}(\lambda, a) \\
O_{k+1, n_{k-1}} & \hat{N}_{k-1}(\lambda, a)
\end{pmatrix}$$

Here $\overline{N}_{k-1}(\lambda, a)$ is the $n_{k-1} \times (k + 1)$ matrix

$$\begin{pmatrix}
O_{n_{k-2}, k+1} \\
\hat{N}_{k-1}
\end{pmatrix}$$
The matrix $N_k$ is defined inductively as follows:

$$N_k(\lambda, a) = \begin{pmatrix} N_{k-1}(\lambda, a) & \overline{N}_{k-1}(\lambda, a) \\ O_{k+1,n_{k-1}} & \hat{N}_{k-1}(\lambda, a) \end{pmatrix}$$

with the $k \times (k + 1)$ matrix

$$\hat{N}_{k-1} = \left( -ay^{k-i}[n]_{t,u} \cdot (\delta_{ij} + \delta_{i+1,j}) \right)_{1 \leq i \leq k, 1 \leq j \leq k+1}.$$
The matrix $N_k$ is defined inductively as follows:

$$
N_k(\lambda, a) = \begin{pmatrix}
N_{k-1}(\lambda, a) & \bar{N}_{k-1}(\lambda, a) \\
O_{k+1,n_{k-1}} & \hat{N}_{k-1}(\lambda, a)
\end{pmatrix}
$$

**Proof**
Find the eigenvector of each eigenvalue.
\[ n, k \begin{array}{c} \n, \ k \end{array}_{qr} = \begin{array}{c} n \end{array}_{qr} - q^{n-k} \begin{array}{c} k \end{array}_{qr}, \]
\[ n, k_{q,r} = [n]_{qr} - q^{n-k}[k]_{qr}, \]

\[ n_{q,r} = \begin{cases} \prod_{i=0}^{k-1} [n, i]_{q,r} / [k]_{qr}! & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \]
\[ n, k_{q,r} = [n]_{qr} - q^{n-k}[k]_{qr}, \]
\[ n_{q,r} = \begin{cases} \prod_{i=0}^{k-1} [n,i]_{q,r} / [k]_{qr}! & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \]

**Example**

\[ 3, 1_{q,r} = 1 + qr + q^2r^2 - q^2 \]
\[ 3, 2_{q,r} = \frac{(1+qr+q^2r^2)(1+qr+q^2r^2-q^2)}{(1+qr)(1+qr)}. \]
Eigenvectors

Define the row vectors $X_{n}^{m,l}$ of degree $n_k$ as follows: For $1 \leq i \leq k + 1$ and $1 \leq j \leq i$, the $\left(\frac{i(i-1)}{2} + j \right)$th entry of $X_{n}^{m,l}$ is equal to

$$X_{i,j}^{m,l} = (-1)^{i+m+l} x^{-(m+l-1)(i-m-l)+\binom{j-l}{2}} y^{\binom{i-m-l}{2}}$$

$$\times \frac{[i-m-l]_{t,u}!}{[i-m-l]_{xy}!} \left[ \begin{array}{c} i-1 \\ \end{array} \right]_{t,u} \left[ \begin{array}{c} m + l - 1 \\ \end{array} \right]_{t,u} m \left[ \begin{array}{c} m + l - j \\ \end{array} \right]_{x,y}.$$

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Let $k$ be a positive integer. Let $m$ and $l$ be positive integers such that $0 \leq m \leq k - 1$ and $1 \leq l \leq k - m$. Then we have

$$X_{m,l}^k N_k(\lambda, a) = (\lambda - ax^{l-1}[m]_x y) X_{m,l}^k.$$
Conjecture

Consider the following two generating functions of ordered partitions with \( k \geq 0 \) blocks:

\[
\xi_k(a; x, y) := \sum_{\pi \in \mathcal{OP}^k} x^{(\text{mak} + b\text{Maj})\pi} y^{\text{cmaj}(\text{LSB})\pi} a^{\mid\pi\mid},
\]

\[
\eta_k(a; x, y) := \sum_{\pi \in \mathcal{OP}^k} x^{(\text{lmak} + b\text{Maj})\pi} y^{\text{cmaj}(\text{LSB})\pi} a^{\mid\pi\mid}.
\]
Conjecture

For \( k \geq 0 \), the following identities would hold:

\[
\xi_k(a; x, y) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{x,y}!}{\prod_{i=1}^{k} (1 - a[i]_{x,y})},
\]

\[
\eta_k(a; x, y) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{x,y}!}{\prod_{i=1}^{k} (1 - a[i]_{x,y})}.
\]


The End of Talk

Thank you!