Enumeration problems of plane partitions and Pfaffian (determinant) expressions

Masao Ishikawa[†]

[†]Department of Mathematics
Tottori University

Combinatorics and Representation Theory,
Graduate School of Mathematics, Nagoya University
September 1 – 5, 2008.

Abstract

Plane parition enumeration is a classical combinatorial problem studyed by MacMahon and have been studied by many people in relations with discrete mathematics, symmetric functions, representation theory and mathematical physics. In this talk we consider certain weighted enumeration problems of two classes of plane partitions, i.e., totally symmetric self-complementary plane partitions (TSSCPP) and cyclically symmetric transpose-complementary plane partions (tc-symmetic PP). We construct one bijection between a subset of TSSCPPs and a class of domino plane partitions and another bijection between tc-symmetic PPs and another class of domino plane partitions. The study of TSSCPPs was started by a paper by Mills, Robbins and Rumsey and they proposed several conjectures in relations with the enumeration problems of alternating sign matrices (ASM). By considering the weighted enumeration of those classes of domino plane partitions we find more mysterious similarities between TSSCPPs (tc-symmetic PPs) and ASMs. We will give Pfaffian (determinant) expressions for those weighted enumeration problems.

Plan of My Talk

1. Plane partitions
2. TSSCPP and tc-symmetric plane partitions
3. Restricted column-strict plane partitions
4. Restricted column-strict domino plane partitions with all rows and columns of even lenth
5. Bender-Knuth type involution
6. Restricted column-strict domino plane partitions with all rows of even lenth
7. Restricted column-strict domino plane partitions with all columns of even lenth

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Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{matrix} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ \vdots & & & & & \\ 0 & 0 & 0 & \ddots & & \end{matrix}$$

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Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

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Example of plane partitions

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- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

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Theorem (MacMahon)

The generating function for plane partitions is

$$\sum_{\pi} q^{|\pi|} = \prod_{k=1}^{\infty} (1 - q^k)^{-k},$$

where the sum runs over all (unrestricted) plane partitions.

Example

$$\prod_{k=1}^{\infty} (1 - q^k)^{-k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$

Generating Function

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Definition

A plane partition is said to be *column-strict* if it is strictly decreasing in coulumns.

Schur functions

Let $x_1, \dots, x_n$ be n variables, and fix a shape λ. The Schur function $s_\lambda(x_1, \dots, x_n)$ is defined to be

$$s_\lambda(x_1, \dots, x_n) = \sum_\pi x^\pi,$$

where π runs over all column-strict plane partitions of shape λ and $x^\pi = \prod_i x_i^{\# \text{ of } i \text{ in } \pi}$.

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An Example of Schur functions

Example

If $\lambda = (22)$ and $\boldsymbol{x} = (x_1, x_2, x_3)$, then the followings are column-strict plane partitions with all parts ≤ 3.

2	2
1	1

3	2
1	1

3	3
1	1

3	2
2	1

3	3
2	1

3	3
2	2

Hence we have

$$s_{(2^2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

Ferrers graph

Definition

The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \left\{(i, j, k) : k \le \pi_{ij}\right\}$$

Example

Ferrers graph

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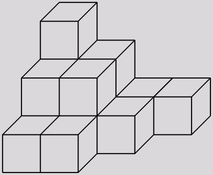


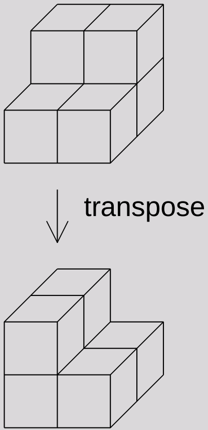
Symmetries of plane paritions

Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example



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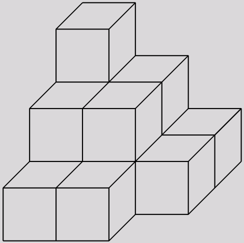
Example

A symmetric PP



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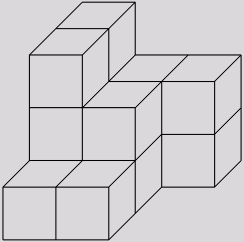
Example

A cyclicaly symmetric PP



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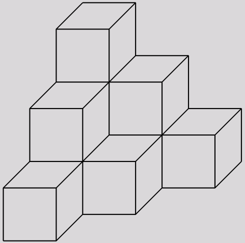
Example

A totally symmetric PP



Complement

Definition

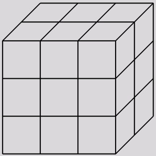
Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r, s, t) = [r] \times [s] \times [t]$.

Define the *complement* π^c of π by

$\pi^c = \{ (r + 1 - i, s + 1 - j, t + 1 - k) : (i, j, k) \notin \pi \}$.

- π is said to be *(r, s, t)-self-complementary* if $\pi = \pi^c$. i.e. $(i, j, k) \in \pi \Leftrightarrow (r + 1 - i, s + 1 - j, t + 1 - k) \notin \pi$.

Example



$B(2, 3, 3)$

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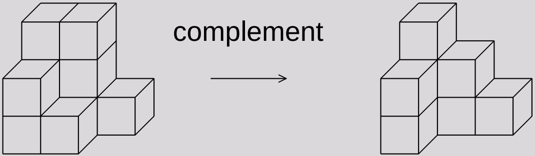

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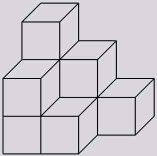

A $(2,3,3)$-self-complementary PP

Transpose-complement

Definition

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- π is said to be *complement=transpose* if $\pi = \pi^{tc}$, i.e. $(i,j,k) \in \pi \Leftrightarrow (r+1-j, r+1-i, t+1-k) \notin \pi$.

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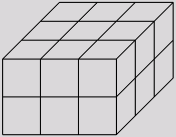

$B(3, 3, 2)$

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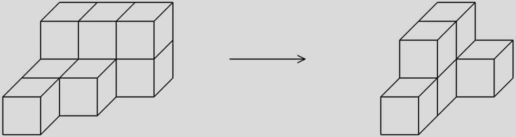

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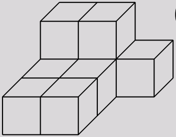

$(3, 3, 2)$-complement=transpoese

Symmetry classes of plane partitions

Symmetry classes (Stanley)

The transformation ^{c} and the group S_3 generate a group T of order 12. The group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems.

Table

1	[illegible]	[illegible]
2	$B(r,r,t)$	*Symmetric*
[illegible]	[illegible]	*Cyclically symmetric*
4	$B(r,r,r)$	*Totally symmetric*
[illegible]	[illegible]	*Self-complementary*
6	$B(r,r,t)$	*Complement = transpose*
[illegible]	[illegible]	*Symmetric and self-complementary*
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
[illegible]	[illegible]	*Cyclically symmetric and self-complementary*
10	$B(r,r,r)$	*Totally symmetric and self-complementary*

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		Self-complementary
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		Symmetric and self-complementary
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
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Table (R. P. Stanley, "Symmetries of Plane Partitions", *J. Combin. Theory Ser.* **A 43**, 103-113 (1986))

1	$B(r,s,t)$	Any
2	$B(r,r,t)$	*Symmetric*
3	$B(r,r,r)$	*Cyclically symmetric*
4	$B(r,r,r)$	*Totally symmetric*
5	$B(r,s,t)$	*Self-complementary*
6	$B(r,r,t)$	*Complement = transpose*
7	$B(r,r,t)$	*Symmetric and self-complementary*
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
9	$B(r,r,r)$	*Cyclically symmetric and self-complementary*
10	$B(r,r,r)$	*Totally symmetric and self-complementary*

Definition

A plane partition contained in $B(2n, 2n, 2n)$ is said to be *totally symmetric self-complementary plane parition of size n* if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.

We denote the set of all self-complementary totally symmetric plane partitions of size n by $\mathscr{T}_n$.

Example

$\mathscr{T}_1$ consists of the single partition

Totally symmetric self-complementary plane partitions

Definition

A plane partition contained in $B(2n, 2n, 2n)$ is said to be *totally symmetric self-complementary plane parition of size n* if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.
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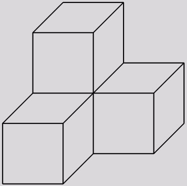

TSSCPPs of size 2

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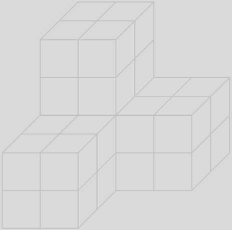
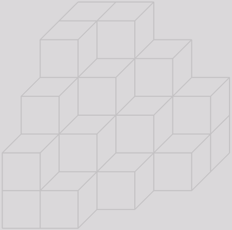
$\mathscr{T}_2$ consists of the following two partitions:





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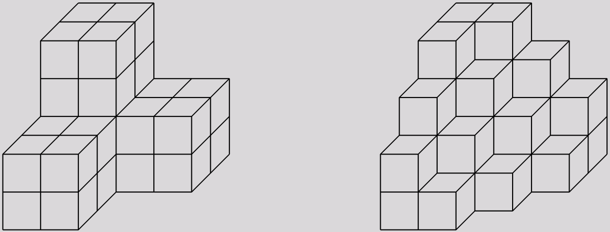
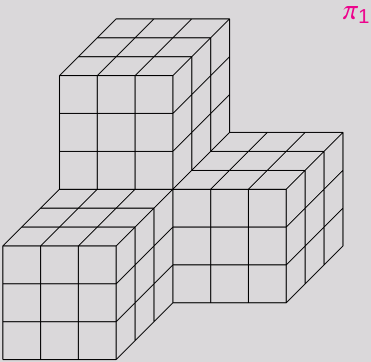
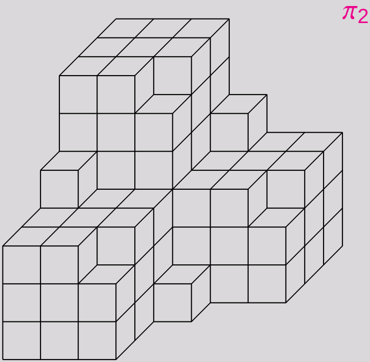


TSSCPPs of size 3

Example

$\mathcal{T}_3$ consists of the following seven partitions:





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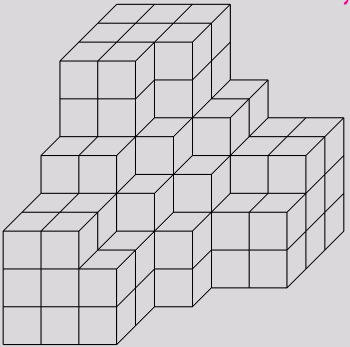
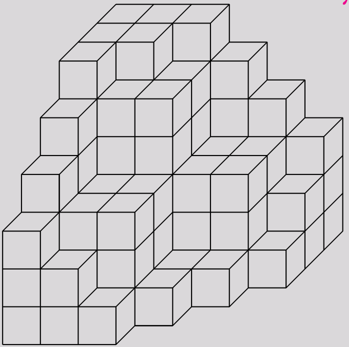


π_3



π_4

TSSCPPs of size 3

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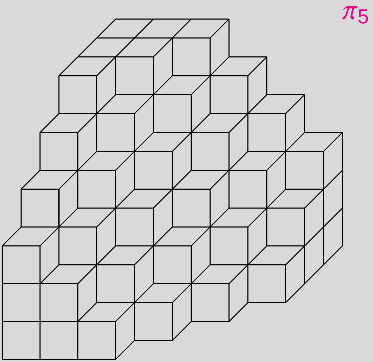

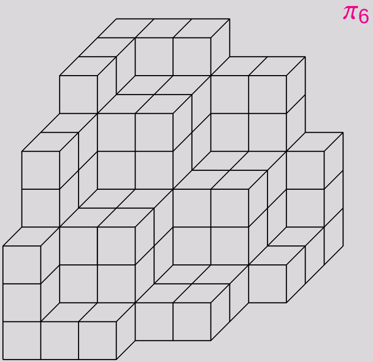

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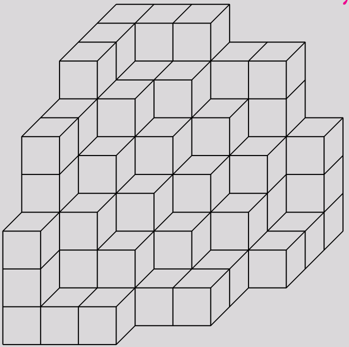

π_7

Tc-symmetric plane partitions

Definition

A plane partition in $B(2n, 2n, 2n)$ is defined to be *tc-symmetric of size n* if it is cyclically symmetric and it is equal to its transpose-complement.

We denote the set of all tc-symmetric plane partitions of size n by $\mathscr{C}_n$.

Example

$\mathscr{C}_1$ consists of the single partition

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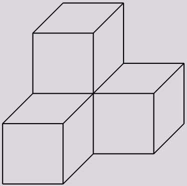
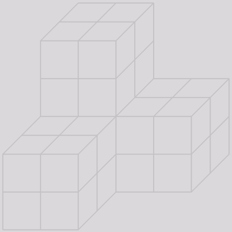
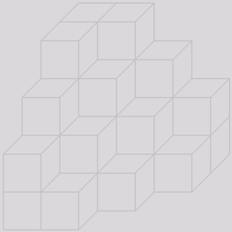


Example

$\mathscr{C}_2$ consists of the following two partitions:





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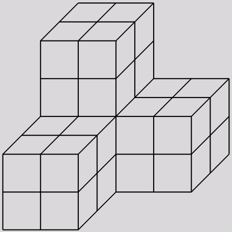
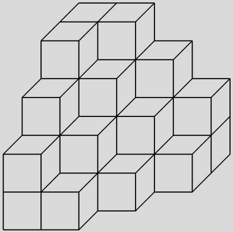
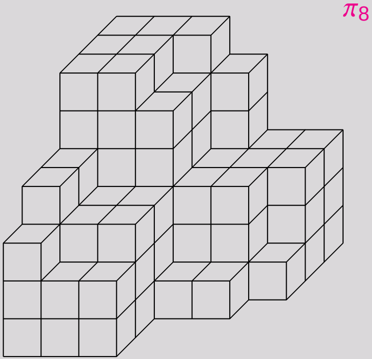


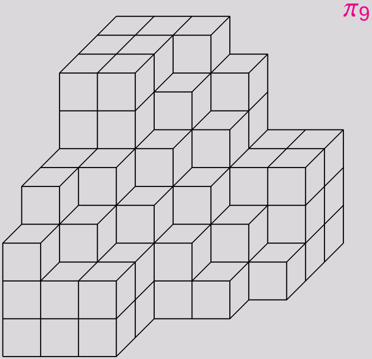


Tc-symmetric PPs of size 3

Example

$\mathscr{C}_3$ consists of the following eleven plane partitions:





Numbers

n	1	2	3	4	5	6	$\cdots$
TSSCPP	1	2	7	42	429	7436	$\cdots$
tc-symmetric PP	1	2	11	170	7429	920460	$\cdots$

Definition

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

$$TC_n = \prod_{i=0}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i)!(4i+1)!}$$

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*. A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

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$$\emptyset \qquad \boxed{1}$$

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Example

$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c|}{1} \\ \hline 1 & \\ \cline{1-1}\end{array}$$

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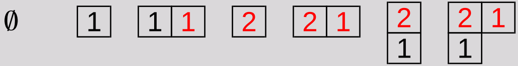

Pairs of Restricted column-strict plane partitions

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Let $\mathscr{Q}_n$ denote the set of all pairs of plane partitions in $\mathscr{P}_n$ of the same shape.

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$$(\emptyset,\emptyset) \quad \left(\boxed{1},\boxed{1}\right) \quad \left(\boxed{2},\boxed{1}\right) \quad \left(\boxed{1},\boxed{2}\right) \quad \left(\boxed{2},\boxed{2}\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array},\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array},\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array},\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array},\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}\right)$$

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Theorem

Let n be a positive integer.
Then we can construct a bijection from $\mathscr{T}_n$ to $\mathscr{P}_n$.

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1	1		2	1

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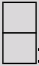
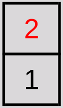
There is 1 RCSPP of shape :



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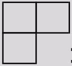
There is 1 RCSPP of shape  :

2	1
1	

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Example ($n = 3$)

This implies

$$1 + 2 + 2 + 1 + 1 = 7$$
$$1^2 + 2^2 + 2^2 + 1^2 + 1^2 = 11$$

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$.

Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, Saturated parts

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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$n = 7$, $c \in \mathscr{P}_3$, $k = 1$, $\overline{U}_1(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 2$, $\overline{U}_2(c) = 5$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 3$, $\overline{U}_3(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 4$, $\overline{U}_4(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \leq i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 5$, $\overline{U}_5(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

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Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 6$, $\overline{U}_6(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$.
Let $\overline{U}_k(c)$ denote the number of parts equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of boxes in π.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 7$, $\overline{U}_7(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Domino plane partitions

Definition

Let $\mathscr{D}_n^{(e)}$ denote the set of column-strict domino plane partitions c subject to the constraints that

1. each number in a domino crossing the $2j-1$st column does not exceed $n-j$,
2. each number in a domino crossing the $2j$th column does not exceed $n-j$,

for $j = 1, \dots, n-1$. If a part in the $2j-1$th or $2j$th column is equal to $n-j$, then we call it a *saturated* part. For a positive integer k and $\pi \in \mathscr{D}_n^{(e)}$, set $\overline{U}_k(\pi)$ denote the number of parts in c equal to k plus the number of saturated parts less than k. Further let $N(\pi)$ denote the number of dominoes in π.

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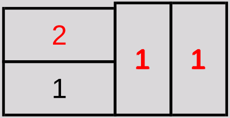
Example

Example

The following domino plane partition π is an element of $\mathscr{D}_3^{(e)}$



since the 1st and 2nd columns ≤ 2, the 3rd and 4th columns ≤ 1. The red numbers stand for saturated parts. Hence we have $\overline{U}_1(\pi) = \overline{U}_2(\pi) = \overline{U}_3(\pi) = 3$. Since π has 4 dominoes, we have $N(\pi) = 4$.

Domino plane partitions

Definition

Let $\mathscr{D}_n^{(o)}$ denote the set of column-strict domino plane partitions c subject to the constraints that

1. each number in a domino crossing the $2j-1$st column does not exceed $n-j$,
2. each number in a domino crossing the $2j$th column does not exceed $n-j-1$,

for $j = 1, \dots, n-1$. The statistics $\overline{U}_k(\pi)$ and can be defined similarly.

Domino plane partitions

Definition

Let $\mathscr{D}_n^{(o)}$ denote the set of column-strict domino plane partitions c subject to the constraints that

1. each number in a domino crossing the $2j-1$st column does not exceed $n-j$,
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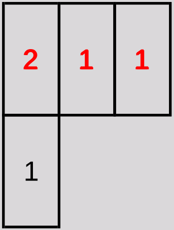
The following domino plane partition π is an element of $\mathscr{D}_3^{(o)}$



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The Stanton-White Bijection

Theorem (Stanton-White)

There are bijections

$$\pi \in \mathscr{D}_n^{(e)} \longleftrightarrow (\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_n,$$

and

$$\pi \in \mathscr{D}_n^{(o)} \longleftrightarrow (\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_{n-1}.$$

By this bijection, we have

$$\overline{U}_k(\pi) = \overline{U}_k(\sigma) + \overline{U}_k(\tau),$$
$$N(\pi) = N(\sigma) + N(\tau).$$

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Tc-symmetric plane partitions and domino plane partitions

Corollary

There is a bijection between domino plane partitions $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row and column lengths are all even and pairs $(\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_n$ (resp. $(\sigma, \tau) \in \mathscr{P}_n \times \mathscr{P}_{n-1}$) such that σ and τ have the same shape. Especially, there is a pijection between tc-symmetric plane partitions and domino plane partitions in $\mathscr{D}_n^{(e)}$ whose row and column lengths are all even.

Tc-symmetric plane partitions and domino plane partitions

Corollary

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Definition

Let $\mathscr{D}_n^{(e,RC)}$ (resp. $\mathscr{D}_n^{(o,RC)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row and column lengths are both all even. We consider the generating functions

$$T_n^{(e)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(e,RC)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)},$$

and

$$T_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,RC)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

We will see the generating functions does not depend on k later.

(τ, t)-enumeration of tc-symmetric plane partitions

Definition

Let $\mathscr{D}_n^{(e,RC)}$ (resp. $\mathscr{D}_n^{(o,RC)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row and column lengths are both all even. We consider the generating functions

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and

$$T_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,RC)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

We will see the generating functions does not depend on k later.

Example

$\mathscr{D}_3^{(e,RC)}$ is composed of the following 11 elements;

$$\emptyset,\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline \color{red}{2} & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{2} \\ \hline\end{array},\quad \begin{array}{|c|}\hline \color{red}{2} \\ \hline 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \color{red}{1} & \color{red}{1} \\ \hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline \color{red}{2} & 1 & \color{red}{1} & \color{red}{1} \\ \hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline \color{red}{2} & \color{red}{2} & \color{red}{1} & \color{red}{1} \\ \hline\end{array},\quad \begin{array}{|c|c|c|}\hline \begin{array}{c}\color{red}{2} \\ \hline 1\end{array} & \color{red}{1} & \color{red}{1} \\ \hline\end{array},$$

$$\begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{2} \\ \hline 1 & 1 \\ \hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline \color{red}{2} & \color{red}{2} & 1 & 1 \\ \hline 1 & 1 & & \\ \cline{1-2}\end{array}.$$

Example

$$T_3^{(e)}(\tau, t) = 1 + (1 + 2t + t^2)\tau^2 + (2t^2 + 2t^3 + t^4)\tau^4 + t^4\tau^6.$$

A determinant expression

Theorem

Let

$$T_{ij}^{e}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-1}{k-j}+t\binom{j-1}{k-j-1}\right\}\tau^{2k-i-j} & \text{if } i,j>0,\\ \delta_{ij} & \text{otherwise,}\end{cases}$$

and

$$T_{ij}^{o}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-2}{k-j}+t\binom{j-2}{k-j-1}\right\}\tau^{2k-i-j} & \text{if } i,j-1>0,\\ \delta_{ij} & \text{otherwise.}\end{cases}$$

Then we have

$$T_n^{(e)}(\tau,t)=\det\left(T_{ij}^{e}(\tau,t)\right)_{0\le i,j\le n-1},$$

and

$$T_n^{(o)}(\tau,t)=\det\left(T_{ij}^{o}(\tau,t)\right)_{0\le i,j\le n-1}.$$

Definition

We define the polynomials $tc_n(t)$ by

$$tc_n(t) = T_n^{(e)}(1, t).$$

Example

A refined enumeration of tc-symmetric plane partitions

Definition

We define the polynomials $tc_n(t)$ by

$$tc_n(t) = T_n^{(e)}(1, t).$$

Example

$$tc_1(t) = 1$$
$$tc_2(t) = 1 + t^2$$
$$tc_3(t) = 2 + 2\,t + 3\,t^2 + 2\,t^3 + 2\,t^4$$
$$tc_4(t) = 11 + 22\,t + 34\,t^2 + 36\,t^3 + 34\,t^4 + 22\,t^5 + 11\,t^6$$
$$tc_5(t) = 170 + 510\,t + 969\,t^2 + 1326\,t^3 + 1479\,t^4 + 1326\,t^5 + 969\,t^6 + 510\,t^7 + 170\,t^8$$

A refined enumeration of tc-symmetric plane partitions

Definition

We define the polynomials $tc_n(t)$ by

$$tc_n(t) = T_n^{(e)}(1, t).$$

Observations

$$tc_n(-1) = 2^{n-1} \prod_{i=1}^{n-1} \frac{(6i-6)!(3i+1)!(2i-1)!}{(4i-3)!(4i)!(3i-3)!}$$

$$tc_n(2) = \prod_{i=1}^{n-1} \frac{(6i-1)!(3i-2)!(2i-1)!}{(4i-2)!(4i-1)!(3i-1)!}$$

Mills-Robbins-Rumsey bijection

Mills, Robbins and Rumsey have constructed a bijection between TSSCPPs and a certain set of shifted plane partitions:

$$\mathscr{T}_n \longleftrightarrow \mathscr{B}_n = \{\text{shifted plane partitions}\}$$

Flips

They also define an involution π_k from this set of shifted plane partitions onto itself:

$$\pi_k : \mathscr{B}_n \to \mathscr{B}_n$$

for $k = 1, 2, \ldots, n$.

Mills-Robbins-Rumsey Conjectures

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Mills-Robbins-Rumsey Conjectures

Definition

They define two important involutions on $\mathscr{B}_n$

$$\rho = \pi_2 \pi_4 \pi_6 \cdots ,$$

$$\gamma = \pi_1 \pi_3 \pi_5 \cdots ,$$

and put $\mathscr{B}_n^{\rho}$ (resp. $\mathscr{B}_n^{\gamma}$) the set of elements $\mathscr{B}_n$ invariant under ρ (resp. γ).

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Conjecture 4 (Conjecture 4 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, $0 \leq r \leq n$ be integers. Then the number of elements c in $\mathscr{B}_n$ with $\rho(c) = c$ and $U_1(c) = r$ would be the same as the number of n by n alternating sign matrices a invariant under the half turn in their own planes (that is $a_{ij} = a_{n+1-i,n+1-j}$ for $1 \leq i, j \leq n$) and satisfying $a_{1,r} = 1$.

Mills-Robbins-Rumsey Conjectures

Definition

They define two important involutions on $\mathscr{B}_n$

$$\rho = \pi_2\pi_4\pi_6\cdots,$$
$$\gamma = \pi_1\pi_3\pi_5\cdots,$$

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Conjecture 6 (Conjecture 6 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 3$ an odd integer and i, $0 \leq i \leq n-1$ be an integer. Then the number of c in $\mathscr{B}_n$ with $\gamma(c) = c$ and $U_2(c) = i$ would be the same as the number of n by n alternating sign matrices with $a_{i1} = 1$ and which are invariant under the vertical flip (that is $a_{ij} = a_{i,n+1-j}$ for $1 \leq i, j \leq n$).

The Numbers of HTSASMs and VSASMs

Definition

$$A_{2n}^{\mathrm{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{\{(n+i)!\}^2} \qquad A_{2n+1}^{\mathrm{HTS}} = \frac{n!(3n)!}{\{(2n)!\}^2} \cdot A_{2n}^{\mathrm{HTS}},$$

$$A_{2n+1}^{\mathrm{VS}} = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

Example

n	1	2	3	4	5	6	7	8	9	$\cdots$
A_n^{HTS}	1	2	3	10	25	140	588	5544	39204	$\cdots$
A_n^{VS}	1		1		3		26		646	$\cdots$

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$$A^{\text{HTS}}_{2n} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{\{(n+i)!\}^2} \qquad A^{\text{HTS}}_{2n+1} = \frac{n!(3n)!}{\{(2n)!\}^2} \cdot A^{\text{HTS}}_{2n},$$

$$A^{\text{VS}}_{2n+1} = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

Example

n	1	2	3	4	5	6	7	8	9	$\cdots$
A^{HTS}_n	1	2	3	10	25	140	588	5544	39204	$\cdots$
A^{VS}_n	1		1		3		26		646	$\cdots$

Enumeration polynomials

Definition

$$A^{\mathrm{VS}}_{2n+1}(t) = \frac{A^{\mathrm{VS}}_{2n-1}}{(4n-2)!} \sum_{r=1}^{2n} t^{r-1} \sum_{k=1}^{r} (-1)^{r+k} \frac{(2n+k-2)!(4n-k-1)!}{(k-1)!(2n-k)!},$$

Example

$$A^{\mathrm{VS}}_{3}(t) = 1$$

$$A^{\mathrm{VS}}_{5}(t) = 1 + t + t^2$$

$$A^{\mathrm{VS}}_{7}(t) = 3 + 6t + 8t^2 + 6t^3 + 3t^4$$

$$A^{\mathrm{VS}}_{9}(t) = 26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$$

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Enumeration polynomials

Definition

$$A_n(t) = \frac{A_n}{\binom{3n-2}{n-1}} \sum_{r=1}^{n} \binom{n+r-2}{n-1}\binom{2n-1-r}{n-1} t^{r-1}$$

$$\frac{\widetilde{A}_{2n}^{\mathrm{HTS}}(t)}{\widetilde{A}_{2n}^{\mathrm{HTS}}} = \frac{(3n-2)(2n-1)!}{(n-1)!(3n-1)!}$$

$$\times \sum_{r=0}^{n} \frac{\{n(n-1)-nr+r^2\}(n+r-2)!(2n-r-2)!}{r!(n-r)!} t^r$$

$$A_{2n}^{\mathrm{HTS}}(t) = \widetilde{A}_{2n}^{\mathrm{HTS}}(t) A_n(t)$$

$$A_{2n+1}^{\mathrm{HTS}}(t) = \frac{1}{3}\left\{A_{n+1}(t)\widetilde{A}_{2n}^{\mathrm{HTS}}(t) + A_n(t)\widetilde{A}_{2n+2}^{\mathrm{HTS}}(t)\right\}$$

where $\widetilde{A}_{2n}^{\mathrm{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{(3i+1)!(n+i)!}$.

Example

Example

$$A_1^{\mathrm{HTS}}(t) = 1$$
$$A_2^{\mathrm{HTS}}(t) = 1 + t$$
$$A_3^{\mathrm{HTS}}(t) = 1 + t + t^2$$
$$A_4^{\mathrm{HTS}}(t) = 2 + 3t + 3t^2 + 2t^3$$
$$A_5^{\mathrm{HTS}}(t) = 3 + 6t + 7t^2 + 6t^3 + 3t^4$$

The Bender-Knuth involution

A classical method to prove that a Schur function is symmetric is to define involutions f_k on column-strict plane partitions c which swaps the number of k's and $(k-1)$'s, for each k. Consider the parts of c equal to k or $k-1$. If both of k and $k-1$ appear in the same column, we say k and $k-1$ paired. The other unpaired k's and $k-1$'s are swaped in each row.

f_k acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	2	2
4	4	3	2	2	2	1	1		
3	2	1	1						
2	1								

The Bender-Knuth involution

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Example

f_2 acts on the following column-strict plane partitions:

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Example

f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	2	2
4	4	3	2	2	2	1	1		
3	2	1	1						
2	1								

The Bender-Knuth involution

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Example

f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	2	2
4	4	3	2	2	2	1	1		
3	2	1	1						
2	1								

The Bender-Knuth involution

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A classical method to prove that a Schur function is symmetric is to define involutions f_k on column-strict plane partitions c which swaps the number of k's and $(k-1)$'s, for each k. Consider the parts of c equal to k or $k-1$. If both of k and $k-1$ appear in the same column, we say k and $k-1$ paired. The other unpaired k's and $k-1$'s are swaped in each row.

Example

f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	1	1
4	4	3	2	2	1	1	1		
3	2	2	1						
1	1								

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Example

f_2 acts on the following column-strict plane partitions:

5	5	4	3	3	3	3	2	2	2
4	4	3	2	2	2	1	1		
3	2	1	1						
2	1								

The Bender-Knuth involution

Remark

f_2 gives a proof of

$$s_\lambda(x_2, x_1, x_3, \ldots, x_n) = s_\lambda(x_1, x_2, x_3, \ldots, x_n).$$

Hence $s_\lambda(x_1, x_2, \ldots, x_n)$ is a symmetric function.

A Bender-Knuth Type involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps k's and $(k-1)$'s where we ignore saturated $(k-1)$ when we perform a swap.

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Example

$n = 7$ Apply $\widetilde{\pi}_3$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

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1				

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Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_3(c) \in \mathscr{P}_3$.

5	5	4	3	2
4	4	3	1	
3	3	2		
2	1			
1				

A Bender-Knuth Type involution

Definition

We define an involution $\widetilde{\pi}_1$ on $\mathscr{P}_n$ similarly assuming the outside of the shape is filled with 0.

Example

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5	5	4	3	2
4	4	3	2	1
3	1			
1				

A Bender-Knuth Type involution

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We define an involution $\widetilde{\pi}_1$ on $\mathscr{P}_n$ similarly assuming the outside of the shape is filled with 0.

Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	3	2	1
4	4	3	2		
3	1	1			

Flips in words of RCSPP

Proposition

If $\sigma \in \mathscr{P}_n$ and $k \geq 2$, then

$$\overline{U}_{k-1}\left(\pi_k(\sigma)\right) = \overline{U}_k\left(\sigma\right)$$
$$N\left(\pi_k(\sigma)\right) = N\left(\sigma\right)$$

Definition

We define involutions on $\mathscr{P}_n$

$$\widetilde{\rho} = \widetilde{\pi}_2\widetilde{\pi}_4\widetilde{\pi}_6\cdots,$$
$$\widetilde{\gamma} = \widetilde{\pi}_1\widetilde{\pi}_3\widetilde{\pi}_5\cdots,$$

and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

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Example

$\mathscr{P}_1^{\widetilde{\rho}} = \{\emptyset\}$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_2^{\widetilde{\rho}} = \left\{ \emptyset, \boxed{1} \right\}$

Example

$\mathscr{P}_3^{\widetilde{\rho}}$ is composed of the following 3 RCSPPs:

$\emptyset$

2	
1	

2	1
1	

Example

$\mathscr{P}_4^{\widetilde{\rho}}$ is composed of the following 10 elements:

$$\emptyset \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 1 & 1 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Example

$\mathscr{P}_5^{\widetilde{\rho}}$ has 25 elements, and $\mathscr{P}_6^{\widetilde{\rho}}$ has 140 elements.

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

1	1

3	2	1
1		

3	3	1
2	2	
1		

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

The following $c \in \mathscr{P}_{11}$ is invariant under $\widetilde{\gamma}$:

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Remove all 1's from $c \in \mathscr{P}_{11}^{\widetilde{\gamma}}$.

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Then we obtain a PP in which each row has even length.

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Identify 3 with 2, 5 with 4, and 7 with 6.

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

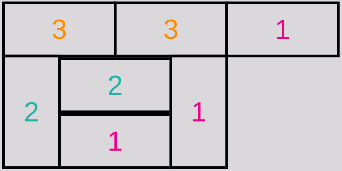
Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Repace 3 and 2 by dominos containing 1, 5 and 4 by dominos containing 2, 7 and 6 by dominos containing 3.



Definition

Let $\mathscr{D}_n^{(e,R)}$ (resp. $\mathscr{D}_n^{(o,R)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row lengths are all even.

Theorem

Let n be a positive integer. Let τ_{2n+1} denote our bijection of $\mathscr{P}_{2n+1}^{\widetilde{\gamma}}$ onto $\mathscr{D}_n^{(e,R)}$. Further we have $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$.

Column-strict domino plane partitions of even rows

Definition

Let $\mathscr{D}_n^{(e,R)}$ (resp. $\mathscr{D}_n^{(o,R)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(o)}$) whose row lengths are all even.

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Example

Example

$\mathscr{D}_1^{(e,R)} = \{\emptyset\}$ is the set of column-strict domino plane partitions with all columns ≤ 0.

Example

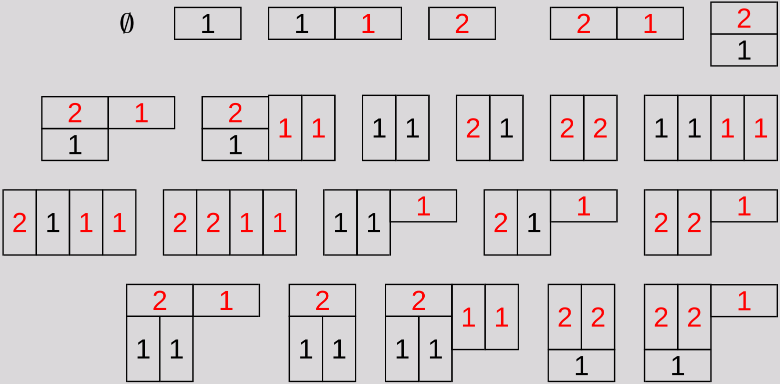
$\mathscr{D}_2^{(e,R)}$ is composed of the following 3 elements:

$\emptyset$, | 1 |, | 1 | 1 |.

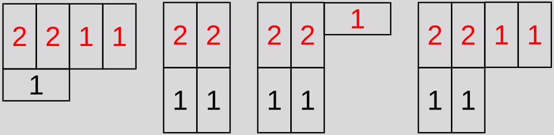
This is the set of column-strict domino plane partitions with the first and second columns ≤ 1, other columns ≤ 0 and each row of even length.

Example

$\mathscr{D}_3^{(e,R)}$ is the set of column-strict domino plane partitions with the 1st and 2nd columns ≤ 2, the 3rd and 4th columns ≤ 1, other columns ≤ 0 and each row of even length (26 elements):



Example



$\mathscr{D}_4^{(e,R)}$ is the set of column-strict domino plane partitions with the 1st and 2nd columns ≤ 3, the 3rd and 4th columns ≤ 2, the 5rd and 6th columns ≤ 1, other columns ≤ 0 and each row of even length (646 elements).

(τ, t)-enumeration

Definition

We consider the generating functions

$$V_n^{(e)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(e,R)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)},$$

and

$$V_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,R)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

Example

$$V_3^{(e)}(\tau, t) = 1 + (1+t)\tau + (1+3t+2t^2)\tau^2 + (2t+3t^2+t^3)\tau^3 + (3t^2+3t^3+t^4)\tau^4 + (2t^3+t^4)\tau^5 + t^4\tau^6$$

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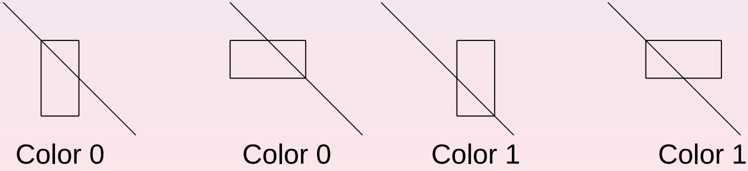
Example

Theorem (Stanton-White, Carré-Leclerc)

We can define a map which associate a pair in $\mathscr{P}_n \times \mathscr{P}_n$ (resp. $\mathscr{P}_n \times \mathscr{P}_{n-1}$) with a domino plane partition in $\mathscr{D}_n^{(e)}$ (resp. $\mathscr{D}_n^{(o)}$).
Let Φ denote the map which associate the pair (c_0, c_1) of column-strict plane partitions with a column-strict domino plane partition d.



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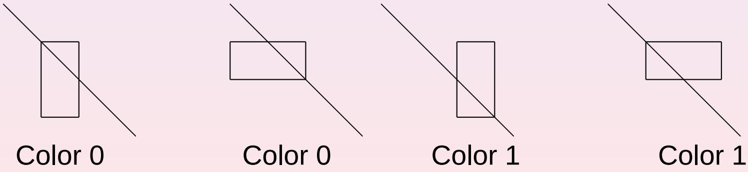
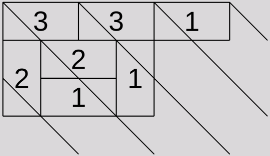


Domino plane partition

Example

For example, we associate the column-strict domino plane partition

$d =$



the pair

$c_0 =$

1	1

$c_1 =$

3	3	1
2	2	

of plane partitions.

Conditions on shape

Theorem

Let d be a column-strict domino plane partition, and let $(c_0, c_1) = \Phi(d)$. Then

(i) All columns of d have even length if, and only if, $\mathrm{sh}\, c_1 \subseteq \mathrm{sh}\, c_0$ and $\mathrm{sh}\, c_0 \setminus \mathrm{sh}\, c_1$ is a vertical strip.

(ii) All rows of d have even length if, and only if, $\mathrm{sh}\, c_0 \subseteq \mathrm{sh}\, c_1$ and $\mathrm{sh}\, c_1 \setminus \mathrm{sh}\, c_0$ is a horizontal strip.

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From RCSPPs to lattce paths

Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \le y \le x\}$ *be the vertex set, and direct an edge from u to v whenever* $v - u = (1, -1)$ *or* $(0, -1)$.
Let $u_j = (n - j, n - j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \ldots, n$*, and let* $\boldsymbol{u} = (u_1, \ldots, u_n)$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n)$. *We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with n-tuples of nonintersecting D-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$.

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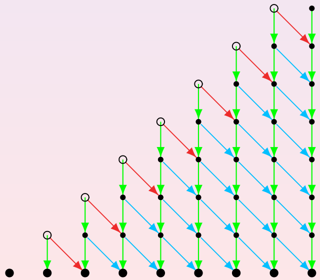


Example of lattice paths

Example

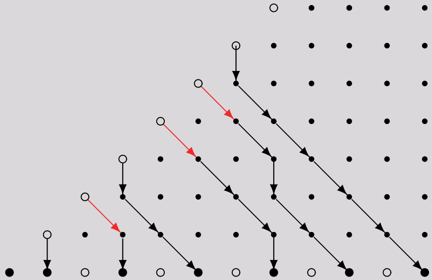
$n = 7$, $c \in \mathscr{P}_7$: RCSPP

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Example of lattice paths

Example

Lattice paths



A determinant expression

Theorem

Let

$$V^{e}_{ij}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-1}{k-j}+t\binom{j-1}{k-j-1}\right\}\tau^{2k-i-j} \\ +\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i-1}+t\binom{i-1}{k-i-2}\right\}\left\{\binom{j-1}{k-j}+t\binom{j-1}{k-j-1}\right\}\tau^{2k-i-j-1} \\ \qquad \text{if } i,j>0, \\ \delta_{ij} \\ \qquad \text{otherwise,}\end{cases}$$

and

$$V^{o}_{ij}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-2}{k-j}+t\binom{j-2}{k-j-1}\right\}\tau^{2k-i-j} \\ +\sum_{k=0}^{\infty}\left\{\binom{i-1}{k-i-1}+t\binom{i-1}{k-i-2}\right\}\left\{\binom{j-2}{k-j}+t\binom{j-2}{k-j-1}\right\}\tau^{2k-i-j-1} \\ \qquad \text{if } i,j-1>0, \\ \delta_{ij} \\ \qquad \text{otherwise.}\end{cases}$$

A determinant expression

Theorem

Then we have

$$V_n^{(e)}(\tau, t) = \det\left(V_{ij}^{e}(\tau, t)\right)_{0 \le i,j \le n-1},$$

and

$$V_n^{(o)}(\tau, t) = \det\left(V_{ij}^{o}(\tau, t)\right)_{0 \le i,j \le n-1}.$$

Conjecture

$$V_n^{(e)}(1, t) = A_{2n+1}^{\mathrm{VS}}(t),$$

A determinant expression

Theorem

Then we have

$$V_n^{(e)}(\tau,t) = \det\left(V_{ij}^{e}(\tau,t)\right)_{0\le,i,j\le n-1},$$

and

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Conjecture

$$V_n^{(e)}(1,t) = A_{2n+1}^{\mathrm{VS}}(t),$$

Observations

We would have

$$V_n^{(e)}(-1,t) = \begin{cases} \left(A_{2m-1}^{\mathrm{VS}}\right)^2 tc_m(t)^2 & \text{if } n = 2m-1, \\ (TC_m)^2 \, (1 - t + t^2) \, A_{2m+1}^{\mathrm{VS}}(t)^2 & \text{if } n = 2m, \end{cases}$$

and

$$V_n^{(o)}(-1,t) = \begin{cases} A_{2m-1}^{\mathrm{VS}} \, TC_{m-1} \, A_{2m-1}^{\mathrm{VS}}(t) \, tc_m(t) & \text{if } n = 2m-1, \\ A_{2m-1}^{\mathrm{VS}} \, TC_m \, A_{2m+1}^{\mathrm{VS}}(t) \, tc_m(t) & \text{if } n = 2m, \end{cases}$$

Generalized domino plane partitions

Generalized domino plane partitions

A *domino* is a special kind of skew shape consists of two squares. A 1×2 domino is called a *horizontal domino* while a 2×1 domino is called a *vertical domino*. A *generalized domino plane partition of shape* λ consists of a tiling of the shape λ by means of ordinary 1×1 squares or dominoes, and a filling of each square or domino with a positive integer so that the integers are weakly decreasing along either rows or columns. Further we call it a *domino plane partition* if the shape λ is tiled with only dominoes.

Generalized domino plane partitions

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A *domino* is a special kind of skew shape consists of two squares. A 1×2 domino is called a *horizontal domino* while a 2×1 domino is called a *vertical domino*. A *generalized domino plane partition of shape* λ consists of a tiling of the shape λ by means of ordinary 1×1 squares or dominoes, and a filling of each square or domino with a positive integer so that the integers are weakly decreasing along either rows or columns. Further we call it a *domino plane partition* if the shape λ is tiled with only dominoes.

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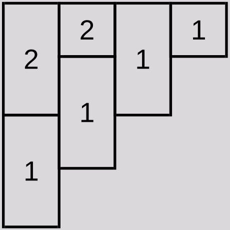
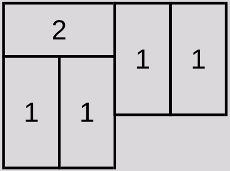
Generalized domino plane partitions

Example

The left-below is a column-strict generalized domino plane partition of shape $(4, 3, 2, 1)$, and the right-below is a column-strict domino plane partition of shape $(4, 4, 2)$.





Twisted domino plane partitions

Definition

Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{P}_n^{\mathrm{HTS}}$ denote the set of column-strict generalized domino plane partitions c subject to the constraints that

(E1) c has at most n columns;

(E2) each part in the jth column does not exceed $\lceil (n-j)/2 \rceil$;

(E3) A domino containing $\lceil (n-j)/2 \rceil$ must not cross the jth column for any j such that $n-j$ is odd.

(E4) A single box can appear only when it contains $\lceil (n-j)/2 \rceil$ and it is in the jth column such that $n-j$ is odd.

We call an element in $\mathscr{P}_n^{\mathrm{HTS}}$ a *twisted domino plane partition*.

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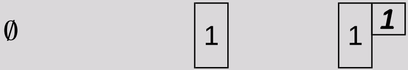
We call an element in $\mathscr{P}_n^{\mathrm{HTS}}$ a *twisted domino plane partition*.

Example

$\mathscr{P}_1^{\text{HTS}} = \{\emptyset\}$

$\mathscr{P}_2^{\text{HTS}} = \left\{\emptyset, \boxed{\boldsymbol{1}}\right\}$

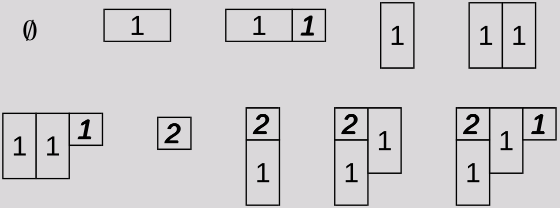
$\mathscr{P}_3^{\text{HTS}}$ is composed of the following 3 elements:



Twisted domino plane partitions

Example

$\mathscr{P}_4^{\mathrm{HTS}}$ is composed of the following 10 elements:



$\mathscr{P}_5^{\mathrm{HTS}}$ has 25 elements and $\mathscr{P}_6^{\mathrm{HTS}}$ has 140 elements.

Twisted domino PPs and RCSDPPs with all columns of even length

Conjecture

For a positive integer n, there would be a bijection between $\mathscr{P}_n^{\mathrm{HTS}}$ (the set of twisted domono PPs) and $\mathscr{D}_n^{(e,C)}$ or $\mathscr{D}_n^{(o,C)}$ (the set of restricted column-strict domino PPs with all columns of even length) which has the following property;

1. the numeber of 1's is kept invariant;
2. the number of columns is kept invariant.

Twisted domino PPs and RCSDPPs with all columns of even length

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For a positive integer n, there would be a bijection between $\mathscr{P}_n^{\mathrm{HTS}}$ (the set of twisted domono PPs) and $\mathscr{D}_n^{(e,C)}$ or $\mathscr{D}_n^{(o,C)}$ (the set of restricted column-strict domino PPs with all columns of even length) which has the following property;

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1. the numeber of 1's is kept invariant;
2. the number of columns is kept invariant.

RCSDPPs with all columns of even length

Example

$\mathscr{D}_1^{(e,C)} = \{\emptyset\}$

$\mathscr{D}_1^{(o,C)} = \left\{\emptyset, \boxed{\mathbf{1}}\right\}$

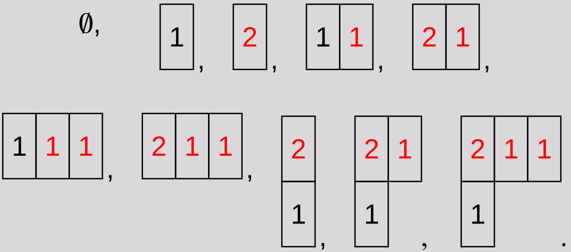
$\mathscr{D}_2^{(e,C)}$ has the following 3 elements:

$$\emptyset, \qquad \boxed{1}, \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}.$$

RCSDPPs with all columns of even length

Example

$\mathscr{D}_3^{(o,C)}$ has the following 10 elements:



$\mathscr{D}_3^{(e,C)}$ has 25 elements, $\mathscr{D}_4^{(e,C)}$ has 140 elements, and $\mathscr{D}_4^{(e,C)}$ has 588 elements.

Definition

Let $\mathscr{D}_n^{(e,C)}$ (resp. $\mathscr{D}_n^{(o,C)}$) denote the set of $\pi \in \mathscr{D}_n^{(e)}$ (resp. $\pi \in \mathscr{D}_n^{(e)}$) whose column lengths are all even. We consider the generating functions

$$H_n^{(e)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(e,C)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)},$$

and

$$H_n^{(o)}(\tau, t) = \sum_{\pi \in \mathscr{D}_n^{(o,C)}} \tau^{N(\pi)} t^{\overline{U}_k(\pi)}.$$

Example

Example

$\mathscr{D}_3^{(o,C)}$ consists of the following 10 elements:

$$\emptyset,\quad \begin{array}{|c|}\hline 1\\\hline\end{array},\quad \begin{array}{|c|}\hline 2\\\hline\end{array},\quad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array},\quad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array},\quad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline\end{array},\quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array},\quad \begin{array}{|c|c|}\hline 2&1\\\hline 1\\\cline{1-1}\end{array},\quad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 1\\\cline{1-1}\end{array}.$$

Thus we have

$$H_3^{(o)}(\tau,t) = 1 + (1+t)\tau + (2t+t^2)\tau^2 + (2t^2+t^3)\tau^3 + t^3\tau^4.$$

A determinant expression

Theorem

Let

$$H^e_{ij}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-1}{l-j}+t\binom{j-1}{l-j-1}\right\}\tau^{k+l-i-j} & \text{if } i,j>0,\\ (1+t\tau)(1+\tau)^{i-1} & \text{if } i>0 \text{ and } j=0,\\ \delta_{0,j} & \text{if } i=0,\end{cases}$$

and

$$H^o_{ij}(\tau,t)=\begin{cases}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\left\{\binom{i-1}{k-i}+t\binom{i-1}{k-i-1}\right\}\left\{\binom{j-2}{l-j}+t\binom{j-2}{l-j-1}\right\}\tau^{k+l-i-j} & \text{if } i,j-1>0,\\ (1+t\tau)(1+\tau)^{i-1} & \text{if } i>0 \text{ and } j=0,1,\\ \delta_{ij} & \text{if } i=0.\end{cases}$$

A determinant expression

Theorem

Then we have

$$H_n^{(e)}(\tau,t) = \det\left(H_{ij}^{e}(\tau,t)\right)_{0\le,i,j\le n-1},$$

and

$$H_n^{(o)}(\tau,t) = \det\left(H_{ij}^{o}(\tau,t)\right)_{0\le,i,j\le n-1}.$$

Conjecture

$$H_n^{(e)}(1,t) = A_{2n-1}^{\mathrm{HTS}}(t),$$
$$H_n^{(o)}(1,t) = A_{2n}^{\mathrm{HTS}}(t),$$

A determinant expression

Observation

We would have

$$H_n^{(e)}(-1,t) = (1 - t + t^2)\,A_{2n-1}^{\mathrm{VS}}(t),$$

and

$$H_n^{(o)}(-1,t) = t(1-t)\,V_{n-2}^{(o)}(1,t) \qquad \text{for } n \geq 3.$$

Thank you!