Enumeration problems of plane partitions and determinant generating functions

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Plan of My Talk

1. Plane partitions
2. TSSCPP and tc-symmetric plane partitions
3. Restricted column-strict plane partitions
4. $\tau = -1$ Conjectures
5. Restricted column-strict domino plane partitions with all rows and columns of even lenth
6. Restricted column-strict domino plane partitions with all rows of even lenth
7. Restricted column-strict domino plane partitions with all columns of even lenth

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Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{matrix} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \ddots & & \end{matrix}$$

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Shape

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Let $\pi = (\pi_{ij})_{i,j \geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

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Example of plane partitions

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- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\boxed{1}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 1\\\cline{1-1}\end{array}$$

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Definition

A plane partition is said to be *column-strict* if it is strictly decreasing in coulumns.

Schur functions

Let $x_1, \ldots, x_n$ be n variables, and fix a shape λ. The Schur function $s_\lambda(x_1, \ldots, x_n)$ is defined to be

$$s_\lambda(x_1, \ldots, x_n) = \sum_\pi x^\pi,$$

where π runs over all column-strict plane partitions of shape λ and $x^\pi = \prod_i x_i^{\#\text{ of } i \text{ in } \pi}$.

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An Example of Schur functions

Example

If $\lambda = (22)$ and $\boldsymbol{x} = (x_1, x_2, x_3)$, then the followings are column-strict plane partitions with all parts ≤ 3.

2	2
1	1

3	2
1	1

3	3
1	1

3	2
2	1

3	3
2	1

3	3
2	2

Hence we have

$$s_{(2^2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

Ferrers graph

Definition

The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \left\{ (i, j, k) : k \leq \pi_{ij} \right\}$$

Example

Ferrers graph

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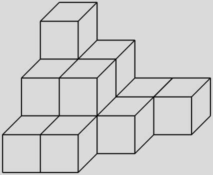
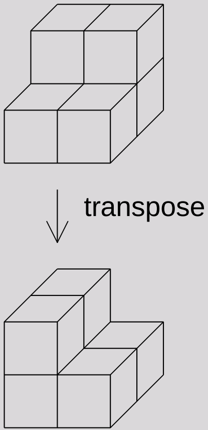


Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example



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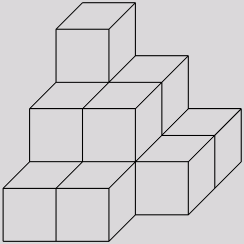
Example

A symmetric PP



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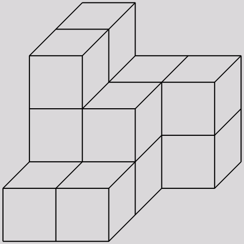
Example

A cyclicaly symmetric PP



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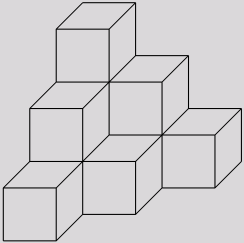
Example

A totally symmetric PP



Complement

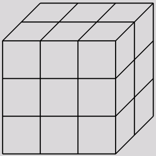
Definition

Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r, s, t) = [r] \times [s] \times [t]$.

Define the *complement* π^c of π by
$\pi^c = \{ (r+1-i, s+1-j, t+1-k) \, : \, (i,j,k) \notin \pi \}$.

- π is said to be *(r, s, t)-self-complementary* if $\pi = \pi^c$. i.e. $(i,j,k) \in \pi \Leftrightarrow (r+1-i, s+1-j, t+1-k) \notin \pi$.

Example



$B(2, 3, 3)$

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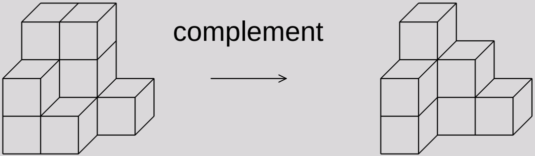

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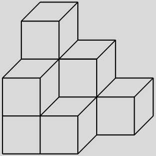

A $(2,3,3)$-self-complementary PP

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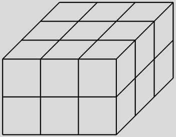

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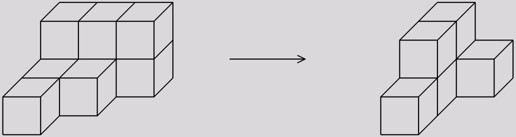

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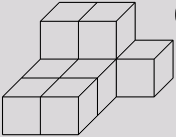

$(3, 3, 2)$-complement=transpoese

Totally symmetric self-complementary plane partitions

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We denote the set of all self-complementary totally symmetric plane partitions of size n by $\mathscr{T}_n$.

Example

$\mathscr{T}_1$ consists of the single partition

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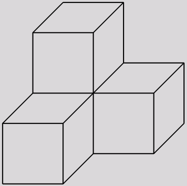
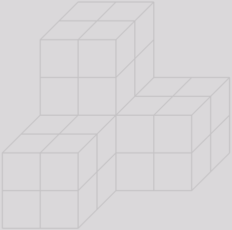
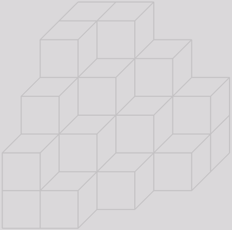


Example

$\mathscr{T}_2$ consists of the following two partitions:





TSSCPPs of size 2

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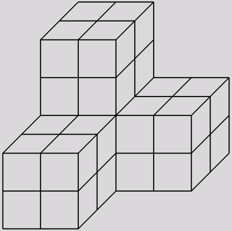

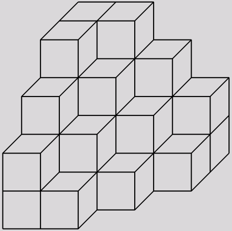

TSSCPPs of size 3

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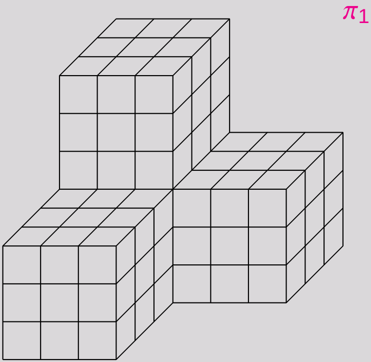
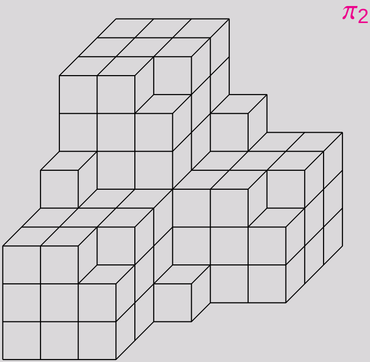
$\mathscr{T}_3$ consists of the following seven partitions:





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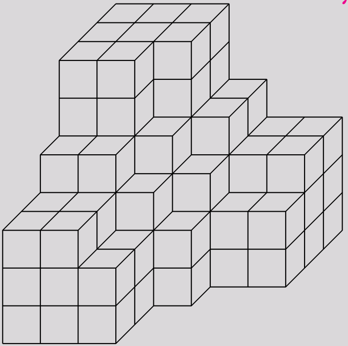
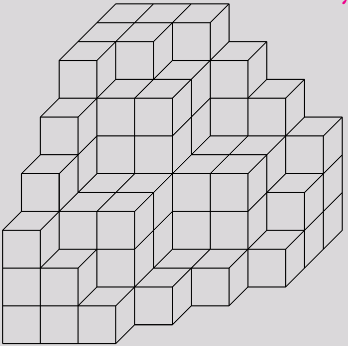


π_3

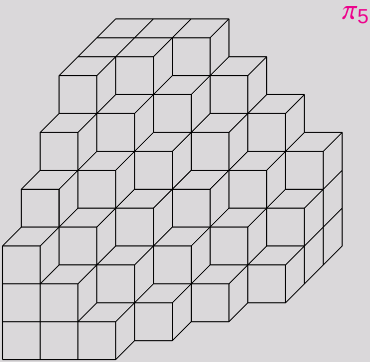


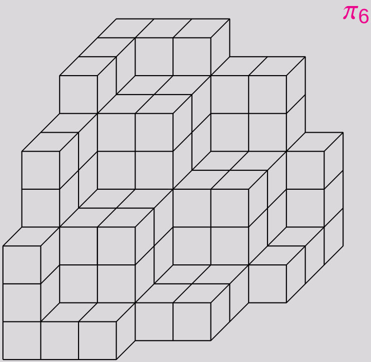
π_4

TSSCPPs of size 3

Example

$\mathscr{T}_3$ consists of the following seven partitions:

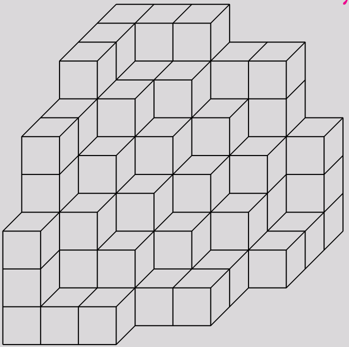




TSSCPPs of size 3

Example

$\mathscr{T}_3$ consists of the following seven partitions:



π_7

Tc-symmetric plane partitions

Definition

A plane partition in $B(2n, 2n, 2n)$ is defined to be *tc-symmetric of size n* if it is cyclically symmetric and it is equal to its transpose-complement.

We denote the set of all tc-symmetric plane partitions of size n by $\mathscr{C}_n$.

Example

$\mathscr{C}_1$ consists of the single partition

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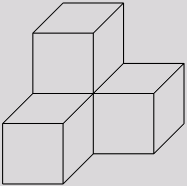
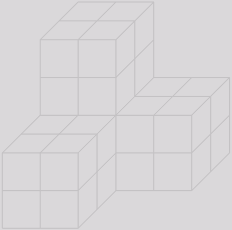
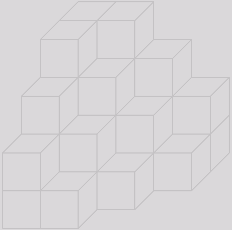


Tc-symmetric PPs of size 2

Example

$\mathscr{C}_2$ consists of the following two partitions:

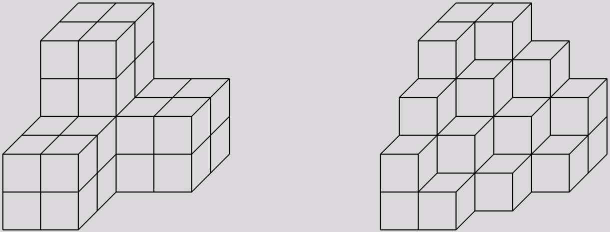




Tc-symmetric PPs of size 2

Example

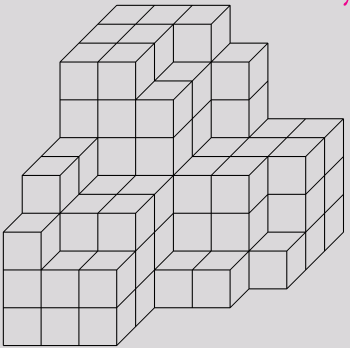
$\mathscr{C}_2$ consists of the following two partitions:



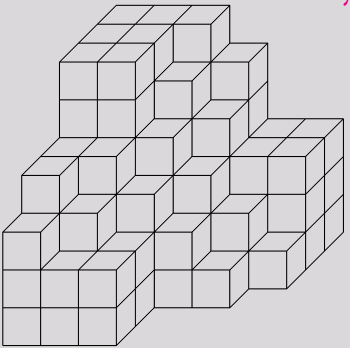
Tc-symmetric PPs of size 3

Example

$\mathscr{C}_3$ consists of the following eleven plane partitions:



π_8



π_9

Numbers

n	1	2	3	4	5	6	$\cdots$
TSSCPP	1	2	7	42	429	7436	$\cdots$
tc-symmetric PP	1	2	11	170	7429	920460	$\cdots$

Definition

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

$$TC_n = \prod_{i=0}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i)!(4i+1)!}$$

Definition

$$A_{2n}^{\mathrm{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{\{(n+i)!\}^2} \qquad A_{2n+1}^{\mathrm{HTS}} = \frac{n!(3n)!}{\{(2n)!\}^2} \cdot A_{2n}^{\mathrm{HTS}},$$

$$A_{2n+1}^{\mathrm{VS}} = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

Example

n	1	2	3	4	5	6	7	8	9	$\cdots$
A_n^{HTS}	1	2	3	10	25	140	588	5544	39204	$\cdots$
A_n^{VS}	1		1		3		26		646	$\cdots$

The Numbers of HTSASMs and VSASMs

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$$A_{2n}^{\mathrm{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{\{(n+i)!\}^2} \qquad A_{2n+1}^{\mathrm{HTS}} = \frac{n!(3n)!}{\{(2n)!\}^2} \cdot A_{2n}^{\mathrm{HTS}},$$

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A_n^{HTS}	1	2	3	10	25	140	588	5544	39204	$\cdots$
A_n^{VS}	1		1		3		26		646	$\cdots$

Definition

$$A^{VS}_{2n+1}(t) = \frac{A^{VS}_{2n-1}}{(4n-2)!} \sum_{r=1}^{2n} t^{r-1} \sum_{k=1}^{r} (-1)^{r+k} \frac{(2n+k-2)!(4n-k-1)!}{(k-1)!(2n-k)!},$$

Example

$$A^{VS}_{3}(t) = 1$$
$$A^{VS}_{5}(t) = 1 + t + t^2$$
$$A^{VS}_{7}(t) = 3 + 6t + 8t^2 + 6t^3 + 3t^4$$
$$A^{VS}_{9}(t) = 26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$$

Enumeration polynomials

Definition

$$A^{\mathrm{VS}}_{2n+1}(t) = \frac{A^{\mathrm{VS}}_{2n-1}}{(4n-2)!} \sum_{r=1}^{2n} t^{r-1} \sum_{k=1}^{r} (-1)^{r+k} \frac{(2n+k-2)!(4n-k-1)!}{(k-1)!(2n-k)!},$$

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Enumeration polynomials

Definition

$$A_n(t) = \frac{A_n}{\binom{3n-2}{n-1}} \sum_{r=1}^{n} \binom{n+r-2}{n-1}\binom{2n-1-r}{n-1} t^{r-1}$$

$$\frac{\widetilde{A}_{2n}^{\mathrm{HTS}}(t)}{\widetilde{A}_{2n}^{\mathrm{HTS}}} = \frac{(3n-2)(2n-1)!}{(n-1)!(3n-1)!} \times \sum_{r=0}^{n} \frac{\{n(n-1)-nr+r^2\}(n+r-2)!(2n-r-2)!}{r!(n-r)!} t^r$$

$$A_{2n}^{\mathrm{HTS}}(t) = \widetilde{A}_{2n}^{\mathrm{HTS}}(t) A_n(t)$$

$$A_{2n+1}^{\mathrm{HTS}}(t) = \frac{1}{3}\left\{A_{n+1}(t)\widetilde{A}_{2n}^{\mathrm{HTS}}(t) + A_n(t)\widetilde{A}_{2n+2}^{\mathrm{HTS}}(t)\right\}$$

where $\widetilde{A}_{2n}^{\mathrm{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{(3i+1)!(n+i)!}$.

Examples

Example

$$A_1(t) = 1$$
$$A_2(t) = 1 + t$$
$$A_3(t) = 2 + 3t + 3t^2$$
$$A_4(t) = 7 + 14t + 14t^2 + 7t^3$$

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Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of plane partitions $c = (c_{ij})_{1 \leq i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*. A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

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Example

$\mathscr{P}_1$ consists of the single PP $\emptyset$.

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Example

$\mathscr{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \boxed{1}$$

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Example

$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline 1 & \\\cline{1-1}\end{array}$$

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$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \qquad \boxed{1} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \boxed{2} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

Bijections

Theorem

Let n be a positive integer.
Then we can construct a bijection from $\mathscr{T}_n$ to $\mathscr{P}_n$.

Definition

Let $c = (c_{ij})_{1 \leq i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.

Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.
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Example

$n = 7$, $c \in \mathscr{P}_3$, Saturated parts

5	5	4	2	2
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3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 1$, $\overline{U}_1(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 2$, $\overline{U}_2(c) = 5$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 3$, $\overline{U}_3(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 4$, $\overline{U}_4(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c=(c_{ij})_{1\le i,j}\in\mathscr{P}_n$ and $k=1,\dots,n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n=7$, $c\in\mathscr{P}_3$, $k=5$, $\overline{U}_5(c)=4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 6$, $\overline{U}_6(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 7$, $\overline{U}_7(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Generating function

Generating function

We consider the generating function

$$f_{k,n}(\tau,t) = \sum_{\pi \in \mathscr{P}_n^{\mathrm{R}}} \tau^{N(\pi)}\, t^{\overline{U}_k(\pi)},$$

where $N(\pi)$ denotes the number of boxes in π.

Example

if $n = 3$, then $\mathscr{P}_3^{\mathrm{R}}$ is composed of the following 7 plane partitions:

$$\emptyset \qquad \boxed{1} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \boxed{2} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

$$f_{k,3}(\tau,t) = 1 + (1+t)\tau + t(2+t)\tau^2 + t^2\tau^3,$$

for $k = 1, 2, 3$.

Generating function

We consider the generating function

$$f_{k,n}(\tau,t)=\sum_{\pi\in\mathscr{P}_n^{\mathrm{R}}}\tau^{N(\pi)}\,t^{\overline{U}_k(\pi)},$$

where $N(\pi)$ denotes the number of boxes in π.

Example

if $n=3$, then $\mathscr{P}_3^{\mathrm{R}}$ is composed of the following 7 plane partitions:

$$f_{k,3}(\tau,t)=1+(1+t)\tau+t(2+t)\tau^2+t^2\tau^3,$$

for $k=1,2,3$.

A Pfaffian expression

Definition

Let $B_{n,N} = (b_{i,j}(\tau,t))_{0\le i\le n-1,\,0\le j\le N-1}$ denote the n by N matrix defined by

$$b_{i,j}(\tau,t) = \begin{cases} \delta_{i,j} & \text{if } i = 0, \\ \left\{\binom{i-1}{j-i} + t\binom{i-1}{j-i-1}\right\}\tau^{j-i} & \text{if } i > 0. \end{cases}$$

Example

If $n = 3$ and $N = 5$, then we have

$$B_{3,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t\tau & 0 & 0 \\ 0 & 0 & 1 & (1+t)\tau & t\tau^2 \end{pmatrix}.$$

A Pfaffian expression

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A Pfaffian expression

Definition

Let S_n denote the anti-symmetric $n \times n$ matrix defined by $S_n = ((-1)^{j-i-1})_{1 \le i < j \le n}$, and let $J_n = (\delta_{i,n+1-j})_{1 \le i,j \le n}$ denote the anti-diagonal matrix of size n.

Example

$$S_4 = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \qquad J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

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$$S_4 = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \qquad J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

A Pfaffian expression

Theorem

For a positive integer n, let N be the least integer such that $N \geq 2n - 1$ and $n + N$ is even. Then we have

$$f_{k,n}(\tau, t) = \operatorname{Pf}\begin{pmatrix} O_n & J_n\, B_{n,N} \\ -B_{n,N}^{T}\, J_n & S_N \end{pmatrix},$$

for $k = 1, \ldots, n$.

Example

If $n = 3$ and $N = 5$, then we obtain

$$\operatorname{Pf}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & (1+t)\tau & t\tau^2 \\ 0 & 0 & 0 & 0 & 1 & t\tau & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\ -1 & -t\tau & 0 & 1 & -1 & 0 & 1 & -1 \\ -(1+t)\tau & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ -t\tau^2 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

equals $1 + (1+t)\tau + t(2+t)\tau^2 + t^2\tau^3$.

A Pfaffian expression

Theorem

For a positive integer n, let N be the least integer such that $N \geq 2n - 1$ and $n + N$ is even. Then we have

$$f_{k,n}(\tau, t) = \mathrm{Pf}\begin{pmatrix} O_n & J_n B_{n,N} \\ -B_{n,N}^T J_n & S_N \end{pmatrix},$$

for $k = 1, \dots, n$.

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equals $1 + (1+t)\tau + t(2+t)\tau^2 + t^2\tau^3$.

Example

If we put $\tau = 1$ into $f_{k,3}(\tau, t) = 1 + (1+t)\tau + t(2+t)\tau^2 + t^2\tau^3$, then we obtain $A_3(t) = 2 + 3t + 2t^2$.

Fact

If we put $\tau = 1$, then

$$f_{k,n}(1, t) = A_n(t),$$

for $n \geq 1$.

$\tau = 1$

Example

If we put $\tau = 1$ into $f_{k,3}(\tau, t) = 1 + (1+t)\tau + t(2+t)\tau^2 + t^2\tau^3$, then we obtain $A_3(t) = 2 + 3t + 2t^2$.

Fact

If we put $\tau = 1$, then

$$f_{k,n}(1, t) = A_n(t),$$

for $n \geq 1$.

$\tau = -1$

Example

If we put $\tau = -1$ into $f_{k,3}(\tau, t) = 1 + (1+t)\tau + t(2+t)\tau^2 + t^2\tau^3$, then we obtain $f_{k,3}(-1, t) = t$.

Example

The first few terms of $f_{k,n}(-1, t)$ looks as follows:

$$f_{k,3}(-1,t) = t$$
$$f_{k,4}(-1,t) = (1-t)(1+t+t^2)$$
$$f_{k,5}(-1,t) = 3t(1+t+t^2)$$
$$f_{k,6}(-1,t) = 3(1-t)(3+6t+8t^2+6t^3+3t^4)$$
$$f_{k,7}(-1,t) = 26t(3+6t+8t^2+6t^3+3t^4)$$

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f_{k,7}(-1,t) &= 26t(3+6t+8t^2+6t^3+3t^4)
\end{aligned}
$$

$\tau = -1$ conjecture

Conjecture

Let n be a positive integer such that $n \geq 3$.

1. If n is even, then we would have

$$f_{k,n}(-1,t) = A_{n-1}^{\mathrm{VS}} \cdot (1-t)\, A_{n+1}^{\mathrm{VS}}(t).$$

2. If n is odd, then we would have

$$f_{k,n}(-1,t) = A_{n}^{\mathrm{VS}} \cdot t\, A_{n}^{\mathrm{VS}}(t).$$

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Link patterns

Definition

A *non-crossing perfect matching* (*link pattern*) of the vertex set $[2n] = \{1, 2, \dots, 2n\}$ is an unordered collection of vertices, or *edges*, which does not contain edges $\{i, j\}$ and $\{k, l\}$ such that $i < k < j < l$. Let $\mathscr{F}_{2n}$ denote the set of all link patterns of $[2n]$. We consider the periodic case by identifying 1 and $2n$.

Example

For $n = 3$,

$$\mathscr{F}_6 = \Big\{\{1,2\}\{3,4\}\{5,6\}, \{1,2\}\{3,4\}\{4,5\}, \{1,4\}\{2,3\}\{5,6\}, \{1,6\}\{2,3\}\{4,5\}, \{1,6\}\{2,5\}\{3,4\}\Big\}.$$

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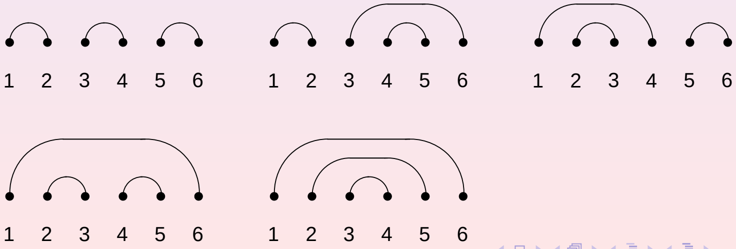
Link patterns

Definition

A convenient typographical notation for non-crossing perfect matching of $[2n]$ is obtained by using parentheses for paired vertices.

Example

$$\mathscr{F}_6 = \Big\{()()(), ()(()), (())(), (()()), ((()))\Big\}.$$



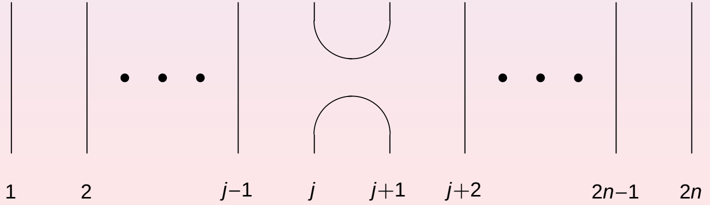
Matchmakers

Definition

Throughout the following we put $\tau = -(q + q^{-1})$. Define generators or *matchmakers* e_j, $j \in [2n]$, acting non-trivially on elements $F \in \mathscr{F}_{2n}$ by

$$e_j \ : \begin{cases} \{j, j+1\} \mapsto \tau \{j, j+1\} \\ \{i, j\}\{j+1, k\} \mapsto \{i, k\}\{j, j+1\}. \end{cases}$$

$e_j =$



Temperley-Lieb Algebras

Temperley-Lieb Algebra

The match makers e_j, $j \in [2n]$ satisfy the following relations:

$$\begin{aligned} e_i^2 &= \tau e_i, \; i = 1, \dots, 2n, \\ e_i e_{i\pm1} e_i &= e_i, \\ e_i e_j &= e_j e_i, \quad |i - j| > 1. \end{aligned}$$

We also have the cyclic operator σ such that $e_{i+1} = \sigma e_i \sigma^{-1}$. This algebra is called the (affine) Temperley-Lieb Algebra and denoted by TL_{2n}.

Example

For $n = 2$ case, we have two link patters. The order of the basis is $(())$, $()()$ (or equivalently label by $0011, 0101$). Explicitly, the generators are written

$$e_1 = e_3 = \begin{pmatrix} 0 & 0 \\ 1 & \tau \end{pmatrix}, \quad e_2 = e_4 = \begin{pmatrix} \tau & 1 \\ 0 & 0 \end{pmatrix}.$$

Example

For $n = 3$ cases. We have five basis. The order of basis is $((())), (()()), (())(), ()(()), ()()()$, (or equivalently label by $000111, 001011, 001101, 010011, 010101$). For example, the generator e_1 is written as

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \tau & 0 \\ 1 & 0 & 1 & 0 & \tau \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Other generators are obtained from $e_{i+1} = \sigma e_i \sigma^{-1}$

Polynomial representations

Definition

We determine a vector

$$\Psi = \sum_{\pi \in \mathscr{F}_n} \psi_\pi(z_i) \, |\pi\rangle$$

by the following manner. The vectors $|\pi\rangle$ are basis vectors on which TL_{2n} acts from left. The $\psi_\pi(z_i) = \psi_\pi(z_1, \dots, z_{2n})$ are polynomials on which TL_{2n} from right by

$$f\overline{E}_i = (qz_i - q^{-1}z_{i+1}) \frac{f(\dots, z_i, z_{i+1}, \dots) - f(\dots, z_{i+1}, z_i, \dots)}{z_i - z_{i+1}}$$

where $E_i = e_i - \tau$.

Polynomial representations

Fact

The polynomials $\psi_\pi(z_i)$ are uniquely determined by

$$\psi_{\pi_0} = \prod_{1 \le i < j \le n} (qz_i - q^{-1}z_j)(qz_{i+n} - q^{-1}z_{j+n}),$$

$$E_i\Psi = \Psi\overline{E}_i \qquad \text{for } i = 1, \dots, 2n,$$

where $\pi_0 = (((\dots()\dots)))$.

Example

If $n = 2$, then we obtain

$$\psi_{(())} = (qx_1 - q^{-1}x_2)(qx_3 - q^{-1}x_4),$$

$$\psi_{()()} = \psi_{(())}\overline{E}_2 = (-q^2x_1 + q^{-2}x_4)(qx_2 - q^{-1}x_3).$$

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Fact

If we substitute $q = e^{2\pi i/3}$ (i.e. $\tau = 1$), then we obtain

$$\sum_{\pi \in \mathscr{F}_n} \psi_\pi(z_i) = s_\lambda(z_1, \ldots, z_{2n})$$

where $\lambda = (n-1, n-1, \ldots, 1, 1, 0, 0)$.

Further, if we substitute $z_1 = \frac{1+qt}{t+q}$, $z_2 = \cdots = z_{2n} = 1$, then we obtain

$$\frac{\sum_{\pi \in \mathscr{F}_n} \psi_\pi(z_i)}{\psi_{\pi_0}(z_i)} = A_n(t).$$

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Example ($\tau = 1$)

Example

If $n = 2$ and $q = e^{2\pi i/3}$, then

$$\psi_{(())} + \psi_{()()} = s_{1^2}(x_1, x_2, x_3, x_4),$$
$$= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

and, when $z_1 = \frac{1+qt}{t+q}$, $z_2 = z_3 = z_4 = 1$, we obtain

$$\frac{\psi_{(())} + \psi_{()()}}{\psi_{(())}} = 1 + t.$$

$\tau = -1$ conjecture

Conjecture

If we substitute $q = e^{\pi i/3}$ (i.e. $\tau = -(q + q^{-1}) = -1$), $z_1 = \frac{1-qt}{t-q}$, $z_2 = \cdots = z_{2n} = 1$, then we would obtain

$$\frac{\sum_{\pi \in \mathscr{F}_n} \psi_\pi(z_i)}{\psi_{\pi_0}(z_i)} = \begin{cases} A_{n-1}^{\mathrm{VS}} \cdot (1-t) A_{n+1}^{\mathrm{VS}}(t) & \text{if } n \text{ is even,} \\ A_n^{\mathrm{VS}} \cdot t\, A_n^{\mathrm{VS}}(t) & \text{if } n \text{ is odd.} \end{cases}$$

Pairs of Restricted column-strict plane partitions

Definition

Let $\mathscr{Q}_n$ denote the set of all pairs of plane partitions in $\mathscr{P}_n$ of the same shape.

Example

Pairs of Restricted column-strict plane partitions

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Example

$\mathscr{P}_1$ consists of the single pair $(\emptyset, \emptyset)$.

Pairs of Restricted column-strict plane partitions

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Let $\mathscr{Q}_n$ denote the set of all pairs of plane partitions in $\mathscr{P}_n$ of the same shape.

Example

$\mathscr{P}_2$ consists of the following 2 pairs:

$$(\emptyset, \emptyset) \qquad \left(\boxed{1}, \boxed{1}\right)$$

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Definition

Let $\mathscr{Q}_n$ denote the set of all pairs of plane partitions in $\mathscr{P}_n$ of the same shape.

Example

$\mathscr{P}_3$ consists of the followng 11 pairs

$$(\emptyset,\emptyset) \quad \left(\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}\right) \quad \left(\begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}\right)$$

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Theorem

Let n be a positive integer.
Then we can construct a bijection from $\mathscr{C}_n$ to $\mathscr{Q}_n$.

Domino plane partitions

Definition

Let $\mathscr{D}_n^{(e)}$ denote the set of column-strict domino plane partitions c subject to the constraints that

1. each number in a domino crossing the $2j-1$st column does not exceed $n-j$,
2. each number in a domino crossing the $2j$th column does not exceed $n-j$,

for $j = 1, \dots, n-1$. If a part in the $2j-1$th or $2j$th column is equal to $n-j$, then we call it a *saturated* part. For a positive integer k and $\pi \in \mathscr{D}_n^{(e)}$, set $\overline{U}_k(\pi)$ denote the number of parts in c equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of dominoes in π.

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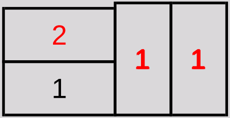
Example

Example

The following domino plane partition π is an element of $\mathscr{D}_3^{(e)}$



since the 1st and 2nd columns ≤ 2, the 3rd and 4th columns ≤ 1. The red numbers stand for saturated parts. Hence we have $\overline{U}_1(\pi) = \overline{U}_2(\pi) = \overline{U}_3(\pi) = 3$. Since π has 4 dominoes, we have $N(\pi) = 4$.

Domino plane partitions

Definition

Let $\mathscr{D}_n^{(o)}$ denote the set of column-strict domino plane partitions c subject to the constraints that

1. each number in a domino crossing the $2j-1$st column does not exceed $n-j$,
2. each number in a domino crossing the $2j$th column does not exceed $n-j-1$,

for $j = 1, \dots, n-1$. The statistics $\overline{U}_k(\pi)$ and can be defined similarly.

Domino plane partitions

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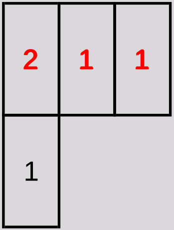
Example

Example

The following domino plane partition π is an element of $\mathscr{D}_3^{(o)}$



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The Stanton-White Bijection

Theorem (Stanton-White)

There are bijections

$$\pi \in \mathscr{D}_n^{(e)} \longleftrightarrow (\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_n,$$

and

$$\pi \in \mathscr{D}_n^{(o)} \longleftrightarrow (\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_{n-1}.$$

By this bijection, we have

$$\overline{U}_k(\pi) = \overline{U}_k(\sigma) + \overline{U}_k(\tau),$$
$$N(\pi) = N(\sigma) + N(\tau).$$

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Tc-symmetric plane partitions and domino plane partitions

Theorem

There is a bijection between domino plane partitions $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row and column lengths are all even and pairs $(\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_n$ (resp. $(\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_{n-1}$) such that σ and τ have the same shape. Especially, there is a pijection between tc-symmetric plane partitions and domino plane partitions in $\mathscr{D}_n^{(e)}$ whose row and column lengths are all even.

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Definition

Let $\mathscr{D}_n^{(e,RC)}$ (resp. $\mathscr{D}_n^{(o,RC)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row and column lengths are both all even. We consider the generating functions

$$T_n^{(e)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(e,RC)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)},$$

and

$$T_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,RC)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

We will see the generating functions does not depend on k later.

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We will see the generating functions does not depend on k later.

Example

$\mathscr{D}_3^{(e,RC)}$ is composed of the following 11 elements;

$$\emptyset,\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline \color{red}{2} & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{2} \\ \hline\end{array},\quad \begin{array}{|c|}\hline \color{red}{2} \\ \hline 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \color{red}{1} & \color{red}{1} \\ \hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline \color{red}{2} & 1 & \color{red}{1} & \color{red}{1} \\ \hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline \color{red}{2} & \color{red}{2} & \color{red}{1} & \color{red}{1} \\ \hline\end{array},\quad \begin{array}{|c|c|c|}\hline \color{red}{2} & \color{red}{1} & \color{red}{1} \\ \cline{1-1} 1 & & \\ \hline\end{array},$$

$$\begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{2} \\ \hline 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline \color{red}{2} & \color{red}{2} & 1 & 1 \\ \hline 1 & 1 \\ \cline{1-2}\end{array}.$$

Example

$$T_3^{(e,RC)}(\tau,t) = 1 + (1 + 2t + t^2)\tau^2 + (2t^2 + 2t^3 + t^4)\tau^4 + t^4\tau^6.$$

A determinant expression

Theorem

Let

$$T^e_{ij}(\tau,t) = \begin{cases} \sum_{k=0}^{\infty} \left\{\binom{i-1}{k-i} + t\binom{i-1}{k-i-1}\right\} \left\{\binom{j-1}{k-j} + t\binom{j-1}{k-j-1}\right\} \tau^{2k-i-j} & \text{if } i,j>0, \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

and

$$T^o_{ij}(\tau,t) = \begin{cases} \sum_{k=0}^{\infty} \left\{\binom{i-1}{k-i} + t\binom{i-1}{k-i-1}\right\} \left\{\binom{j-2}{k-j} + t\binom{j-2}{k-j-1}\right\} \tau^{2k-i-j} & \text{if } i,j-1>0, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

Then we have

$$T^{(e)}_n(\tau,t) = \det\left(T^e_{ij}(\tau,t)\right)_{0\le i,j\le n-1},$$

and

$$T^{(o)}_n(\tau,t) = \det\left(T^o_{ij}(\tau,t)\right)_{0\le i,j\le n-1}.$$

Definition

We define the polynomials $tc_n(t)$ by

$$tc_n(t) = T_n^{(e)}(1, t).$$

Example

A refined enumeration of tc-symmetric plane partitions

Definition

We define the polynomials $tc_n(t)$ by

$$tc_n(t) = T_n^{(e)}(1,t).$$

Example

$$\begin{aligned}
tc_1(t) &= 1\\
tc_2(t) &= 1 + t^2\\
tc_3(t) &= 2 + 2\,t + 3\,t^2 + 2\,t^3 + 2\,t^4\\
tc_4(t) &= 11 + 22\,t + 34\,t^2 + 36\,t^3 + 34\,t^4 + 22\,t^5 + 11\,t^6\\
tc_5(t) &= 170 + 510\,t + 969\,t^2 + 1326\,t^3 + 1479\,t^4 + 1326\,t^5\\
&\quad + 969\,t^6 + 510\,t^7 + 170\,t^8
\end{aligned}$$

Definition

We define the polynomials $tc_n(t)$ by

$$tc_n(t) = T_n^{(e)}(1, t).$$

Observations

$$tc_n(-1) = 2^{n-1} \prod_{i=1}^{n-1} \frac{(6i-6)!(3i+1)!(2i-1)!}{(4i-3)!(4i)!(3i-3)!}$$

$$tc_n(2) = \prod_{i=1}^{n-1} \frac{(6i-1)!(3i-2)!(2i-1)!}{(4i-2)!(4i-1)!(3i-1)!}$$

Column-strict domino plane partitions of even rows

Definition

Let $\mathscr{D}_n^{(e,R)}$ (resp. $\mathscr{D}_n^{(o,R)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row lengths are all even.

Theorem

Let n be a positive integer. We can construct an explicit bijection of $\mathscr{D}_n^{(e,R)}$ onto a subset of TSSCPPs which is defined by Mills, Robbins and Rumsey and conjectured to have the same cardinality with VSASMs. Further we have $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$.

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Example

$\mathscr{D}_1^{(e,R)} = \{\emptyset\}$ is the set of column-strict domino plane partitions with all columns ≤ 0.

Example

Example

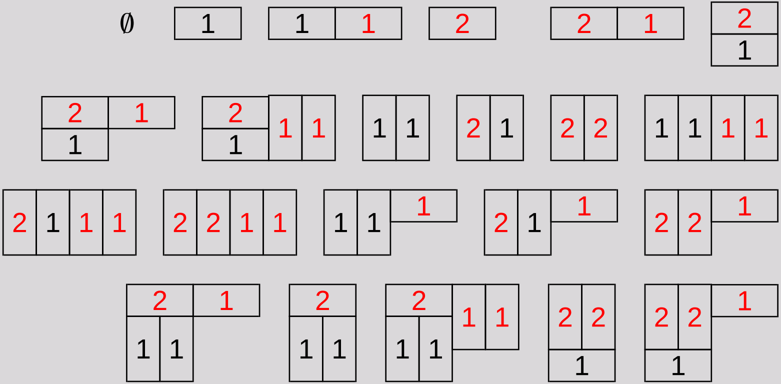
$\mathscr{D}_2^{(e,R)}$ is composed of the following 3 elements:

$\emptyset$,

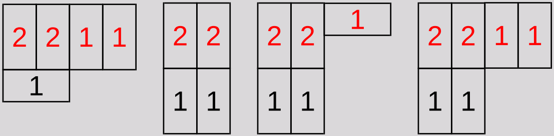
This is the set of column-strict domino plane partitions with the first and second columns ≤ 1, other columns ≤ 0 and each row of even length.

Example

$\mathscr{D}_3^{(e,R)}$ is the set of column-strict domino plane partitions with the 1st and 2nd columns ≤ 2, the 3rd and 4th columns ≤ 1, other columns ≤ 0 and each row of even length (26 elements):



Example



$\mathscr{D}_4^{(e,R)}$ is the set of column-strict domino plane partitions with the 1st and 2nd columns ≤ 3, the 3rd and 4th columns ≤ 2, the 5rd and 6th columns ≤ 1, other columns ≤ 0 and each row of even length (646 elements).

(τ, t)-enumeration

Definition

We consider the generating functions

$$V_n^{(e)}(\tau,t) = \sum_{\pi\in\mathscr{D}_n^{(e,R)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)},$$

and

$$V_n^{(o)}(\tau,t) = \sum_{\pi\in\mathscr{D}_n^{(o,R)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

Example

$$V_3^{(e)}(\tau,t) = 1 + (1+t)\tau + (1+3t+2t^2)\tau^2 + (2t+3t^2+t^3)\tau^3 + (3t^2+3t^3+t^4)\tau^4 + (2t^3+t^4)\tau^5 + t^4\tau^6$$

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$$V_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,R)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

Example

$$V_3^{(e)}(\tau, t) = 1 + (1 + t)\tau + (1 + 3t + 2t^2)\tau^2 + (2t + 3t^2 + t^3)\tau^3 + (3t^2 + 3t^3 + t^4)\tau^4 + (2t^3 + t^4)\tau^5 + t^4\tau^6$$

A determinant expression

Theorem

Let

$$V^e_{ij}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-1}{k-j}+t\binom{j-1}{k-j-1}\right\}\tau^{2k-i-j} \\ +\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i-1}+t\binom{i-1}{k-i-2}\right\}\left\{\binom{j-1}{k-j}+t\binom{j-1}{k-j-1}\right\}\tau^{2k-i-j-1} \\ \qquad \text{if } i,j>0, \\ \delta_{ij} \\ \qquad \text{otherwise,}\end{cases}$$

and

$$V^o_{ij}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-2}{k-j}+t\binom{j-2}{k-j-1}\right\}\tau^{2k-i-j} \\ +\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i-1}+t\binom{i-1}{k-i-2}\right\}\left\{\binom{j-2}{k-j}+t\binom{j-2}{k-j-1}\right\}\tau^{2k-i-j-1} \\ \qquad \text{if } i,j-1>0, \\ \delta_{ij} \\ \qquad \text{otherwise.}\end{cases}$$

Theorem

Then we have

$$V_n^{(e)}(\tau,t) = \det\left(V_{ij}^{e}(\tau,t)\right)_{0\le i,j\le n-1},$$

and

$$V_n^{(o)}(\tau,t) = \det\left(V_{ij}^{o}(\tau,t)\right)_{0\le i,j\le n-1}.$$

Theorem

$$V_n^{(e)}(1,1) = \frac{1}{2^n}\prod_{i=0}^{n-1}\frac{(6i+4)!(2i+1)!}{(4i+2)!(4i+3)!},$$

Conjecture

$$V_n^{(e)}(1,t) = A_{2n+1}^{\mathrm{VS}}(t),$$

A determinant expression

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Theorem

$$V_n^{(e)}(1, 1) = \frac{1}{2^n} \prod_{i=0}^{n-1} \frac{(6i+4)!(2i+1)!}{(4i+2)!(4i+3)!},$$

Conjecture

$$V_n^{(e)}(1, t) = A_{2n+1}^{\mathrm{VS}}(t),$$

Observations

Observations

$$V_n^{(e)}(1,-1) = \left(\frac{3}{4}\right)^{n-1} \prod_{i=1}^{n-1} \frac{(6i-2)!(3i+2)!(2i)!}{(4i-1)!(4i+1)!(3i)!},$$

$$V_n^{(e)}(1,2) = 2^{n-1} \prod_{i=1}^{n-1} \frac{(6i-5)!(2i-2)!}{(4i-4)!(4i-3)!},$$

$$V_n^{(o)}(1,1) = \prod_{i=0}^{n-1} \frac{(6i+4)!(3i+5)!(2i+1)!(2i+3)!i!}{(4i+3)!(4i+6)!(3i+2)!(2i)!(i+2)!},$$

$$V_{n+1}^{(o)}(1,2) = 2^{n-1} \prod_{i=1}^{n-1} \frac{(6i-2)!(2i-1)!}{(4i-3)!(4i)!}.$$

$\tau = -1$ Conjecture

Conjectures

We would have

$$V_n^{(e)}(-1,t) = \begin{cases} \left(A^{\mathrm{VS}}_{2m-1}\right)^2 \, tc_m(t)^2 & \text{if } n = 2m-1, \\ (TC_m)^2 \, (1-t+t^2) \, A^{\mathrm{VS}}_{2m+1}(t)^2 & \text{if } n = 2m, \end{cases}$$

and

$$V_n^{(o)}(-1,t) = \begin{cases} A^{\mathrm{VS}}_{2m-1} \, TC_{m-1} \, A^{\mathrm{VS}}_{2m-1}(t) \, tc_m(t) & \text{if } n = 2m-1, \\ A^{\mathrm{VS}}_{2m-1} \, TC_m \, A^{\mathrm{VS}}_{2m+1}(t) \, tc_m(t) & \text{if } n = 2m, \end{cases}$$

Column-strict domino plane partitions of even columns

Definition

Let $\mathscr{D}_n^{(e,C)}$ (resp. $\mathscr{D}_n^{(o,C)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose column lengths are all even.

Problem

Let n be a positive integer. Can we construct an explicit bijection of $\mathscr{D}_n^{(e,R)}$ onto a subset of TSSCPPs which is defined by Mills, Robbins and Rumsey and conjectured to have the same cardinality with HTSASMs?

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RCSDPPs with all columns of even length

Example

$\mathscr{D}_1^{(e,C)} = \{\emptyset\}$

$\mathscr{D}_1^{(o,C)} = \left\{\emptyset, \boxed{\mathbf{1}}\right\}$

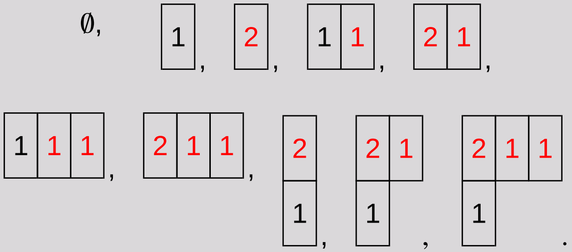
$\mathscr{D}_2^{(e,C)}$ has the following 3 elements:

$$\emptyset, \quad \boxed{1}, \quad \boxed{1\,|\,1}.$$

RCSDPPs with all columns of even length

Example

$\mathscr{D}_3^{(o,C)}$ has the following 10 elements:



$\mathscr{D}_3^{(e,C)}$ has 25 elements, $\mathscr{D}_4^{(e,C)}$ has 140 elements, and $\mathscr{D}_4^{(e,C)}$ has 588 elements.

Definition

Let $\mathscr{D}_n^{(e,C)}$ (resp. $\mathscr{D}_n^{(o,C)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(e)}$) whose column lengths are all even. We consider the generating functions

$$H_n^{(e)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(e,C)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)},$$

and

$$H_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,C)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

Example

$\mathscr{D}_3^{(o,C)}$ consists of the following 10 elements:

$$\emptyset,\quad \begin{array}{|c|}\hline 1 \\ \hline\end{array},\quad \begin{array}{|c|}\hline 2 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 1 & & \\ \cline{1-1}\end{array}.$$

Thus we have

$$H_3^{(o)}(\tau, t) = 1 + (1+t)\tau + (2t + t^2)\tau^2 + (2t^2 + t^3)\tau^3 + t^3\tau^4.$$

A determinant expression

Theorem

Let

$$H^e_{ij}(\tau,t) = \begin{cases} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left\{\binom{i-1}{k-i} + t\binom{i-1}{k-i-1}\right\} \left\{\binom{j-1}{l-j} + t\binom{j-1}{l-j-1}\right\} \tau^{k+l-i-j} & \text{if } i,j > 0, \\ (1+t\tau)(1+\tau)^{i-1} & \text{if } i > 0 \text{ and } j = 0, \\ \delta_{0,j} & \text{if } i = 0, \end{cases}$$

and

$$H^o_{ij}(\tau,t) = \begin{cases} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left\{\binom{i-1}{k-i} + t\binom{i-1}{k-i-1}\right\} \left\{\binom{j-2}{l-j} + t\binom{j-2}{l-j-1}\right\} \tau^{k+l-i-j} & \text{if } i,j-1 > 0, \\ (1+t\tau)(1+\tau)^{i-1} & \text{if } i > 0 \text{ and } j = 0,1, \\ \delta_{ij} & \text{if } i = 0. \end{cases}$$

A determinant expression

Theorem

Then we have

$$H_n^{(e)}(\tau, t) = \det\left(H_{ij}^{e}(\tau, t)\right)_{0\leq,i,j\leq n-1},$$

and

$$H_n^{(o)}(\tau, t) = \det\left(H_{ij}^{o}(\tau, t)\right)_{0\leq,i,j\leq n-1}.$$

Theorem and Conjecture

Theorem

$$H_n^{(e)}(1,1) = \frac{3^n}{2^{2n}} \prod_{i=0}^{n-1} \frac{\{(3i+2)!\, i!\}^2}{\{(2i+1)!\}^4},$$

$$H_n^{(o)}(1,1) = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!\,(i!)^2}{\{(2i)!(2i+1)!\}^2}.$$

Conjecture

$$H_n^{(e)}(1,t) = A_{2n-1}^{\mathrm{HTS}}(t),$$

$$H_n^{(o)}(1,t) = A_{2n}^{\mathrm{HTS}}(t),$$

Theorem and Conjecture

Theorem

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Conjecture

$$H_n^{(e)}(1,t) = A_{2n-1}^{\mathrm{HTS}}(t),$$

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$\tau = -1$ Conjecture

Conjecture

We would have

$$H_n^{(e)}(-1,t) = (1-t+t^2)\,A_{2n-1}^{\mathrm{VS}}(t),$$

and

$$H_n^{(o)}(-1,t) = t(1-t)\,V_{n-2}^{(o)}(1,t) \qquad \text{for } n \geq 3.$$

More General Definition

Definition

Let $\mathscr{P}_{n,m}$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \leq i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $m + n - j$.

(C3) c has at most n columns.

Example

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Example

$\mathscr{P}_{0,4}$ consists of the followng 1 element:

$$\emptyset$$

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(C3) c has at most n columns.

Example

$\mathscr{P}_{1,3}$ consists of the followng 8 elements:

$$\emptyset \quad \boxed{1} \quad \boxed{2} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \quad \boxed{3} \quad \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$$

More General Definition

Example

$\mathscr{P}_{2,2}$ consists of the followng 25 elements:

$$\emptyset \quad \begin{matrix}1\end{matrix} \quad \begin{matrix}1 & 1\end{matrix} \quad \begin{matrix}2\end{matrix} \quad \begin{matrix}2 & 1\end{matrix} \quad \begin{matrix}2 & 2\end{matrix} \quad \begin{matrix}2\\1\end{matrix} \quad \begin{matrix}2 & 1\\1 & \end{matrix}$$

$$\begin{matrix}2 & 2\\1 & \end{matrix} \quad \begin{matrix}2 & 2\\1 & 1\end{matrix} \quad \begin{matrix}3\end{matrix} \quad \begin{matrix}3 & 1\end{matrix} \quad \begin{matrix}3 & 2\end{matrix} \quad \begin{matrix}3\\1\end{matrix} \quad \begin{matrix}3 & 1\\1 & \end{matrix} \quad \begin{matrix}3 & 2\\1 & \end{matrix}$$

$$\begin{matrix}3 & 2\\1 & 1\end{matrix} \quad \begin{matrix}3\\2\end{matrix} \quad \begin{matrix}3 & 1\\2 & \end{matrix} \quad \begin{matrix}3 & 2\\2 & \end{matrix} \quad \begin{matrix}3 & 2\\2 & 1\end{matrix} \quad \begin{matrix}3\\2\\1\end{matrix} \quad \begin{matrix}3 & 1\\2 & \\1 & \end{matrix} \quad \begin{matrix}3 & 2\\2 & \\1 & \end{matrix} \quad \begin{matrix}3 & 2\\2 & 1\\1 & \end{matrix}$$

$\mathscr{P}_{3,1} = \mathscr{P}_{4,0}$ consists of 42 elements.

Using Binet-Caucy formula, we obatin the following theorem:

Theorem

Let $\mathscr{Q}_{n,x,y}$ denote the set of pairs (c_1, c_2) such that $c_1 \in \mathscr{P}_{n,x}$, $c_2 \in \mathscr{P}_{n,y}$, and c_1 and c_2 have the same shape. Then we have

$$\sum_{(c_1,c_2)\in\mathscr{Q}_{n,x,y}} \tau^{|\mathrm{sh}\,c_1|+|\mathrm{sh}\,c_2|} = \det\left[\sum_k \binom{i+x}{k-i}\binom{j+y}{k-j}\tau^{2k-i-j}\right]_{0\le i,j\le n-1}.$$

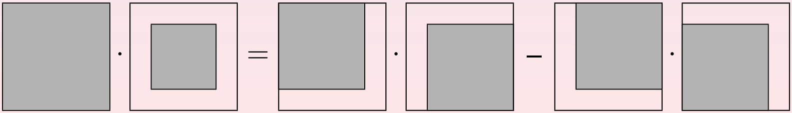
Theorem (Desnanot–Jacobi formula)

Given a matrix M, let

$$M^i_j = \text{the submatrix of } M \text{ obtained by removing row } i \text{ and column } j,$$
$$M^{i,k}_{j,l} = \text{the submatrix of } M \text{ obtained by removing row } i, \text{ row } k, \text{ column } j, \text{ and column } l.$$

Then the Desnanot–Jacobi formula is

$$\det M \cdot \det M^{1,n}_{1,n} = \det M^n_n \cdot \det M^1_1 - \det M^n_1 \cdot \det M^1_n.$$



Hirota-Miwa type equation

Definition

Let

$$f_{n,x,y} = \det\left[\sum_k \binom{i+x}{k-i-x}\binom{j+y}{k-j-y}\tau^{2k-i-j-x-y}\right]_{0\le i,j\le n-1}.$$

Theorem (Hirota-Miwa type equation)

Then $f_{n,x,y}$ satisfies the following equation:

$$f_{n,x,y}f_{n-2,x+1,y+1} = f_{n-1,x,y}f_{n-1,x+1,y+1} - f_{n-1,x+1,y}f_{n,x,y+1},$$

$$f_{0,x,y} = 1, \qquad f_{1,x,y} = \sum_k \binom{x}{k-x}\binom{y}{k-y}\tau^{2k-x-y}.$$

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Hirota-Miwa type equation

Definition

Let

$$g_{n,x,y} = \det\left[\sum_k \left\{\binom{i+x}{k-i-x}\tau^{k-i-x} + \binom{i+x}{k-i-x-1}\tau^{k-i-x-1}\right\} \times \binom{j+y}{k-j-y}\tau^{k-j-y}\right]_{0\le i,j\le n-1}.$$

Theorem (Hirota-Miwa type equation)

Then $g_{n,x,y}$ satisfies the following equation:

$$g_{n,x,y}g_{n-2,x+1,y+1} = g_{n-1,x,y}g_{n-1,x+1,y+1} - g_{n-1,x+1,y}g_{n,x,y+1},$$

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Definition

Let

$$h_{n,x,y} = \det\left[\sum_{k}\sum_{l=0}^{k}\binom{i+x}{k-i-x}\binom{j+y}{l-j-y}\tau^{k+l-i-j-x-y}\right]_{0\le i,j\le n-1}.$$

Theorem (Hirota-Miwa type equation)

Then $h_{n,x,y}$ satisfies the following equation:

$$h_{n,x,y}h_{n-2,x+1,y+1} = h_{n-1,x,y}h_{n-1,x+1,y+1} - h_{n-1,x+1,y}h_{n,x,y+1},$$

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Then $h_{n,x,y}$ satisfies the following equation:

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Thank you!