Euler-Mahonian Statistics of Ordered Partitions

Transfer matrix method and determinant evaluation

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[a] joint work with Anisse Kasraoui and Jiang Zeng

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An ordered partition of a set S into k *blocks* is a sequence $B_1 - B_2 - \cdots - B_k$ such that:

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Set $[n] := \{1, \ldots, n\}$.

$$\pi = \{2,9\} - \{3\} - \{1,4,8\} - \{5,6\} - \{7\}$$

is an ordered partition of $[9]$ with 5 blocks.

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Definition

$\mathcal{OP}_n^k :=$ {ordered partitions of $[n]$ with k blocks}.

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cardinal$(\mathcal{OP}_n^k) = k!S(n,k)$.

q-Stirling numbers

q-integers and q-factorials

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- $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$,
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The q-Stirling number $S_q(n,k)$ of the second kind satisfy:

$$S_q(n,k) = q^{k-1}S_q(n-1,k-1) + [k]_q S_q(n-1,k).$$

where $S_q(n,k) = \delta_{n\,k}$ if $n = 0$ or $k = 0$. (Carlitz)

Table

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$n \setminus k$	0	1	2	3
1	1			
2	1	q		
3	1	$2q + 2q^2$	q^3	
4	1	$3q + 5q^2 + 3q^3$	$3q^3 + 5q^4 + 3q^5$	q^6

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Steingrimsson:

Find Euler-Mahonian statistics on ordered partitions.

Steingrímsson's Conjecture

Steingrímsson defines a system of statistics:

ros, rob, rcs, rcb, lob, los, lcs, lcb, lsb, rsb, bInv, inv, cinv.

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Conjecture 2 (Steingrímsson, 1997) *The following combinations of SYSTEM*

$$\mathrm{mak} + \mathrm{bInv}\,, \quad \mathrm{lmak}' + \mathrm{bInv}, \quad \mathrm{cinvLSB}, \quad \mathrm{mak}' + \mathrm{bInv}\,, \quad \mathrm{lmak} + \mathrm{bInv}\,,$$

are Euler-mahonian on $\mathcal{OP}$.

Singleton, Opener, Closer, Transient

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The above sets are denoted by $\mathcal{O}(\pi)$, $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$ and $\mathcal{T}(\pi)$, respectively.

Example

We can classify each entry of an ordered partition into four categories.
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$\mathcal{S}(\pi) = \{1\}, \mathcal{O}(\pi) = \{2, 3, 7\}, \mathcal{C}(\pi) = \{5, 6, 8\}$,
$\mathcal{T}(\pi) = \{4\}$.

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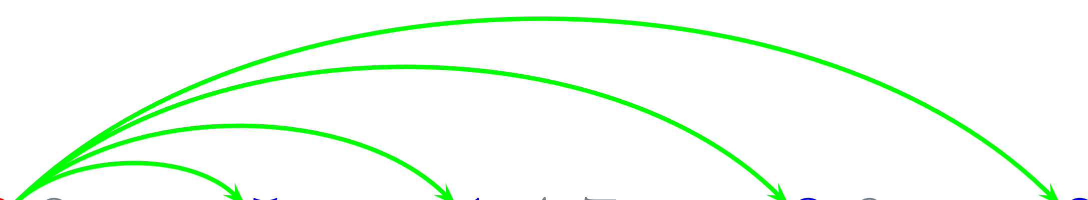

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$$\mathrm{ros}(\pi) = 17$$

Other Statistics

rob (right-opener-big)

$$\mathrm{rob}_i(\pi) = \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid i < j,\, w_j > w_i\},$$

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$$\begin{array}{rcccccccccccccc} \pi = & 6 & 8 & - & 5 & - & 1 & 4 & 7 & - & 3 & 9 & - & 2 \\ \mathrm{rob}_i : & 0 & 0 & / & 0 & / & 2 & 0 & 0 & / & 0 & 0 & / & 0 \end{array}$$

$$\mathrm{rob}(\pi) = 2$$

Other Statistics

rcs (right-closer-small)

$$\mathrm{rcs}_i(\pi) = \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \,|\, i > j,\, w_j > w_i\},$$

where $(\mathcal{C} \cup \mathcal{S})(\pi) = \mathcal{C}(\pi) \cup \mathcal{S}(\pi)$.

$$\begin{array}{rccccccccc} \pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\ \mathrm{rcs}_i : & 2\,3 & / & 1 & / & 0\,1\,1 & / & 1\,1 & / & 0 \\ \mathrm{rcs}(\pi) & = & 10 & & & & & & & \end{array}$$

Other Statistics

rcb (right-closer-big)

$$\text{rcb}_i(\pi) = \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \mid i < j,\, w_j > w_i\},$$

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Other Statistics

los (left-opener-small)

$$\mathrm{los}_i(\pi) = \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \,|\, i > j,\ w_j < w_i\},$$

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Other Statistics

lob (left-opener-big)

$$\mathrm{lob}_i(\pi) = \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid i < j,\, w_j < w_i\},$$

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lcs (left-closer-small)

$$\mathrm{lcs}_i(\pi) = \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \,|\, i > j,\, w_j < w_i\},$$

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$$\mathrm{lcs}(\pi) = 4$$

Other Statistics

lcb (left-closer-big)

$$\mathrm{lcb}_i(\pi) = \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \mid i < j,\, w_j < w_i\},$$

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$$\begin{array}{lccccccccccccc} \pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\ \mathrm{rsb}_i : & 2\,1 & / & 2 & / & 0\,1 & / & 0\,0 & / & 0 \end{array}$$

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$$\pi = 6\,8 - 5 - 1\,4\,7 - 3\,\mathbf{9} - 2$$

$$\mathrm{rsb}_i: \; 2\,1 \,/\, 2 \,/\, 0\,1 \,/\, 0\,0 \,/\, 0$$

$$\mathrm{rsb}(\pi) = 7$$

lsb (left-small-big)

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$$
\begin{array}{lccccccccc}
\pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\
\mathrm{lsb}_i : & 0\,0 & / & 0 & / & 0\,0\,1 & / & 1\,0 & / & 1 \\
\mathrm{lsb}(\pi) & = & 3 & & & & & & &
\end{array}
$$

inv, cinv

If $\pi \in \mathcal{OP}_k^n$, there is a unique permutation σ in S_k such that

$$\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)},$$

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$$\boxed{\mathrm{perm}(\pi) = 54132}$$

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$$\mathrm{perm}(\pi) = 54132$$

$$\mathrm{inv}(\pi) = 8$$

$$\mathrm{cinv}(\pi) = \binom{5}{2} - 8 = 2$$

Block Operations

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

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$$\{3, 9\} > \{2\}.$$

Block Operations

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

Block inversion:

A **block inversion** in π is a pair (i, j) such that $i < j$ and $B_i > B_j$. We denote by $\mathrm{bInv}\,\pi$ *the number of block inversions in* π. We also set $\mathrm{cbInv} = \binom{k}{2} - \mathrm{bInv}$.

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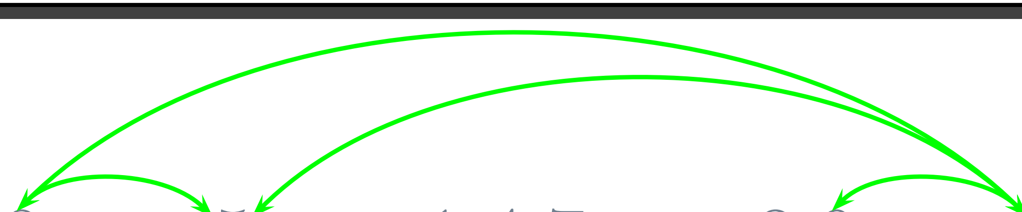

$$\pi = 6\,8 \;-\; 5 \;-\; 1\,4\,7 \;-\; 3\,9 \;-\; 2$$

$\mathrm{bInv}\ \pi = 4$, $\mathrm{cbInv}\ \pi = \binom{5}{2} - 4 = 6$.

Block Operations

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

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> A block descent in π is a block B_i such that i and $B_i > B_{i+1}$.

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$$\pi = \; 6\,8 \; - \; 5 \; - \; 1\,4\,7 \; - \; 3\,9 \; - \; 2$$

$$\{6\ 8\} > \{5\}, \{3\ 9\} > \{2\}.$$

Block Operations

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

Block descent:

The block block major index, denote by $\mathrm{bMaj}\,\pi$, is *the sum of indices of block descents in* π. We also set $\mathrm{cbMaj} = \binom{k}{2} - \mathrm{bMaj}$.

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The block block major index, denote by $\mathrm{bMaj}\,\pi$, is *the sum of indices of block descents in* π. We also set $\mathrm{cbMaj} = \binom{k}{2} - \mathrm{bMaj}$.

$$\pi = \begin{matrix} 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\ 1 & & 2 & & 3 & & 4 & & 5 \end{matrix}$$

$\mathrm{bMaj}\,\pi = 1 + 4 = 5$, $\mathrm{cbMaj}\,\pi = \binom{5}{2} - 5 = 5$.

mak and lmak

Definition [Steingrímsson (Foata $\&$ Zeilberger)]

$$\mathrm{mak} = \mathrm{ros} + \mathrm{lcs},$$

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$$\mathrm{lmak}' = n(k-1) - [\mathrm{lcb} + \mathrm{rob}].$$

Proposition 3 *(Ksavrelof & Zeng)*

$$\mathrm{mak} = \mathrm{lmak}' \quad \textit{and} \quad \mathrm{mak}' = \mathrm{lmak}\,.$$

cinvLSB, cmajLSB

Definition

Let $\mathcal{OP}^k$ be *the set of all ordered partitions with* k *blocks*.

$$\mathrm{cinvLSB} := \mathrm{lsb} + \mathrm{cbInv} + \binom{k}{2}$$

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$$\pi = \ 6\ 8\ -\ 5\ -\ 1\ 4\ 7\ -\ 3\ 9\ -\ 2$$

$\mathrm{lsb}\,\pi = 3$, $\mathrm{cbInv}\,\pi = 6$, $\mathrm{cbMaj}\,\pi = 5$.

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$$\mathrm{cinvLSB}\,\pi = 3 + 6 + \binom{5}{2} = 19,$$

$$\mathrm{cmajLSB}\,\pi = 3 + 5 + \binom{5}{2} = 18.$$

Generating Functions

Consider the following generating functions of $\mathcal{OP}^k$:

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$$\varphi_k(a; x, y, t, u) = \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{mak}+\mathrm{bInv})\pi}\, y^{\mathrm{cinvLSB}\ \pi}\, t^{\mathrm{inv}\ \pi}\, u^{\mathrm{cinv}\ \pi} a^{|\pi|},$$

where $|\pi| = n$ if π is an ordered partition of $[n]$.

Generating Functions

Consider the following generating functions of $\mathcal{OP}^k$:

$$\psi_k(a;x,y,t,u) = \sum_{\pi\in\mathcal{OP}^k} x^{(\mathrm{lmak}+\mathrm{bInv})\pi}\, y^{\mathrm{cinvLSB}\ \pi}\, t^{\mathrm{inv}\ \pi}\, u^{\mathrm{cinv}\ \pi}\, a^{|\pi|},$$

where $|\pi| = n$ if π is an ordered partition of $[n]$.

Main Result

Definition

$[n]_{p,q} = \frac{p^n - q^n}{p - q}$: p, q-integer

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$[n]_{p,q}! = [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q}$: p, q-factorial

$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$: p, q-binomial coefficient

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$$\varphi_k(a;x,y,t,u) = \frac{a^k\,(xy)^{\binom{k}{2}}\,[k]_{tx,uy}!}{\prod_{i=1}^{k}(1-a[i]_{x,y})},$$

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Theorem We have

$$\varphi_k(a;x,y,t,u) = \frac{a^k\,(xy)^{\binom{k}{2}}[k]_{tx,uy}!}{\prod_{i=1}^{k}(1-a[i]_{x,y})},$$

$$\psi_k(a;x,y,t,u) = \frac{a^k(xy)^{\binom{k}{2}}\,[k]_{tx,uy}!}{\prod_{i=1}^{k}(1-a[i]_{x,y})}.$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

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The restriction $B_j \cap [i]$ of a block B_j on $[i]$ is said to be **active** if $B_j \neq [i]$ and $B_j \cap [i] \neq \emptyset$.

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

The restriction $B_j \cap [i]$ of a block B_j on $[i]$ is said to be **complete** if $B_j \subseteq [i]$.

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the **trace** of the ordered partition π.

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where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\pi = \ 6\,8 \ - \ 5 \ - \ 1\,4\,7 \ - \ 3\,9 \ - \ 2$$

Trace

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Example

$$\pi = \quad 6\,8 \;-\; 5 \;-\; 1\,4\,7 \;-\; 3\,9 \;-\; 2$$
$$T_1(\pi) = \quad 1$$

Trace

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Example

$$\pi = \quad 6\ 8 \ - \ 5 \ - \ 1\ 4\ 7 \ - \ 3\ 9 \ - \ 2$$
$$T_2(\pi) = \quad 1 \ - \ 2$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\begin{array}{rcccccccccc} \pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\ T_3(\pi) = & & & & & 1 & - & 3 & - & 2 \end{array}$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\pi = \quad 6\,8 - 5 - 1\,4\,7 - 3\,9 - 2$$
$$T_4(\pi) = \quad 1\,4 - 3 - 2$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\pi = \quad 6\,8 \;-\; 5 \;-\; 1\,4\,7 \;-\; 3\,9 \;-\; 2$$
$$T_5(\pi) = \quad 5 \;-\; 1\,4 \;-\; 3 \;-\; 2$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1 \leq i \leq n}$ is called the **trace** of the ordered partition π.

Example

$$\pi = \quad 6\,8 - 5 - 1\,4\,7 - 3\,9 - 2$$
$$T_6(\pi) = \quad 6 - 5 - 1\,4 - 3 - 2$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\begin{array}{rccccccccc} \pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\ T_7(\pi) = & 6 & - & 5 & - & 1\,4\,7 & - & 3 & - & 2 \end{array}$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\pi = \quad 6\,8 - 5 - 1\,4\,7 - 3\,9 - 2$$
$$T_8(\pi) = 6\,8 - 5 - 1\,4\,7 - 3 - 2$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the **trace** of the ordered partition π.

Example

$$\pi = \quad 6\,8 - 5 - 1\,4\,7 - 3\,9 - 2$$
$$T_9(\pi) = 6\,8 - 5 - 1\,4\,7 - 3\,9 - 2$$

Trace

Let $\pi = B_1 - B_2 - \cdots - B_k$ be in $\mathcal{OP}_n^k$.

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \cdots - B_k(\leq i),$$

where $B_j(\leq i) = B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the trace of the ordered partition π.

Definition

$x_i = \sharp$complete blocks of $T_i(\pi)$: abscissa
$y_i = \sharp$active blocks of $T_i(\pi)$: height
Let us call $\{(x_i, y_i)\}_{1\leq i\leq n}$ the form of π.

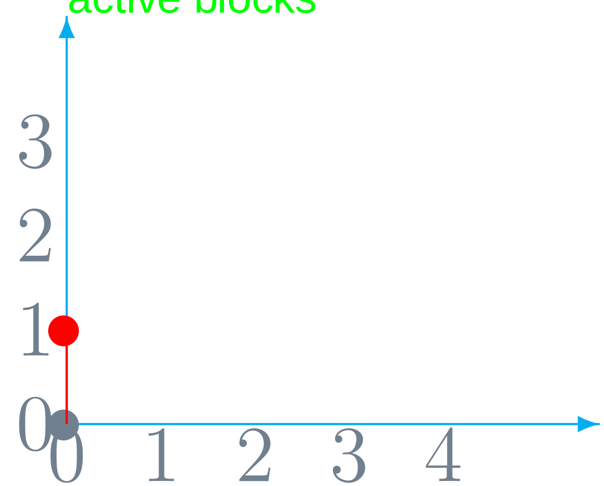
Path

$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$

i-th trace of π —surjection→ form of π

1-th trace of π

$\{1, \cdots\}$



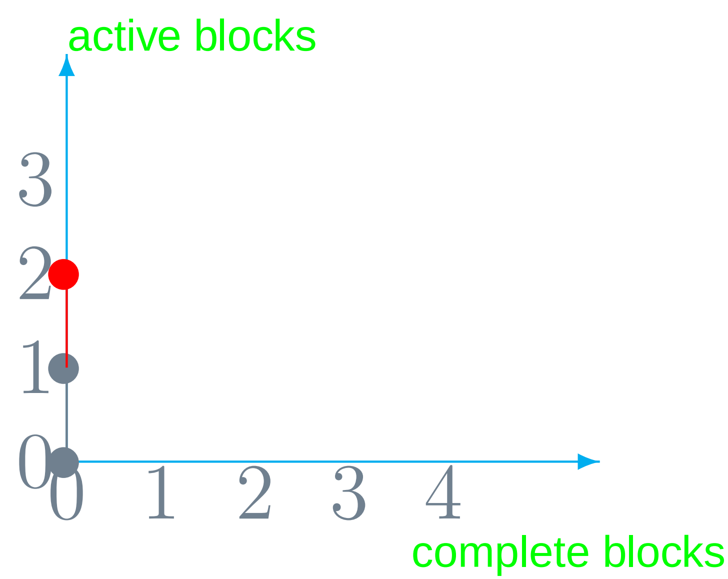
Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π $\xrightarrow{\text{surjection}}$ form of π

2-th trace of π

$$\{1, \cdots\} - \{2, \cdots\}$$



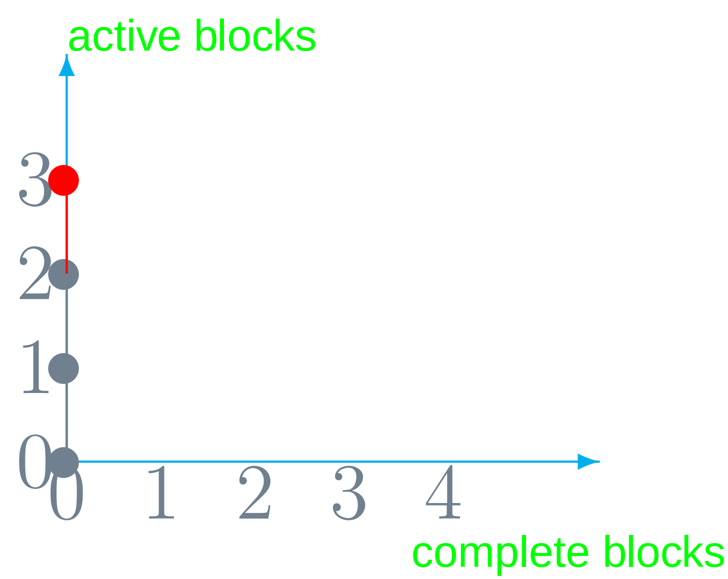
Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π $\xrightarrow{\text{surjection}}$ form of π

3-th trace of π

$$\{3,\cdots\} - \{1,\cdots\} - \{2,\cdots\}$$



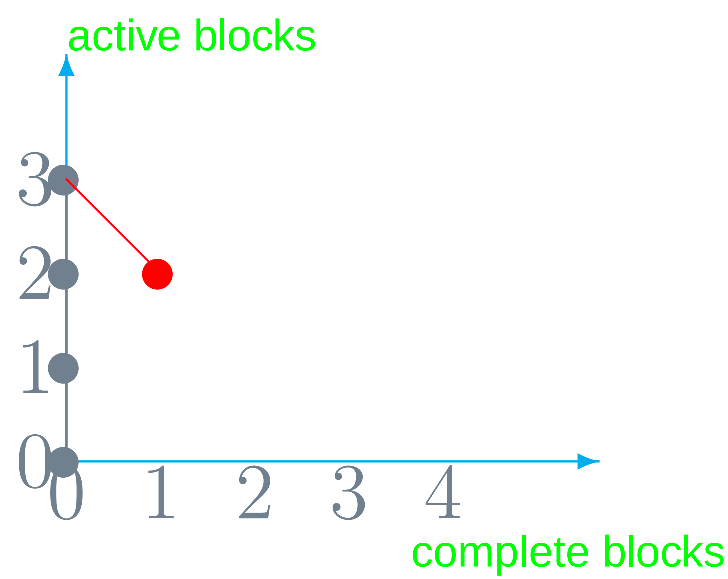
Path

$$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$$

i-th trace of π $\xrightarrow{\text{surjection}}$ form of π

4-th trace of π

$$\{3, \cdots\} - \{1, 4\} - \{2, \cdots\}$$



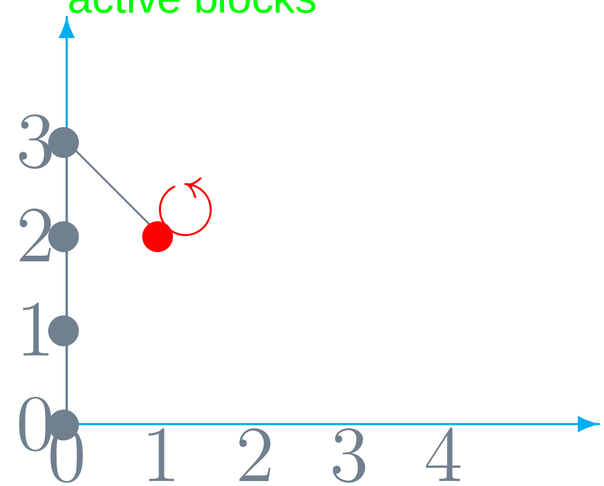
Path

$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$

i-th trace of π —surjection→ form of π

5-th trace of π

$\{3, 5, \cdots\} - \{1, 4\} - \{2, \cdots\}$



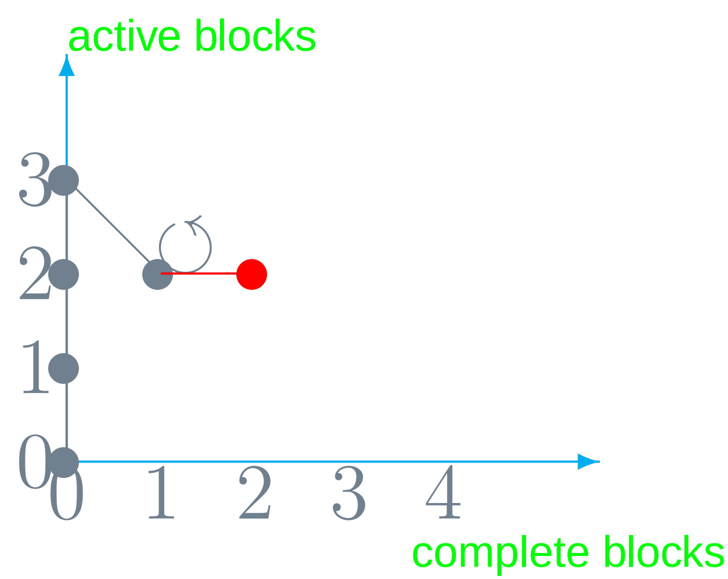
Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π $\xrightarrow{\text{surjection}}$ form of π

6-th trace of π

$$\{6\} - \{3,5,\cdots\} - \{1,4\} - \{2,\cdots\}$$



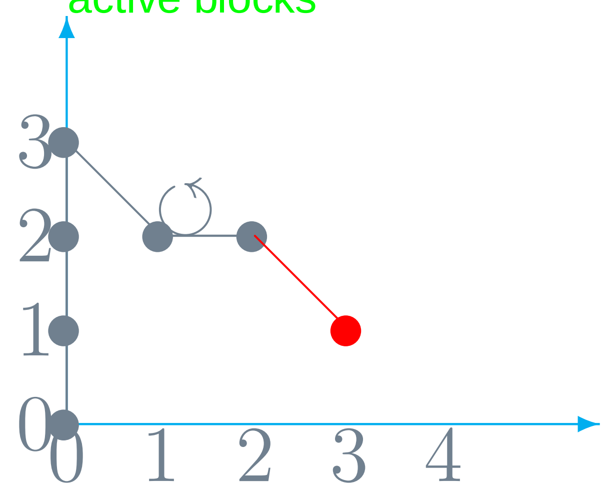
Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π —surjection→ form of π

7-th trace of π

$$\{6\} - \{3,5,7\} - \{1,4\} - \{2,\cdots\}$$



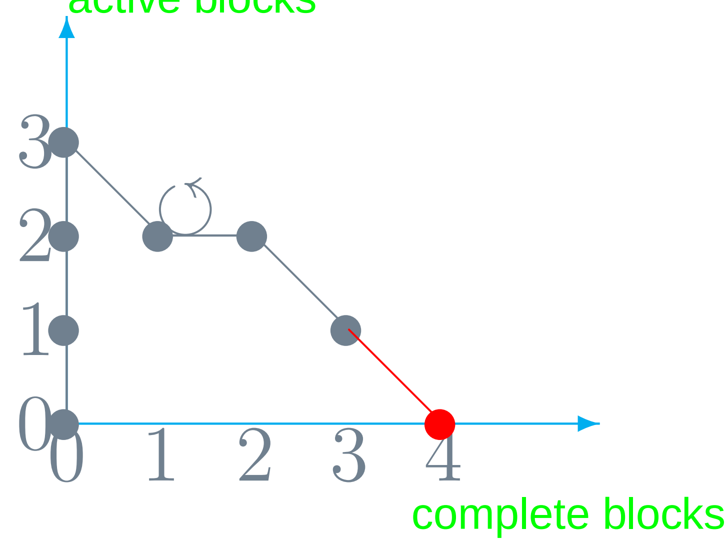
Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π $\xrightarrow{\text{surjection}}$ form of π

8-th trace of π

$\{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$

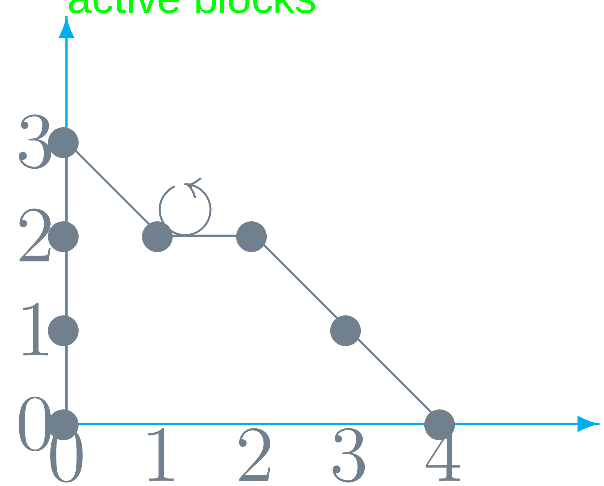


Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π $\xrightarrow{\text{surjection}}$ form of π

Thus the following path correspond to the orderd partition $\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$.



Choice

$$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}.$$

Choice

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}.$$

$$T_6(\pi) = \{6\} - \{3,5,\cdots\} - \{1,4\} - \{2,\cdots\}.$$

Choice

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}.$$

$$T_6(\pi) = \{6\} - \{3,5,\cdots\} - \{1,4\} - \{2,\cdots\}.$$

Form of $T_6(\pi) = (2,2)$

Choice

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}.$$

$$T_6(\pi) = \{6\} - \{3,5,\cdots\} - \{1,4\} - \{2,\cdots\}.$$

Form of $T_6(\pi) = (2,2)$

$2+2+1=5$ possibilities to open a new block or insert a singleton into $T_6(\pi)$.

$$\begin{array}{ccccccccc} & \{6\} & - & \{3,5,\cdots\} & - & \{1,4\} & - & \{2,\cdots\} & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Choice

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}.$$

$$T_6(\pi) = \{6\} - \{3,5,\cdots\} - \{1,4\} - \{2,\cdots\}.$$

Form of $T_6(\pi) = (2,2)$

2 possibilities to close an active block or add a transient into $T_6(\pi)$.

$$\{6\} \quad - \quad \{3,5,\cdots\} \quad - \quad \{1,4\} \quad - \quad \{2,\cdots\}$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$1 \qquad\qquad\qquad\qquad 2$$

Path Diagram

Definition

A path diagram of depth k and length n

Path Diagram

Definition

A path diagram of depth k and length n is a pair (ω, ξ):

- ⋆ ω is a path in $\mathbb{N}^2$ of length n from $(0,0)$ to $(k,0)$, whose steps are

North, East, South-East or Null.

Path Diagram

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- $\star$ $\xi = (\xi_i)_{1\leq i\leq n}$ is a sequence of integers

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A path diagram of depth k and length n is a pair (ω, ξ):

⋆ $\xi = (\xi_i)_{1 \le i \le n}$ is a sequence of integers such that:

♠ $\boxed{1 \le \xi_i \le q}$ if the i-th step is Null or South-East, of height q,

♠ $\boxed{1 \le \xi_i \le p + q + 1}$ if the i-th step is North or East, of abscissa p and height q.

Path

$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$

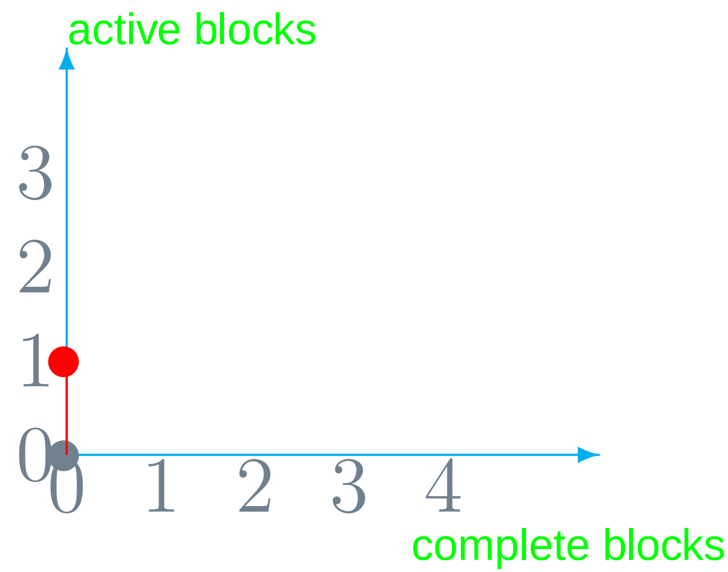
bijection

i-th trace of π ⟷ path diagram of π

1-th trace of π

$\{1, \cdots\}$

$\xi_1 = 1$



Path

$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$

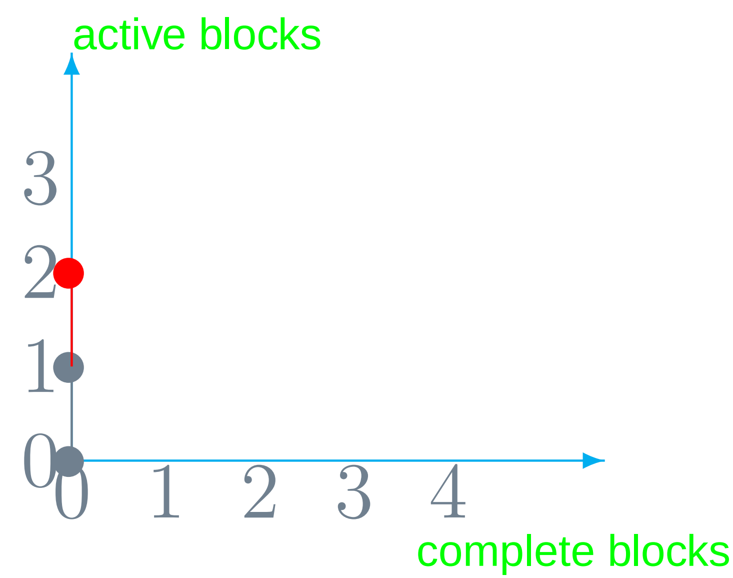
bijection

i-th trace of π ⟷ path diagram of π

2-th trace of π

$\{1, \cdots\} - \{2, \cdots\}$

$\xi_2 = 2$



Path

$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$

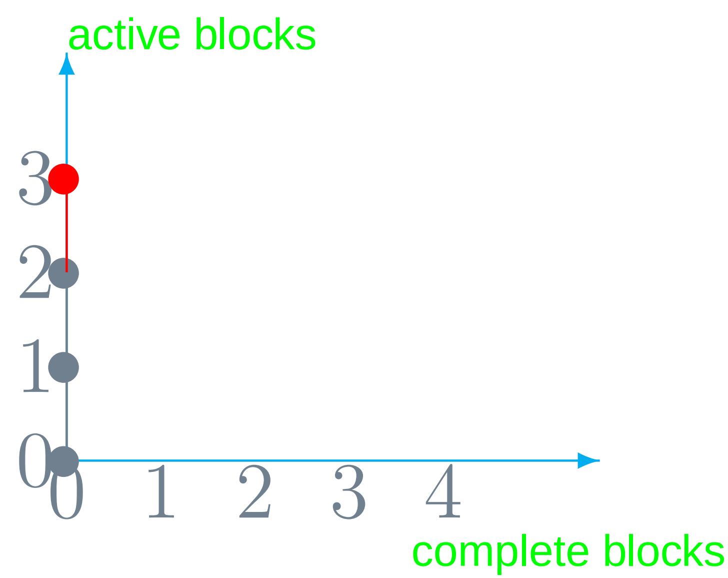
bijection

i-th trace of π $\longleftrightarrow$ path diagram of π

3-th trace of π

$\{3, \cdots\} - \{1, \cdots\} - \{2, \cdots\}$

$\xi_3 = 1$



Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

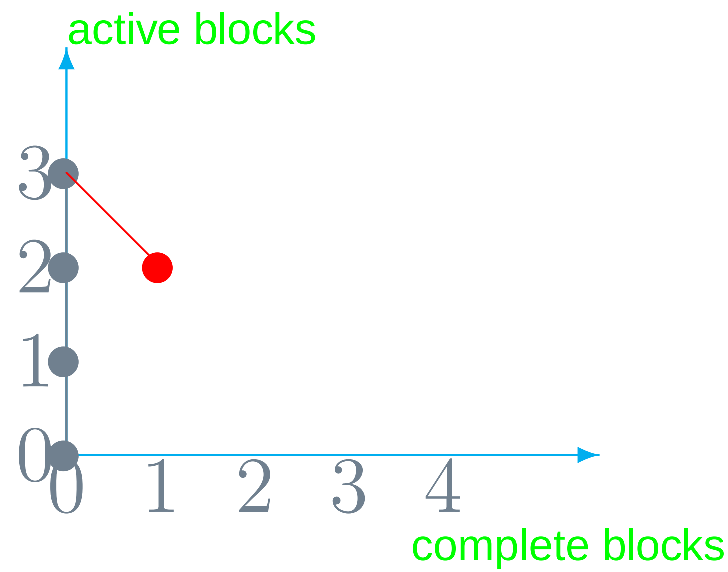
bijection

i-th trace of π ⟷ path diagram of π

4-th trace of π

$\{3,\cdots\} - \{1,4\} - \{2,\cdots\}$

$\xi_4 = 2$



Path

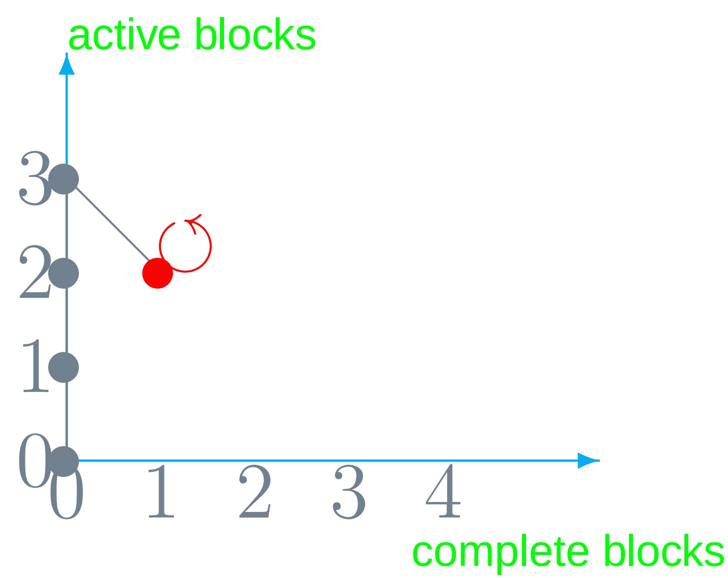
$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

i-th trace of π ⟷ bijection ⟷ path diagram of π

5-th trace of π

$$\{3, 5, \cdots\} - \{1,4\} - \{2, \cdots\}$$

$$\xi_5 = 1$$



Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

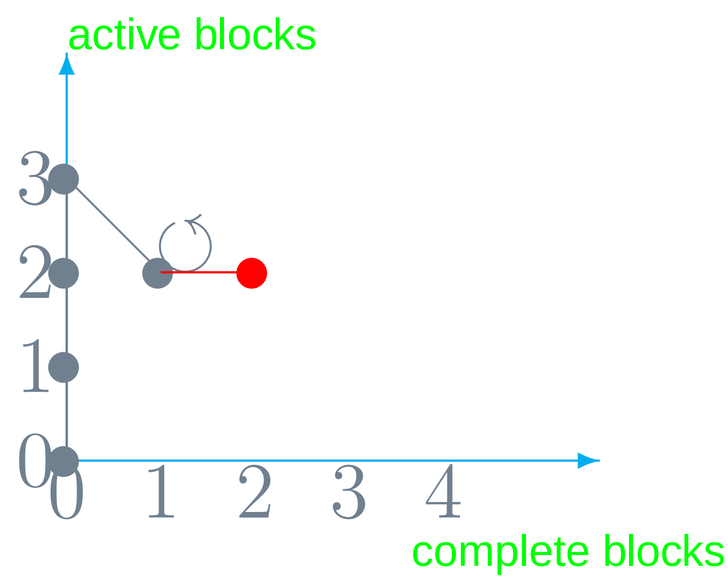
bijection

i-th trace of π ⟷ path diagram of π

6-th trace of π

$$\{6\} - \{3,5,\cdots\} - \{1,4\} - \{2,\cdots\}$$

$$\xi_6 = 1$$



Path

$$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$$

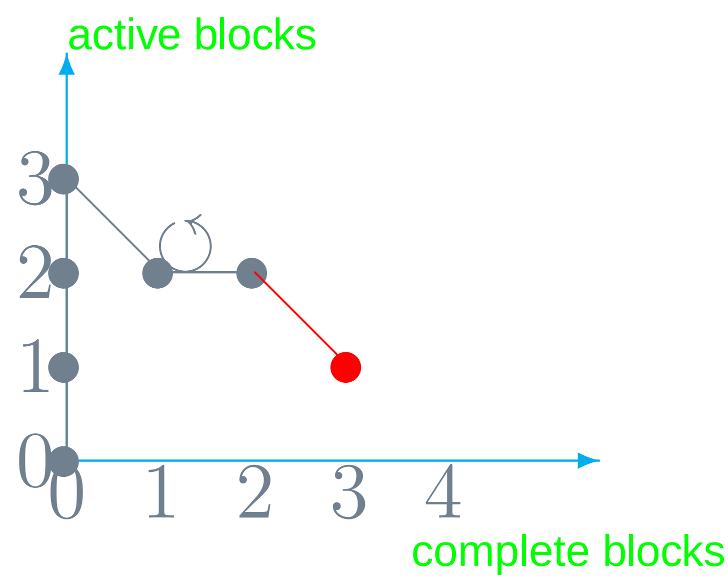
bijection

i-th trace of π ⟷ path diagram of π

7-th trace of π

$$\{6\} - \{3,5,7\} - \{1,4\} - \{2,\cdots\}$$

$$\xi_7 = 1$$



Path

$\pi = \{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$

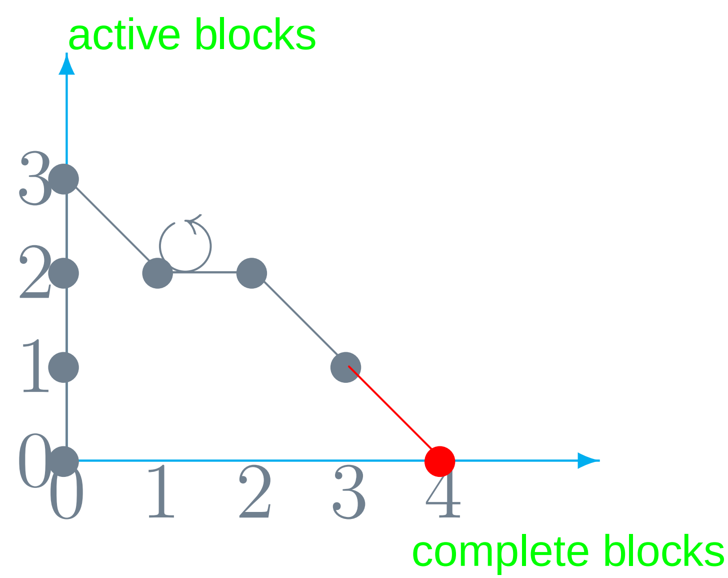
bijection

i-th trace of π ⟷ path diagram of π

8-th trace of π

$\{6\} - \{3,5,7\} - \{1,4\} - \{2,8\}$

$\xi_8 = 1$



Path

$\pi = \{6\} - \{3, 5, 7\} - \{1, 4\} - \{2, 8\}$

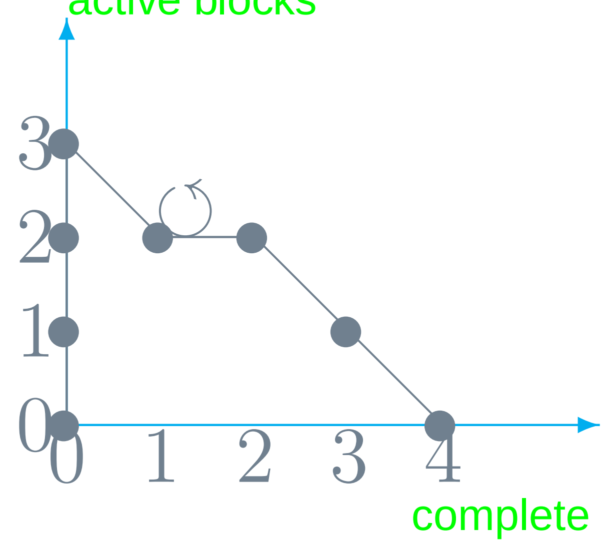
bijection

i-th trace of π ⟷ path diagram of π

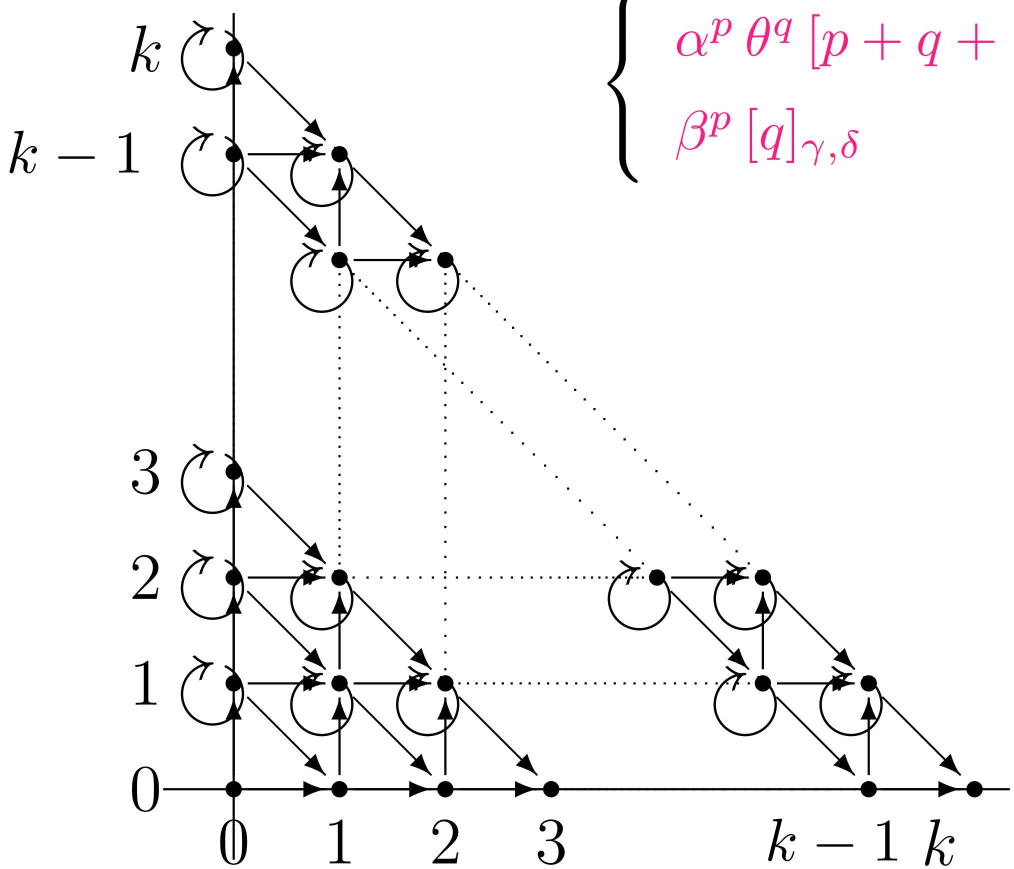
Thus we obitain

$\omega = (\mathrm{N}, \mathrm{N}, \mathrm{N}, \text{S-E}, \mathrm{Null}, \mathrm{E}, \text{S-E}, \text{S-E})$.

$\xi = (1, 2, 1, 2, 1, 1, 1, 1)$

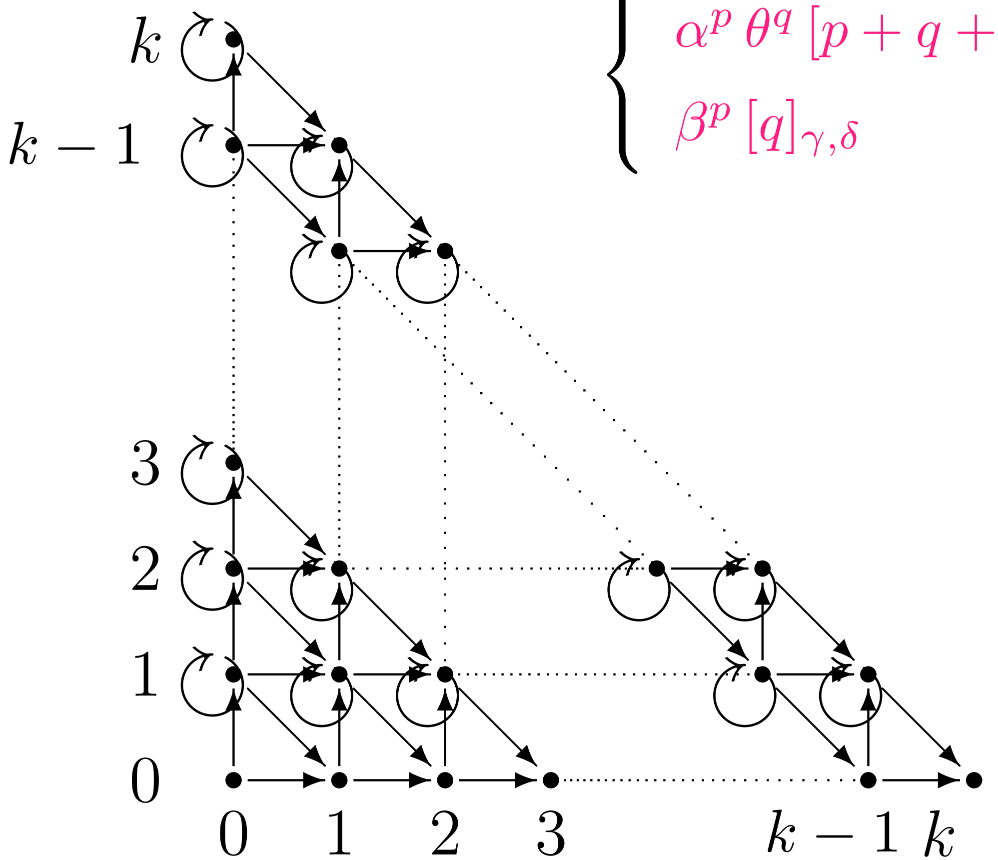


The digraph D_k



$$\begin{cases} \alpha^p\,\theta^q\,[p+q+1]_{\varepsilon,\eta} & \text{if N or E;} \\ \beta^p\,[q]_{\gamma,\delta} & \text{if Null or S-E.} \end{cases}$$

The digraph D_k



$$\begin{cases} \alpha^p \, \theta^q \, [p+q+1]_{\varepsilon,\eta} & \text{if N or E;} \\ \beta^p \, [q]_{\gamma,\delta} & \text{if Null or S-E.} \end{cases}$$

$$n_k = 1 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$$

The digraph D_k

$$\pi \longleftrightarrow (\omega, \xi)$$

(a) if the i-th step of ω is North (resp. East), then $i \in \mathcal{O}(\pi)$ (resp. $i \in \mathcal{S}(\pi)$) and

$$(\mathrm{lcs} + \mathrm{rcs})_i(\pi) = p_{i-1}, \quad \mathrm{los}_i(\pi) = \xi_i - 1,$$

$$(\mathrm{lsb} + \mathrm{rsb})_i(\pi) = q_{i-1}, \quad \mathrm{ros}_i(\pi) = p_{i-1} + q_{i-1} + 1 - \xi_i;$$

The digraph D_k

$$\pi \longleftrightarrow (\omega, \xi)$$

(b) if the i-th step of ω is South-East (resp. Null), then $i \in \mathcal{C}(\pi)$ (resp. $i \in \mathcal{T}(\pi)$) and

$$(\mathrm{lcs} + \mathrm{rcs})_i(\pi) = p_{i-1}, \quad \mathrm{lsb}_i(\pi) = \xi_i - 1,$$

$$(\mathrm{lsb} + \mathrm{rsb})_i(\pi) = q_{i-1} - 1, \quad \mathrm{rsb}_i(\pi) = q_{i-1} - \xi_i.$$

The digraph D_k

$$Q_k(a;\alpha,\beta,\gamma,\delta,\varepsilon,\eta,\theta) :=$$

$$\sum_{\pi\in\mathcal{OP}^k} \alpha^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{O}\cup\mathcal{S})\pi}\,\beta^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{T}\cup\mathcal{C})\pi}\,\gamma^{\mathrm{rsb}(\mathcal{T}\cup\mathcal{C})\pi}$$

$$\times\ \delta^{\mathrm{lsb}(\mathcal{T}\cup\mathcal{C})\pi}\,\varepsilon^{\mathrm{ros}(\mathcal{O}\cup\mathcal{S})\pi}\,\eta^{\mathrm{los}(\mathcal{O}\cup\mathcal{S})\pi}\,\theta^{(\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})\pi}a^{|\pi|}$$

$$= \sum_{w\in D_k:(0,0)\to(0,k)} val(w)a^{|w|}$$

Transfer-Matrix Method

- $D = (V, E)$ a digraph.

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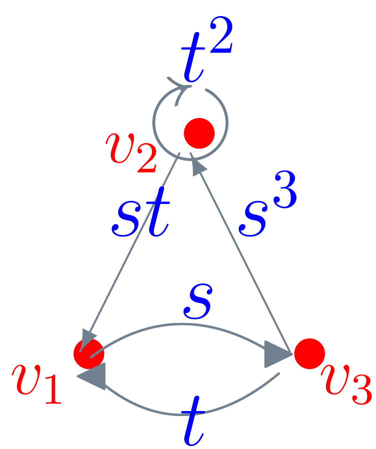
Example



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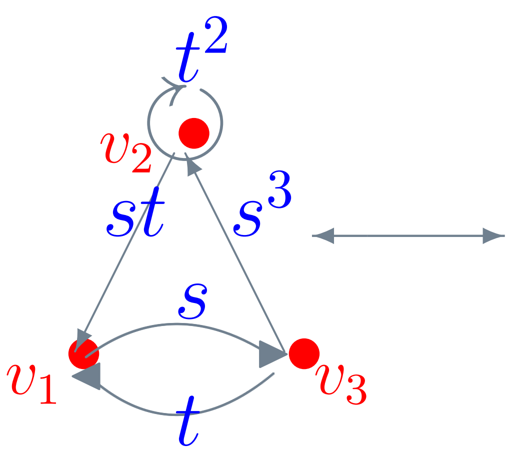

$$A = \begin{pmatrix} 0 & 0 & s \\ st & t^2 & 0 \\ t & s^3 & 0 \end{pmatrix}$$

Transfer-Matrix Method

A **walk** of length k is a sequence $w = v_{i_0} v_{i_1} \ldots v_{i_k}$ of points of D such that $(v_{i_r}, v_{i_{r+1}}) \in E$.

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Theorem

$$\sum_{w: v_i \to v_j} val(w) z^{|w|} = (-1)^{i+j} \frac{\det(I - zA; j, i)}{\det(I - zA)}.$$

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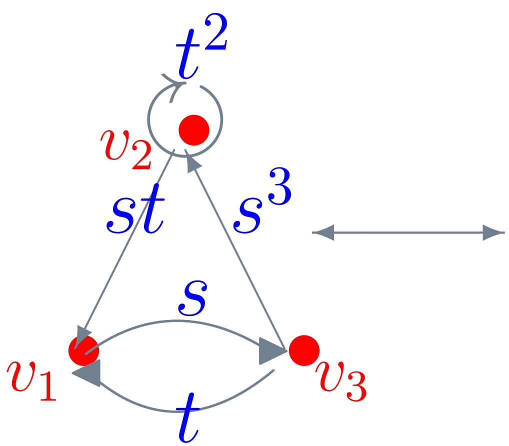
Example



$$A = \begin{pmatrix} 0 & 0 & s \\ st & t^2 & 0 \\ t & s^3 & 0 \end{pmatrix}$$

$w_0 = v_3 v_2 v_2 v_1 v_3 v_1$ walk of length $|w_0| = 5$ and $val(w_0) = s^3 \times t^2 \times st \times s \times t = s^5 t^4$.

Transfer-Matrix Method

A **walk** of length k is a sequence $w = v_{i_0} v_{i_1} \ldots v_{i_k}$ of points of D such that $(v_{i_r}, v_{i_{r+1}}) \in E$.

Example

$$\sum_{w: v_1 \mapsto v_3} val(w) z^{|w|} = \frac{\det(I_2 - z\,A_2; 3, 1)}{\det(I_2 - z\,A_2)}$$

$$= \frac{zs(1 - zt^2)}{1 - zt^2 + z^3 s^5 t + z^2 ts - z^3 t^3 s}$$

Determinant Expression

$$Q_k(a;t_1,t_2,t_3,t_4,t_5,t_6,t_7) = \sum_{w\in D_k:(0,0)\to(0,k)} val(w)a^{|w|}$$

Transfer-matrix method $\Longrightarrow$

$$= (-1)^{1+n_k}\frac{\det(I - aA_k;\, n_k, 1)}{\det(I - aA_k)}.$$

Determinant Expression

For instance, when $k = 2$, we have

$$A_2 = \left(\begin{array}{c|cc|ccc} 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & t_7\,[2]_{t_5,t_6} & t_7\,[2]_{t_5,t_6} & 0 \\ 0 & 0 & 0 & 0 & t_1\,[2]_{t_5,t_6} & t_1\,[2]_{t_5,t_6} \\ \hline 0 & 0 & 0 & [2]_{t_3,t_4} & [2]_{t_3,t_4} & 0 \\ 0 & 0 & 0 & 0 & t_2 & t_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Determinant Expression

$$Q_2(a;\boldsymbol{t}) = -\frac{\det(I_2 - aA_2; 6, 1)}{\det(I_2 - aA_2)}$$

$$= \frac{a^2[2]_{t_5,t_6}(at_2t_7 + t_1(1 - a[2]_{t_3,t_4}))}{(1-a)(1-a[2]_{t_3,t_4})(1-at_2)}.$$

Generating Function

In order to prove Steingrímsson's conjecture, it is sufficient to evaluate the following special cases of $Q_k(a;\boldsymbol{t})$:

$$f_k(a;x,y,t,u)=Q_k(a;x,x,x,y,t,u,y),$$
$$g_k(a;x,y,t,u)=Q_k(a;1,x,1,xy,t,u,y).$$

Generating Function

The goal of our proof is the following identity:

$$f_k(a;x,y,t,u) = \frac{a^k x^{\binom{k}{2}} [k]_{t,u}!}{\prod_{i=1}^{k}(1 - a[i]_{x,y})},$$

$$g_k(a;x,y,t,u) = \frac{a^k\, [k]_{t,u}!}{\prod_{i=1}^{k}(1 - ax^{k-i}[i]_{xy})}.$$

Generating Function

Let A'_k and A''_k be the matrices obtained from A_k by making the substitutions. Let

$$M_k = I_k - aA'_k \quad \text{and} \quad N_k = I_k - aA''_k.$$

Then we derive from the above formula that

$$f_k(a;x,y,t,u) = \frac{(-1)^{1+n_k}\det(M_k;n_k,1)}{\det M_k},$$
$$g_k(a;x,y,t,u) = \frac{(-1)^{1+n_k}\det(N_k;n_k,1)}{\det N_k}.$$

Matrix M_k

Example

$k = 1$

$$M_1 = \left(\begin{array}{c|cc} 1 & -a & -a \\ \hline 0 & 1-a & -a \\ 0 & 0 & 1 \end{array}\right)$$

Matrix M_k

Example $k = 2$

$$M_2 = \left(\begin{array}{ccc|ccc} 1 & -a & -a & 0 & 0 & 0 \\ 0 & 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ 0 & 0 & 1 & 0 & -ax(t+u) & -ax(t+u) \\ \hline 0 & 0 & 0 & 1-a(x+y) & -a(x+y) & 0 \\ 0 & 0 & 0 & 0 & 1-ax & -ax \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Matrix M_k

The matrix M_k is defined inductively as follows:

$$M_k = \left(\begin{array}{c|c} M_{k-1} & \overline{M}_{k-1} \\ \hline O_{k+1,n_{k-1}} & \widehat{M}_{k-1} \end{array} \right).$$

Here $\widehat{M}_{k-1}$ is the $(k+1) \times (k+1)$ matrix

$$\widehat{M}_{k-1} = \left(\delta_{ij} - ax^{i-1}[n+1-i]_{x,y}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \le i,j \le k+1}$$

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Here $\overline{M}_{k-1}$ is the $n_{k-1} \times (k+1)$ matrix

$$\overline{M}_{k-1} = \left(\begin{array}{c} O_{n_{k-2},k+1} \\ \hline \check{M}_{k-1} \end{array} \right)$$

Matrix M_k

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$$M_k = \left(\begin{array}{c|c} M_{k-1} & \overline{M}_{k-1} \\ \hline O_{k+1,n_{k-1}} & \widehat{M}_{k-1} \end{array} \right).$$

with the $k \times (k+1)$ matrix

$$\check{M}_{k-1} = \left(-ax^{i-1}y^{k-i}[k]_{t,u}(\delta_{ij} + \delta_{i+1,j})\right)_{1\le i\le k,\, 1\le j\le k+1}.$$

Matrix M_k

The matrix M_k is defined inductively as follows:

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<u>Theorem</u>

$$\det(M_k; n_k, 1) = (-1)^{\binom{k}{2}} a^k x^{\binom{k}{2}} [k]_{t,u}!$$
$$\times \prod_{m=1}^{k-1} \prod_{i=1}^{m} (1 - ax^i [m-i+1]_{x,y}).$$

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Proof

Use

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det A \cdot \det \left(D - CA^{-1}B \right).$$

Matrix N_k

Example $k = 2$

$$N_2(\lambda, a) = \begin{pmatrix} \lambda & -a & -a & 0 & 0 & 0 \\ 0 & \lambda - a & -a & -ay[2]_{t,u} & -ay[2]_{t,u} & 0 \\ 0 & 0 & \lambda & 0 & -a[2]_{t,u} & -a[2]_{t,u} \\ 0 & 0 & 0 & \lambda - a(1+xy) & -a(1+xy) & 0 \\ 0 & 0 & 0 & 0 & \lambda - ax & -ax \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Matrix N_k

The matrix N_k is defined inductively as follows:

$$N_k(\lambda, a) = \left(\begin{array}{c|c} N_{k-1}(\lambda, a) & \overline{N}_{k-1}(\lambda, a) \\ \hline O_{k+1, n_{k-1}} & \widehat{N}_{k-1}(\lambda, a) \end{array} \right)$$

Here $\widehat{N}_{k-1}(\lambda, a)$ is the $(k+1) \times (k+1)$ matrix

$$\widehat{N}_{n-1}(\lambda, a)$$
$$= \left(\lambda\delta_{ij} - ax^{i-1}[n+1-i]_{xy}(\delta_{ij} + \delta_{i+1,j}) \right)_{1 \le i,j \le n+1}$$

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Here $\overline{N}_{k-1}(\lambda, a)$ is the $n_{k-1} \times (k+1)$ matrix

$$\left(\begin{array}{c} O_{n_{k-2}, k+1} \\ \hline \check{N}_{k-1} \end{array} \right)$$

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$$\check{N}_{k-1} = \left(-ay^{k-i}[n]_{t,u} \cdot (\delta_{ij} + \delta_{i+1,j})\right)_{1 \le i \le k,\, 1 \le j \le k+1}.$$

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Proof

Find the eigenvector of each eigenvalue.

Eigenvectors

- $]n,k[_{q,r} = [n]_{qr} - q^{n-k}[k]_{qr},$

Eigenvectors

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- $\left]\begin{matrix} n \\ k \end{matrix}\right[_{q,r} = \begin{cases} \frac{\prod_{i=0}^{k-1}]n,i[_{q,r}}{[k]_{qr}!} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$

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Example

- $]3,1[_{q,r} = 1 + qr + q^2r^2 - q^2$
- $\left]\begin{matrix} 3 \\ 2 \end{matrix}\right[_{q,r} = \frac{(1+qr+q^2r^2)(1+qr+q^2r^2-q^2)}{(1+qr)}$.

Eigenvectors

Define the row vectors $\boldsymbol{X}_n^{m,l}$ of degree n_k as follows: For $1 \le i \le k+1$ and $1 \le j \le i$, the $\left(\frac{i(i-1)}{2}+j\right)$th entry of $\boldsymbol{X}_n^{m,l}$ is equal to

$$X_{i,j}^{m,l} = (-1)^{i+m+l} x^{-(m+l-1)(i-m-l)+\binom{j-l}{2}} y^{\binom{i-m-l}{2}} \times \frac{[i-m-l]_{t,u}!}{[i-m-l]_{xy}!} \begin{bmatrix} i-1 \\ m+l-1 \end{bmatrix}_{t,u} \begin{bmatrix} m \\ m+l-j \end{bmatrix}_{x,y}.$$

Eigenvectors

Let k be a positive integer. Let m and l be positive integers such that $0 \leq m \leq k-1$ and $1 \leq l \leq k-m$. Then we have

$$\boldsymbol{X}_k^{m,l}\, N_k(\lambda, a) = (\lambda - a x^{l-1}[m]_{xy})\, \boldsymbol{X}_k^{m,l}.$$

Conjecture

Consider the following two generating functions of ordered partitions with $k \geq 0$ blocks:

$$\xi_k(a;x,y) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{mak}+\mathrm{bMaj})\pi}\, y^{\mathrm{cmajLSB}\,\pi}\, a^{|\pi|},$$

$$\eta_k(a;x,y) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{lmak}+\mathrm{bMaj})\pi}\, y^{\mathrm{cmajLSB}\,\pi}\, a^{|\pi|}.$$

Conjecture

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For $k \geq 0$, the following identities would hold:

$$\xi_k(a;x,y) = \frac{a^k\,(xy)^{\binom{k}{2}}\,[k]_{x,y}!}{\prod_{i=1}^{k}(1-a[i]_{x,y})},$$

$$\eta_k(a;x,y) = \frac{a^k(xy)^{\binom{k}{2}}\,[k]_{x,y}!}{\prod_{i=1}^{k}(1-a[i]_{x,y})}.$$

Reference

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The End of Talk

Thank you!