Hankel determinants of Catalan, Motzkin and Schröder numbers and its q-analogues

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Abstract

In this talk we try to generalize Catalan Hankel determinants and make a q-analogue. The Catalan numbers are well-know to be the number of Dyck paths. We replace the Catalan numbers with Motzkin numbers, Schröder numbers and etc. paths, which counts certain paths in the plane.

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Hankel matrix

Let $a_0, a_1, a_2, \ldots$ be any sequence of integers. We consider the Hankel matrix

$$A_n^{(t)} = (a_{i+j+t})_{0 \le i,j \le n-1} = \begin{pmatrix} a_t & a_{t+1} & \ldots & a_{t+n-1} \\ a_{t+1} & a_{t+2} & \ldots & a_{t+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t+n-1} & a_{t+n} & \ldots & a_{t+2n-2} \end{pmatrix}$$

of degree n.

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How can we compute $\det A_n^{(t)}$?

Hankel Determinants

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Catalan numbers

Definition

For $n = 0, 1, 2, \ldots,$ The Catalan number C_n is defined to be

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan number C_n counts the Dyck paths from $(0, 0)$ to $(2n, 0)$.

Example

The generating function for the Catalan numbers is given by

$$\frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} C_n t^n = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots.$$

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Fact (Mays-Wojciechowski 2000)

For the Catalan numbers C_0, C_1, C_2,..., let

$$C_n^{(t)} = (C_{i+j+t})_{0\le i,j\le n-1}$$

denote the Hankel matrix. Then, for $n \geq 0$, the following identities hold:

$$\det C_n^{(0)} = \det C_n^{(1)} = 1,$$
$$\det C_n^{(2)} = n+1,$$
$$\det C_n^{(3)} = \frac{1}{6}(n+1)(n+2)(2n+3).$$

Catalan Hankel determinants

Theorem (Desainte-Catherine-Viennot 1986)

In general

$$\det C_n^{(t)} = \prod_{0 \le i \le j \le t-1} \frac{i+j+2n}{i+j}.$$

holds for $t, n \ge 0$.

Theorem (Krattenthaler 2007)

$$\det\left(C_{k_{i+1}+j}\right)_{0 \le i,j \le n-1} = \prod_{1 \le i < j \le n} (k_i - k_j) \prod_{i=1}^{n} \frac{(i+n)!(2k_i)!}{(2i)!k_i!(k_i+n)!}$$

for a positive integer n and non-negative integers $k_1, k_2, \ldots, k_n$.

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Fibonacci Numbers

Let $n \geq 0$. The sequence $\{F_n\}$ integers defined by $F_0 = F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2}$$

is called the Fibonacci sequence.

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The generating function for the Fibonacci numbers is given by

$$\frac{1}{1-t-t^2} = \sum_{n\geq 0} F_n t^n = 1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5 + \cdots .$$

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Fact (Cvetković-Rajković-Ivković 2002)

We consider the Hankel matrix

$$\widetilde{C}_n^{(t)} = (C_{i+j+t} + C_{i+j+t+1})_{0\le i,j\le n-1}$$
$$= \begin{pmatrix} C_t + C_{t+1} & C_{t+1} + C_{t+2} & \dots & C_{t+n-1} + C_{t+n} \\ C_{t+1} + C_{t+2} & C_{t+2} + C_{t+3} & \dots & C_{t+n-} + C_{t+n+1} \\ \vdots & \vdots & \dots & \vdots \\ C_{t+n-1} + C_{t+n} & C_{t+n} + C_{t+n+1} & \dots & C_{t+2n-2} + C_{t+2n-1} \end{pmatrix}.$$

Then the follwoing identities hold for $n \geq 1$:

$$\widetilde{C}_n^{(0)} = F_{2n},$$
$$\widetilde{C}_n^{(1)} = F_{2n+1}.$$

Notation

q-shifted factorials

We use the notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

for $n = 0, 1, 2, \dots$. $(a;q)_n$ is called the *q-shifted factorial*.
Frequently used compact natation:

$$(a_1, a_2, \dots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty,$$
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More Notation

Raising factorials

If we put $a = q^{\alpha}$, then

$$\lim_{q\to 1} \frac{(q^{\alpha};q)_n}{(1-q)^n} = \lim_{q\to 1} \frac{(1-q^{\alpha})}{(1-q)} \frac{(1-q^{\alpha+1})}{(1-q)} \cdots \frac{(1-q^{\alpha+n-1})}{(1-q)}$$
$$= (\alpha)(\alpha+1)\cdots(\alpha+n-1).$$

We write $(\alpha)_n = \prod_{k=0}^{n-1}(\alpha+k)$, which is called the *raising factorial*.

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Basic hypergeomrtric series

We shall define the ${}_{r+1}\phi_r$ *basic hypergeomrtric series* by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. $$

Hypergeomrtric series

If we put $a_i = q^{\alpha_i}$ and $b_i = q^{\beta_i}$ in the above series and let $q \to 1$, then we obtain the ${}_{r+1}F_r$ *hypergeomrtric series*

$$ {}_{r+1}F_r\left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{r+1} \\ \beta_1, \dots, \beta_r \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_{r+1})_n}{n! (\beta_1)_n \dots (\beta_r)_n} z^n. $$

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Motzkin numbers

Definition (Motzkin numbers)

The Motzkin number M_n is defined to be

$$M_n = {}_2F_1\left[\begin{matrix}(1-n)/2, -n/2 \\ 2\end{matrix}; 4\right].$$

The Motzkin number M_n counts the number of Moztkin paths from $(0,0)$ to $(n,0)$.

The generating function

The generating function for Motzkin numbers is

$$\sum_{n=0}^{\infty} M_n t^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

$$= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + \cdots$$

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Motzkin Hankel determinants

Definition

For the Motzkin numbers $M_0, M_1, M_2, \ldots$, define the Hankel matrix $M_n^{(t)}$ to be

$$M_n^{(t)} = \left(M_{i+j+t}\right)_{0 \le i,j \le n-1} .$$

Theorem (Aigner 1998)

We have

1. $\det M_n^{(0)} = 1$ for $n \ge 1$,
2. $\det M_n^{(1)} = 1, 0, -1$ for $n \equiv 0, 1 \pmod{6}$, $n \equiv 2, 5 \pmod{6}$, $n \equiv 3, 4 \pmod{6}$, respectively.

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$$\sum_{n=0}^{\infty} S_n t^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x}$$
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Moments

Here we consider the series

$$\mu_n = \frac{(aq;q)_n}{(abq^2;q)_n} \quad (n = 0, 1, 2, \dots).$$

Specializations

If we put $a = q^\alpha$, $b = q^\beta$ and let $q \to 1$, then

$$\mu_n \to \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

Note that

$$\frac{(\frac{1}{2})_n}{(2)_n} = \frac{C_n}{2^{2n}}, \quad \frac{(\frac{1}{2})_n}{(1)_n} = \frac{1}{2^{2n}}\binom{2n}{n}, \quad \frac{(\frac{3}{2})_n}{(2)_n} = \frac{1}{2^{2n}}\binom{2n+1}{n}.$$

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Purpose of this talk

Theorem

Let n be a positive integer. Then

$$\det\left(\mu_{i+j}\right)_{0\le i,j\le n-1} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \times \prod_{k=1}^{n} \frac{(q,aq,bq;q)_{n-k}}{(abq^{n-k+1};q)_{n-k}(abq^2;q)_{2(n-k)}}.$$

Corollary

Let n be a positive integer and t non-negative integer. Then

$$\det\left(\mu_{i+j+t}\right)_{0\le i,j\le n-1} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \left\{\frac{(aq;q)_t}{(abq^2;q)_t}\right\}^n \times \prod_{k=1}^{n} \frac{(q,aq^{t+1},bq;q)_{n-k}}{(abq^{n-k+t+1};q)_{n-k}(abq^{t+2};q)_{2(n-k)}}.$$

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Main Theorem

Theorem

Let n be a positive integer and $k_1, k_2, \dots, k_n$ non-negative integers. Then we have

$$\det\left(\mu_{k_{i+1}+j}\right)_{0\le i,j\le n-1} = a^{\binom{n}{2}} q^{\binom{n+1}{3}} \prod_{i=1}^{n} \frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \times \prod_{1\le i<j\le n} (q^{k_i} - q^{k_j}) \prod_{i=1}^{n} (bq;q)_{n-i}.$$

Methods to prove theorem

Proof methods

The methods to prove the theorems

- Lattice path method (the Lindström-Gessel-Viennot theorem)
- Orthogonal polynomials and continued fractions (the little q-Jacobi polynomials)
- LU-decompositions (q-Dougall's formula)
- Desnanot-Jacobi adjoint matrix theorem (Dodgson's formula)

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Non-intersecting lattice paths

Definition

Let $D = (V, E)$ be an acyclic digraph.

- $\mathscr{P}(u, v)$: the set of all directed paths from u to v.
- An *n-vertex* $\boldsymbol{v} = (v_1, \ldots, v_n)$ is an n-tuple of vertices of D.
- An *n-path* from $\boldsymbol{u} = (u_1, \ldots, u_n)$ to $\boldsymbol{v} = (v_1, \ldots, v_n)$ is an n-tuple $\boldsymbol{P} = (P_1, \ldots, P_n)$ such that $P_i \in \mathscr{P}(u_i, v_i)$.
- The n-path $\boldsymbol{P}$ is said to be *non-intersecting* if any two different paths P_i and P_j have no vertex in common.
- $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$ (resp. $\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})$) : the set of all (resp. non-intersecting) n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$
- $\boldsymbol{u}$ is said to be *D-compatible* with $\boldsymbol{v}$ if every path $P \in \mathscr{P}(u_i, v_l)$ intersects with every path $Q \in \mathscr{P}(u_j, v_k)$ whenever $i < j$ and $k < l$.

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Lidström-Gessel-Viennot Theorem

Lemma (Lidström-Gessel-Viennot)

Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ be two n-vertices in an acyclic digraph D. Then

$$\sum_{\pi \in S_n} \operatorname{sgn}\pi \, F_0(\boldsymbol{u}^\pi, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \le i,j \le n}.$$

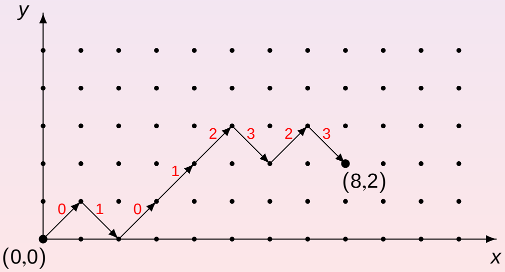
In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then

$$F_0(\boldsymbol{u}, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \le i,j \le n}.$$

Dyck path

Definition (Dyck path)

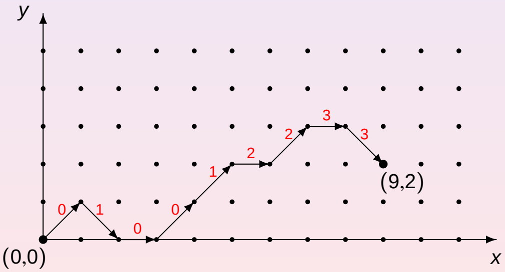
A *Dyck path* is, by definition, a lattice path in the plane lattice $\mathbb{Z}^2$ consisting of two types of steps: rise vectors $(1, 1)$ and fall vectors $(1, -1)$, which never passes below the x-axis. We say a rise vector (resp. fall vector) whose origin is (x, y) and ends at $(x + 1, y + 1)$ (resp. $(x + 1, y - 1)$) has *height* y.



Moztkin path

Definition (Moztkin path)

A *Moztkin path* is, by definition, a lattice path in $\mathbb{Z}^2$ consisting of three types of steps: rise vectors $(1, 1)$, fall vectors $(1, -1)$, and (short) level vectors $(1, 0)$ which never passes below the x-axis.



Schröder path

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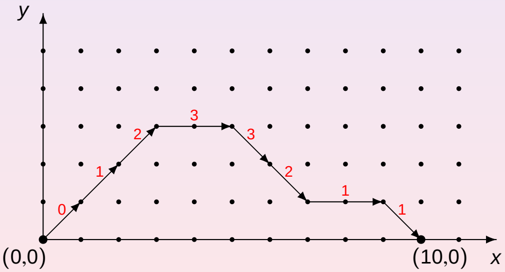
A *Schröder path* is, by definition, a lattice path in $\mathbb{Z}^2$ consisting of three types of steps: rise vectors $(1,1)$, fall vectors $(1,-1)$, and long level vectors $(2,0)$ which never passes below the x-axis.



Paths and weights

Definition

- $\mathscr{D}_{m,n}$: the set of Dyck paths starting from $(0,0)$ and ending at (m,n).
- $\mathscr{M}_{m,n}$: the set of Moztkin paths starting from $(0,0)$ and ending at (m,n).
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Definition (weights)

Assign the weight a_h, b_h, c_h to each rise vector, fall vector, (short or long) level vector of height h, respectively.

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Theorem (Flajolet 1980)

Stieltjes type continued fraction (Dyck paths):

$$\sum_{n\geq 0} \mathrm{GF}\left[\mathscr{D}_{(2n,0)}\right] t^{2n} = \cfrac{1}{1 - \cfrac{a_0 b_1 t^2}{1 - \cfrac{a_1 b_2 t^2}{1 - \cfrac{a_2 b_3 t^2}{\ddots}}}}.$$

Jacobi type continued fraction (Moztkin paths):

$$\sum_{n\geq 0} \mathrm{GF}\left[\mathscr{M}_{(n,0)}\right] t^{n} = \cfrac{1}{1 - c_0 t - \cfrac{a_0 b_1 t^2}{1 - c_1 t - \cfrac{a_1 b_2 t^2}{1 - c_2 t - \cfrac{a_2 b_3 t^2}{\ddots}}}}.$$

Paths and Weights

Proposition

Schröder path:

$$\sum_{n\geq 0} \mathrm{GF}\left[\mathscr{S}_{(2n,0)}\right] t^{2n} = \cfrac{1}{1 - c_0 t^2 - \cfrac{a_0 b_1 t^2}{1 - c_1 t^2 - \cfrac{a_1 b_2 t^2}{1 - c_2 t^2 - \cfrac{a_2 b_3 t^2}{\ddots}}}}$$

Definition (λ_n)

For non-negative integer n, let

$$\lambda_n = \begin{cases} \dfrac{q^{k-1}(1-aq^k)(1-abq^k)}{(1-abq^{2k-1})(1-abq^{2k})} & \text{if } n = 2k-1 \text{ is odd,} \\ \dfrac{aq^k(1-q^k)(1-bq^k)}{(1-abq^{2k})(1-abq^{2k+1})} & \text{if } n = 2k \text{ is even.} \end{cases}$$

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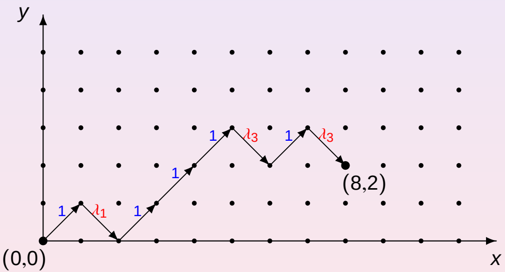
Definition (Weights of Dyck paths)

Let $P \in \mathscr{D}_{m,n}$ be a Dyck path. We assign the weight 1 to each rise vector, and λ_h to each fall vector of height h.

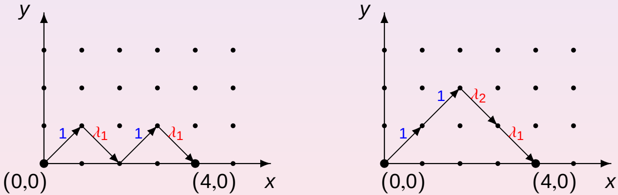


A Dyck Path of weight $\lambda_1\lambda_3^2$

Example

The generating function of $\mathscr{D}_{4,0}$ equals

$$\mathrm{GF}(\mathscr{D}_{4,0}) = \lambda_1^2 + \lambda_1\lambda_2 = \frac{(1-aq)(1-aq^2)}{(1-abq^2)(1-abq^3)} = \mu_2.$$



Dyck Paths in $\mathscr{D}_{4,0}$

Dyck paths and the generating function

Lemma

Let m and n be non-negative integers such that $m \equiv n \pmod 2$. Put $m = 2r$ and $n = 2s$ if m and n are both even, $m = 2r + 1$ and $n = 2s + 1$ if m and n are both odd. Then we have

$$\mathrm{GF}\left[\mathscr{D}_{m,n}\right] = \begin{bmatrix} r \\ s \end{bmatrix}_q \frac{(aq^{1+s};q)_{r-s}}{(abq^{2+2s};q)_{r-s}},$$

where

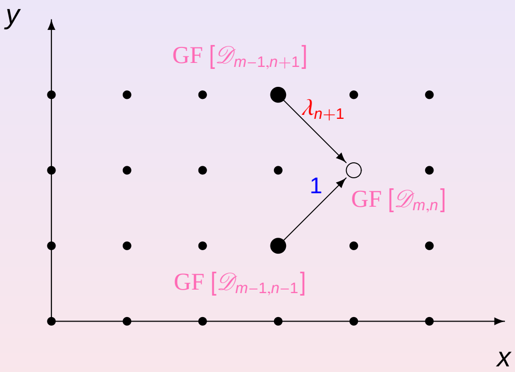
$$\begin{bmatrix} r \\ s \end{bmatrix}_q = \begin{cases} \frac{(q;q)_r}{(q;q)_s (q;q)_{r-s}} & \text{if } 0 \le s \le r, \\ 0 & \text{otherwise.} \end{cases}$$

Especially

$$\mathrm{GF}\left[\mathscr{D}_{2m,0}\right] = \frac{(aq;q)_m}{(abq^2;q)_m} = \mu_m.$$

Proof

Proceed by induction on m.



$$\mathrm{GF}\left[\mathscr{D}_{m,n}\right] = \mathrm{GF}\left[\mathscr{D}_{m-1,n-1}\right] + \lambda_{n+1}\mathrm{GF}\left[\mathscr{D}_{m-1,n+1}\right]$$

Stieltjes type continued fraction

Corollary (Stieltjes type continued fraction)

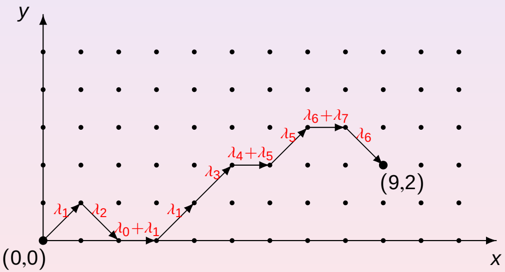
As a corollary of this lemma, we obtain the Stieltjes type continued fraction

$$\sum_{m\geq 0}\mu_m t^m = \cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{1-\cfrac{\lambda_3 t}{\ddots}}}}.$$

Weights to realize the moment sequence

Definition (Weights of Moztkin paths)

Let $P \in \mathscr{M}_{m,n}$ be a Moztkin path. We assign the weight λ_{2h+1} to each rise vector of height h, λ_{2h} to each fall vector of height h, and $\lambda_{2h} + \lambda_{2h+1}$ to each level vector of height h.



A Moztkin Path of weight $\lambda_1^2\lambda_2\lambda_3\lambda_5\lambda_6(\lambda_0+\lambda_1)(\lambda_4+\lambda_5)(\lambda_6+\lambda_7)$

Dyck paths and the generating function

Lemma

Let m and n be non-negative integers. Then we have

$$\mathrm{GF}\left[\mathscr{M}_{m,n}\right] = q^{\binom{n}{2}} \begin{bmatrix} m \\ n \end{bmatrix}_q \frac{(aq;q)_m (1 - abq^{2n+1})}{(abq^{n+1};q)_{m+1}}.$$

Especially

$$\mathrm{GF}\left[\mathscr{M}_{m,0}\right] = \frac{(aq;q)_m}{(abq^2;q)_m} = \mu_m.$$

Corollary (Jacobi type continued fraction)

$$\sum_{m\geq 0} \mu_m t^m = \cfrac{1}{1 - \lambda_1 t - \cfrac{\lambda_1\lambda_2 t^2}{1-(\lambda_2+\lambda_3)t-\cfrac{\lambda_3\lambda_4 t^2}{1-(\lambda_4+\lambda_5)t-\cfrac{\lambda_5\lambda_6 t^2}{\ddots}}}}.$$

Proof by Induction

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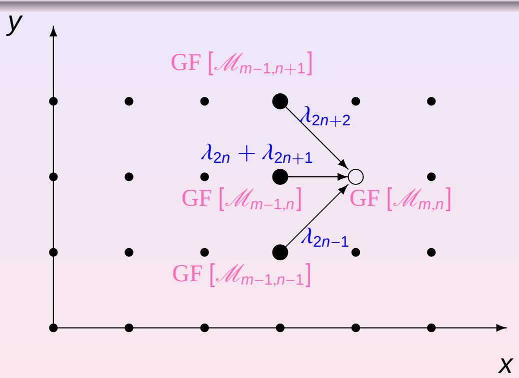

$$\mathrm{GF}\left[\mathscr{M}_{m,n}\right] = \lambda_{2n-1}\mathrm{GF}\left[\mathscr{M}_{m-1,n-1}\right] + (\lambda_{2n} + \lambda_{2n+1})\mathrm{GF}\left[\mathscr{M}_{m-1,n}\right] + \lambda_{2n+2}\mathrm{GF}\left[\mathscr{M}_{m-1,n+1}\right]$$

Hankel determinants

Genetral weights

Assign the weight a_h, b_h to each rise vector, fall vector of height h, respectively.

Theorem

Let $G_m = \mathrm{GF}\left[\mathscr{D}_{2m,0}\right]$ for non-negative integer m.

1. $\det\left(G_{m+n}\right)_{0\le i,j\le n-1} = \prod_{i=1}^{n}\left(a_{2i-2}b_{2i-1}a_{2i-1}b_{2i}\right)^{n-i}$

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3. $\det\left(G_{m+n+2}\right)_{0\le i,j\le n-1}$ equals

$$\sum_{k=0}^{n}\prod_{i=1}^{k}\left(a_0a_1\cdots a_{2i-3}a_{2i-2}^2b_1b_2\cdots b_{2i-1}b_{2i-1}^2\right)$$

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Let $G_m = \mathrm{GF}\left[\mathscr{D}_{2m,0}\right]$ for non-negative integer m.

1. $\det\left(G_{m+n}\right)_{0\le i,j\le n-1} = \prod_{i=1}^{n}\left(a_{2i-2}b_{2i-1}a_{2i-1}b_{2i}\right)^{n-i}$
2. $\det\left(G_{m+n+1}\right)_{0\le i,j\le n-1} = \prod_{i=1}^{n}\left(a_{2i-2}b_{2i-1}\right)^{n-i+1}\left(a_{2i-1}b_{2i}\right)^{n-i}$
3. $\det\left(G_{m+n+2}\right)_{0\le i,j\le n-1}$ equals

$$\sum_{k=0}^{n}\prod_{i=1}^{k}\left(a_0a_1\cdots a_{2i-3}a_{2i-2}^2b_1b_2\cdots b_{2i-1}b_{2i-1}^2\right)$$

$$\times\prod_{i=1}^{k}\left(a_0a_1\cdots a_{2i-1}b_1b_2\cdots b_{2i}\right)$$

Hankel determinants

Genetral weights

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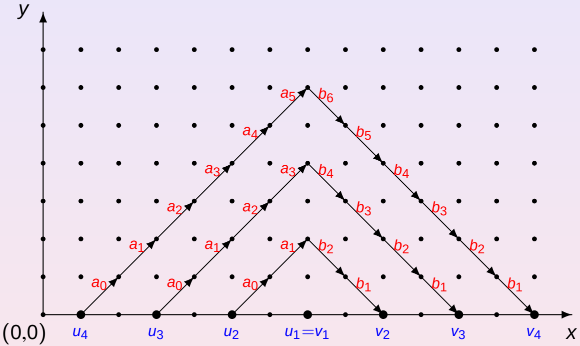
Proof by Lindström-Gessel-Viennot theorem

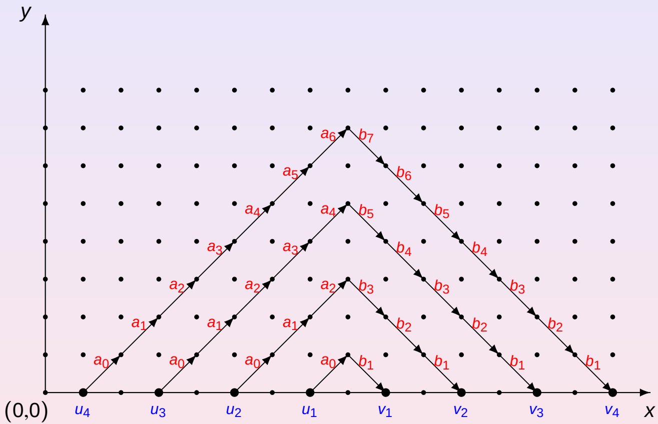
$t = 0, n = 4$



$$\det\left(G_{m+n}\right)_{0\le i,j\le n-1} = \prod_{i=1}^{n}\left(a_{2i-2}b_{2i-1}a_{2i-1}b_{2i}\right)^{n-i}$$

Proof by Lindström-Gessel-Viennot theorem

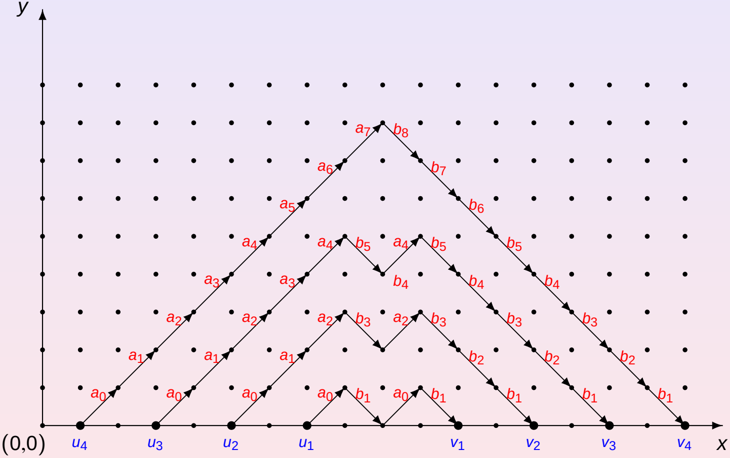
$t = 1, n = 4$



$$\det \left(G_{m+n+1}\right)_{0 \le i,j \le n-1} = \prod_{i=1}^{n} \left(a_{2i-2} b_{2i-1}\right)^{n-i+1} \left(a_{2i-1} b_{2i}\right)^{n-i}$$

Proof by Lindström-Gessel-Viennot theorem

$t = 2, n = 4$



Proof of Theorem

Theorem

Let n be a positive integer. Then

$$\det\left(\mu_{i+j}\right)_{0\le i,j\le n-1} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \times \prod_{k=1}^{n} \frac{(q,aq,bq;q)_{n-k}}{(abq^{n-k+1};q)_{n-k}(abq^2;q)_{2(n-k)}}.$$

Proof of Theorem

The determinant equals

$$\prod_{k=1}^{n}(\lambda_{2k-1}\lambda_{2k})^{n-k} = \prod_{k=1}^{n}\prod_{i=1}^{n-k}(\lambda_{2i-1}\lambda_{2i})$$

where

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Corollary

Let n be a positive integer and t non-negative integer. Then

$$\det\left(\mu_{i+j+t}\right)_{0\le i,j\le n-1} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \left\{\frac{(aq;q)_t}{(abq^2;q)_t}\right\}^n \times \prod_{k=1}^{n} \frac{(q,aq^{t+1},bq;q)_{n-k}}{(abq^{n-k+t+1};q)_{n-k}(abq^{t+2};q)_{2(n-k)}}.$$

Proof of Corollary

Use

$$\mu_{n+t} = \frac{(aq;q)_{n+t}}{(abq^2;q)_{n+t}} = \frac{(aq;q)_t(aq^{t+1};q)_n}{(abq^2;q)_t(abq^{t+2};q)_n}.$$

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The little q-Jacobi polynomials

Definition (Andrews-Askey 1977)

For $n = 0, 1, 2, \ldots$, the little q-Jacobi polynomials $p_n(x; a, b; q)$ is defined to be

$$p_n(x; a, b; q) = \frac{(aq; q)_n}{(abq^{n+1}; q)_n} (-1)^n q^{\binom{n}{2}} {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{n+1} \\ aq \end{matrix} ; q, xq \right].$$

Theorem (Three term relation)

$p_n(x) = p_n(x; a, b; q)$ satisfies the three term relation

$$p_{n+1}(x) = \{x - (\lambda_{2n} + \lambda_{2n+1})\} p_n(x) - \lambda_{2n-1}\lambda_{2n} p_{n-1}(x)$$

where $p_{-1}(x) = 0$ and $p_0(x) = 1$. Especially, $p_n(x)$ is a monic polynimial of degree n for $n = 0, 1, 2, \ldots$.

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Moments

Definition (Linear functional)

For a polynomial $f(x)$, we define the linear functional $\mathscr{L} : f \mapsto \mathscr{L}[f]$ by

$$\mathscr{L}[f] = \frac{(aq;q)_\infty}{(abq^2;q)_\infty} \sum_{j=0}^{\infty} \frac{(bq;q)_j}{(q;q)_j} (aq)^j f(q^j).$$

Proposition (the moment sequence)

From the q-binomial thereom $\sum_{j=0}^{\infty} \frac{(a;q)_j}{(q;q)_j} x^j = \frac{(ax;q)_\infty}{(x;q)_\infty}$, we obtain

$$\mathscr{L}[x^n] = \frac{(aq;q)_n}{(abq^2;q)_n} = \mu_n.$$

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Orthogonality

Proposition (Andrews-Askey 1977)

We have

$$\mathscr{L}\left[p_n(x)p_m(x)\right] = \begin{cases} 0 & \text{if } m \neq n, \\ a^n q^{n^2} \frac{(q,aq,bq;q)_n}{(abq;q)_{2n}(abq^{n+1};q)_{n+1}} & \text{if } m = n. \end{cases}$$

Proposition (the continued fraction)

The above three term equation

$$p_{n+1}(x) = \{x - (\lambda_{2n} + \lambda_{2n+1})\} p_n(x) - \lambda_{2n-1}\lambda_{2n} p_{n-1}(x)$$

is equvalent to the Jacobi type continued fraction

$$\sum_{m\geq 0} \mu_m t^m = \cfrac{1}{1 - \lambda_1 t - \cfrac{\lambda_1\lambda_2 t^2}{1-(\lambda_2+\lambda_3)t - \cfrac{\lambda_3\lambda_4 t^2}{1-(\lambda_4+\lambda_5)t - \cfrac{\lambda_5\lambda_6 t^2}{\ddots}}}}.$$

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The Hankel determinants

Theorem

Let

$$d_n^{(0)} = \det A_n^0 = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}$$

Then

$$\frac{d_{n+1}^{(0)}}{d_n^{(0)}} = \prod_{i=1}^{n} (\lambda_{2i-1}\lambda_{2i}) = a^n q^{n^2} \frac{(q, aq, bq; q)_n}{(abq; q)_{2n} (abq^{n+1}; q)_{n+1}}$$

A proof by the orthogonal polynomials

Proof

Put

$$D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} & \mu_{2n-1} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix}$$

for $n = 0, 1, 2, \dots$. Then $\frac{1}{d_n^{(0)}} D_n(x)$ is a monic polynomial of degree n. Because $\mathscr{L}[x^m D_n(x)] = 0$ for $m < n$, we have $\mathscr{L}[D_m(x) D_n(x)] = 0$ if $m \neq n$. From the uniqueness of the orthogonal polynomials with respect to $\mathscr{L}$, we obatin

$$p_n(x) = \frac{1}{d_n^{(0)}} D_n(x).$$

Proof

Thus we have

$$\mathscr{L}[x^n p_n(x)] = \frac{1}{d_n^{(0)}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n} \end{vmatrix} = \frac{d_{n+1}^{(0)}}{d_n^{(0)}}$$

Proof

From

$$p_{n+1}(x) = \{x - (\lambda_{2n} + \lambda_{2n+1})\} p_n(x) - \lambda_{2n-1}\lambda_{2n} p_{n-1}(x),$$

we have

$$\mathscr{L}[x^{n-1}p_{n+1}(x)] = \mathscr{L}[x^n p_n(x)] \\ - (\lambda_{2n} + \lambda_{2n+1})\mathscr{L}[x^{n-1}p_n(x)] - \lambda_{2n-1}\lambda_{2n}\mathscr{L}[x^{n-1}p_{n-1}(x)]$$

which implies

$$\frac{\mathscr{L}[x^n p_n(x)]}{\mathscr{L}[x^{n-1}p_{n-1}(x)]} = \lambda_{2n-1}\lambda_{2n}$$

Thus we obtain

$$\frac{d_{n+1}^{(0)}}{d_n^{(0)}} = \prod_{i=1}^{n} \lambda_{2i-1}\lambda_{2i}.$$

Reduction to an identity on hypergeometric series

q-Dougall's formula

The LU-decomposition of the Hankel matrix is obatined from the following q-Dougall's formula:

$$
{}_6\phi_5\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d \end{matrix}; q, \frac{aq}{bcd}\right]
= \frac{(qa, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}.
$$

(See Gasper-Rahman 1990.)

Thank you!