Schur 関数に関する

R. P. Stanley の予想

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文献

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Partitions

A partition is any (finite or infinite) sequence

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_r, \ldots)$$

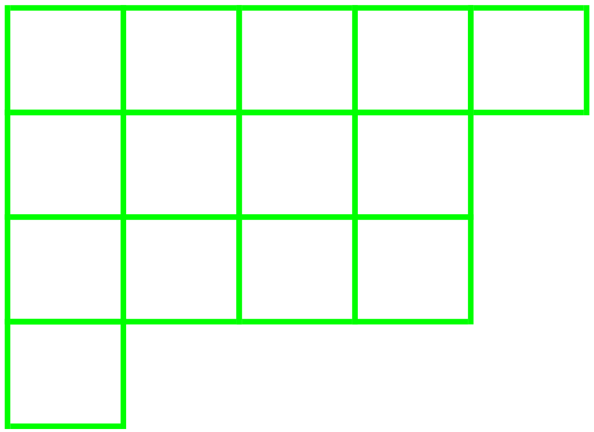
of non-negative integers in decreasing order:

$$\boldsymbol{\lambda}_1 \geq \boldsymbol{\lambda}_2 \geq \cdots \geq \boldsymbol{\lambda}_r \geq \ldots$$

and containing only finitely many non-zero terms.

Example

$\lambda = (5441)$ is a partition of 14 with length 4.



Tableaux

Given a partition λ, A tableaux T of shape λ is a filling of the diagram with numbers whereas the numbers must strictly increase down each column and weakly from left to right along each row. Let $x = (x_1, x_2, \dots)$ be variables. The weight of tableaux T is

$$\mathrm{wt}(T) = x_1^{\sharp 1's} x_2^{\sharp 2's} \cdots$$

Example

A Tableau T of shape (5441).

5	5	4	4	2
3	3	3	2	
2	2	1	1	
1				

The weight of T is $\mathrm{wt}\,(T) = x_1^2 x_2^4 x_3^3 x_4^2 x_5^2$.

Schur functions

The Schur function $s_\lambda(x)$ is, by definition,

$$s_\lambda(x) = \sum_T \mathrm{wt}(T),$$

where the sum runs over all tableaux of shape λ.

Example

When $\lambda = (22)$,

1	1
2	2

1	1
2	3

1	1
3	3

1	2
2	3

1	2
3	3

2	2
3	3

$$s_\lambda(x) = x_1^2 x_2^2 + x_1^2 x_3^2 + \ldots$$

Power Sum Symmetric Functions

Let r denote a positive integer.

$$p_r(x) = x_1^r + x_2^r + \dots$$

is called the rth power sum symmetric functions.

$$p_1(x) = x_1 + x_2 + \dots$$
$$p_2(x) = x_1^2 + x_2^2 + \dots$$

Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ.

a	b	a	b	a
c	d	c	d	
a	b	a	b	
c				

Theorem

Let

$$z = \sum_{\lambda} \omega(\lambda) s_{\lambda}.$$

Here the sum runs over all partitions λ.

Then we have

$$
\begin{aligned}
\log z &- \sum_{n\geq 1}\frac{1}{2n}a^n(b^n - c^n)p_{2n} \\
&- \sum_{n\geq 1}\frac{1}{4n}a^n b^n c^n d^n p_{2n}^2 \\
&\in \mathbb{Q}[[p_1, p_3, p_5, \dots]].
\end{aligned}
$$

Corollary

Let

$$y = \sum_{\substack{\lambda \\ \lambda, \lambda' \text{ even}}} s_\lambda(x).$$

Here the sum runs over all partitions λ such that λ and λ' are even partitions (i.e. with all parts even).

Then we have

$$\log y - \sum_{n \geq 1} \frac{1}{4n} p_{2n}^2$$

$$\in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

Pfaffians

Assume we are given a $2n$ by $2n$ skew-symmetric matrix

$$A = (a_{ij})_{1 \le i,j \le 2n},$$

(i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring.

The Pfaffian of A is, by definition,

$$\mathrm{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n}) \times a_{\sigma_1\sigma_2} \cdots a_{\sigma_{2n-1}\sigma_{2n}}.$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-elements blocks, and where $\epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n})$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n} \end{pmatrix}.$$

Example

When $n = 2$,

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{21} & 0 & a_{23} & a_{24} \\ -a_{31} & -a_{32} & 0 & a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{pmatrix}$$

$$= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Theorem

Let n be a positive integer. Let

$$z_n = \sum_{\ell(\lambda)\leq 2n} \omega(\lambda) s_\lambda(X_{2n})$$

be the sum restricted to $2n$ variables. Then we have

$$z_n = \frac{1}{\prod_{1 \le i < j \le 2n} (x_i - x_j)}$$
$$\times (abcd)^{-\binom{n}{2}}$$
$$\times \operatorname{Pf} \left(p_{ij} \right)_{1 \le i < j \le 2n},$$

where p_{ij} is defined by

$$\frac{\begin{vmatrix} x_i + a x_i^2 & 1 - a(b+c)x_i - abcx_i^3 \\ x_j + a x_j^2 & 1 - a(b+c)x_j - abcx_j^3 \end{vmatrix}}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}.$$

Minor Summation Formula

Let m, n and r be integers such that $r \leq m, n$. Let T be an m by n matrix. For any index sets

$$I = \{i_1, \dots, i_r\}_< \subseteq [m],$$
$$J = \{j_1, \dots, j_r\}_< \subseteq [n],$$

let $\Delta^I_J(A)$ denote the submatrix obtained by selecting the rows indexed by

I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write $\Delta_J(A)$ for $\Delta_J^{[m]}(A)$. Similarly, if $r = n$ and $J = [n]$, we write $\Delta^I(A)$ for $\Delta_{[n]}^I(A)$. For any finite set S and a non-negative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S.

Theorem

Let n and N be non-negative integers such that $2n \leq N$. Let $T = (t_{ij})_{1 \leq i \leq 2n, 1 \leq j \leq N}$ be a $2n$ by N rectangular matrix, and let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a skew-symmetric matrix of size N.

Then

$$\sum_{I\in\binom{[N]}{2n}} \mathrm{Pf}\left(\Delta_I^I(A)\right) \det\left(\Delta_I(T)\right) = \mathrm{Pf}\left(T A\, {}^tT\right).$$

If we put $Q = (Q_{ij})_{1\le i,j\le 2n} = T A\, {}^tT$, then its entries are given by

$$Q_{ij} = \sum_{1\le k<l\le N} a_{kl} \det\left(\Delta_{kl}^{ij}(T)\right),$$

$(1 \leq i, j \leq 2n)$. Here we write $\Delta^{ij}_{kl}(T)$ for

$$\Delta^{\{ij\}}_{\{kl\}}(T) = \begin{vmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl} \end{vmatrix}.$$

Lemma

Let x_i and y_j be indeterminates, and let n is a non-negative integer. Then

$$\mathrm{Pf}\left[x_i y_j\right]_{1\le i<j\le 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}.$$

Lemma

Let n be a non-negative integer. Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{2n})$ be a partition such that $\ell(\boldsymbol{\lambda}) \leq 2n$. Put

$$\boldsymbol{l} = (\boldsymbol{l}_1, \dots, \boldsymbol{l}_{2n}) = \boldsymbol{\lambda} + \boldsymbol{\delta}_{2n}.$$

Define a $2n$ by $2n$ skew-symmetric matrix $A = (\alpha_{ij})_{1 \leq i,j \leq 2n}$ by

$$\alpha_{ij} = a^{\lceil (l_i - 1)/2 \rceil} b^{\lfloor (l_i - 1)/2 \rfloor} c^{\lceil l_j/2 \rceil} d^{\lfloor l_j/2 \rfloor}$$

for $i < j$. Then we have

$$\mathbf{Pf}\left[\boldsymbol{A}\right]_{1\le i,j\le 2n} = (\boldsymbol{abcd})^{\binom{n}{2}}\boldsymbol{\omega}(\boldsymbol{\lambda}).$$

Cauchy type idenitity

Cauchy determinant

$$\det\left[\frac{1}{x_i + y_j}\right]_{1\le i,j\le n} = \frac{\Delta(X)\Delta(Y)}{\prod_{1\le i,j\le n}(x_i + y$$

Pfaffian version

$$\det\left[\frac{x_i - x_j}{x_i + x_j}\right]_{1\le i,j\le 2n} = \frac{\Delta(X)}{\prod_{1\le i<j\le n}(x_i +$$

Notation.

Let $X = (x_1, \ldots, x_{2n})$, $Y = (y_1, \ldots, y_{2n})$, $A = (a_1, \ldots, a_{2n})$ and $B = (b_1, \ldots, b_{2n})$ be $2n$-tuples of variables. Set

$$\boldsymbol{V}_{ij}^{n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{A}, \boldsymbol{B})$$

to be

$$\begin{cases} a_i x_i^{n-j} y_i^{j-1} & \text{if } 1 \leq j \leq n, \\ b_i x_i^{2n-j} y_i^{j-n-1} & \text{if } n+1 \leq j \leq 2n, \end{cases}$$

for $1 \leq i \leq 2n$.

Define $V^n(X, Y; A, B)$ by

$$\det\left(V_{ij}^n(X, Y; A, B)\right)_{1 \leq i,j \leq 2n}.$$

Example.

When $n = 1$,

$$V^1(X, Y; A, B) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

When $n = 2$,

$$V^2(X, Y; A, B) = \begin{vmatrix} a_1x_1 & a_1y_1 & b_1x_1 & b_1 \\ a_2x_2 & a_2y_2 & b_2x_2 & b_2 \\ a_3x_3 & a_3y_3 & b_3x_3 & b_3 \\ a_4x_4 & a_4y_4 & b_4x_4 & b_4 \end{vmatrix}$$

When $n = 3$, $V^2(X, Y; A, B)$ is

$$\begin{vmatrix} a_1x_1^2 & a_1x_1y_1 & a_1y_1^2 & b_1x_1^2 & b_1x_1y_1 & b_1y_1^2 \\ a_2x_2^2 & a_2x_2y_2 & a_2y_2^2 & b_2x_2^2 & b_2x_2y_2 & b_2y_2^2 \\ a_3x_3^2 & a_3x_3y_3 & a_3y_3^2 & b_3x_3^2 & b_3x_3y_3 & b_3y_3^2 \\ a_4x_4^2 & a_4x_4y_4 & a_4y_4^2 & b_4x_4^2 & b_4x_4y_4 & b_4y_4^2 \\ a_5x_5^2 & a_5x_5y_5 & a_5y_5^2 & b_5x_5^2 & b_5x_5y_5 & b_5y_5^2 \\ a_6x_6^2 & a_6x_6y_6 & a_6y_6^2 & b_6x_6^2 & b_6x_6y_6 & b_6y_6^2 \end{vmatrix}.$$

Theorem

Let

$$\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n}), \qquad \boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}$$
$$\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2n}), \qquad \boldsymbol{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_2$$
$$\boldsymbol{C} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{2n}), \qquad \boldsymbol{D} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_2$$

be $2n$-tuples of variables.

Then

$$\operatorname{Pf}\left[\frac{\begin{vmatrix} \boldsymbol{a}_i & \boldsymbol{b}_i \\ \boldsymbol{a}_j & \boldsymbol{b}_j \end{vmatrix} \cdot \begin{vmatrix} \boldsymbol{c}_i & \boldsymbol{d}_i \\ \boldsymbol{c}_j & \boldsymbol{d}_j \end{vmatrix}}{\begin{vmatrix} \boldsymbol{x}_i & \boldsymbol{y}_i \\ \boldsymbol{x}_j & \boldsymbol{y}_j \end{vmatrix}}\right]_{1\le i<j\le 2n} = \frac{\boldsymbol{V}^n(\boldsymbol{X},\boldsymbol{Y};\boldsymbol{A},\boldsymbol{B})\boldsymbol{V}^n(\boldsymbol{X},\boldsymbol{Y};\boldsymbol{C},\boldsymbol{D})}{\prod_{1\le i<j\le 2n}\begin{vmatrix} \boldsymbol{x}_i & \boldsymbol{y}_i \\ \boldsymbol{x}_j & \boldsymbol{y}_j \end{vmatrix}}.$$

Corollary

Let

$$\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n}),$$
$$\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2n}),$$
$$\boldsymbol{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2n})$$

be $2n$-tuples of variables.

Then

$$\mathrm{Pf}\left[\frac{a_i b_j - a_j b_i}{1 - t x_i x_j}\right]_{1 \le i < j \le 2n}$$

$$= (-1)^{\binom{n}{2}} t^{\binom{n}{2}} \frac{V^n(X, 1 + tX^2; A, B)}{\prod_{1 \le i < j \le 2n}(1 - t x_i x_j)}.$$

Proposition

Let $f(x_1, x_2, \ldots)$ be a symmetric function with infinite variables. Then

$$f \in \mathbb{Q}[p_\lambda : \text{ all parts } \lambda_i > 0 \text{ are odd}]$$

if and only if

$$f(t, -t, x_1, x_2, \ldots) = f(x_1, x_2, \ldots).$$

Strategy:

If we set $v_n(\boldsymbol{X}_{2n})$ to be

$$\log z_n(\boldsymbol{X}_{2n}) - \sum_{k\geq 1}\frac{1}{2k}a^k(b^k - c^k)p_{2k}(\boldsymbol{X}_{2n}) - \sum_{k\geq 1}\frac{1}{4k}a^k b^k c^k d^k p_{2k}(\boldsymbol{X}_{2n})^2,$$

then we claim it satisfies

$$v_{n+1}(t, -t, \boldsymbol{X}_{2n}) = v_n(\boldsymbol{X}_{2n}).$$

Theorem

Let $X = (x_1, \ldots, x_{2n})$ be a $2n$-tuple of variables. Then

$$z_n(\boldsymbol{X}_{2n}) = (-1)^{\binom{n}{2}} \frac{\boldsymbol{V}^n(\boldsymbol{X}^2, \boldsymbol{1} + abcd\boldsymbol{X}^4; \boldsymbol{X} + a\boldsymbol{X}^2, \boldsymbol{1} - a(b+c)\boldsymbol{X}^2 - abc\ldots}{\prod_{i=1}^{2n}(1 - abx_i^2) \prod_{1 \le i < j \le 2n}(x_i - x_j)(1 - abcdx_i^2x_j^2)\ldots}$$

where $X^2 = (x_1^2, \ldots, x_{2n}^2)$, $\boldsymbol{1} + abcdX^4 = (1 + abcdx_1^4, \ldots, 1 + abcdx_{2n}^4)$, $X + aX^2 = (x_1 + ax_1^2, \ldots, x_2\ldots$ $ax_{2n}^2)$ and $\boldsymbol{1} - a(b+c)X^2 - abcX^3 =$

$$(1 - a(b+c)x_1^2 - abcx_1^3, \ldots, 1 - a(b+c)x_{2n}^2 - abcx_{2n}^3).$$

Propsition

Let $X = (x_1, \ldots, x_{2n})$ be a $2n$-tuple of variables. Put

$$f_n(X_{2n}) = V^n(X^2, 1+abcdX^4; X+aX^2, 1-a(b+c)X^2-abcX^3).$$

Then $f_n(\boldsymbol{X}_{2n})$ satisfies

$$f_{n+1}(t, -t, \boldsymbol{X}_{2n})$$
$$= (-1)^n 2t$$
$$\times (1 - abt^2)(1 - act^2) \prod_{i=1}^{2n} (t^2 - x_i^2)$$
$$\times \prod_{i=1}^{2n} (1 - abcdt^2 x_i^2) \cdot f_n(\boldsymbol{X}_{2n}).$$

Corollary

Let

$$Z(x;t) = \sum_{\lambda} \omega(\boldsymbol{\lambda}) S_{\lambda}(x;t),$$

Here the sum runs over all partitions $\boldsymbol{\lambda}$.

Then we have

$$\log Z(x;t) = -\sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n)(1-t^{2n}) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n (1-t^{2n})^2 p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

Definition

Define $T_\lambda(x;q,t)$ by

$$\det\left(Q_{(\lambda_i-i+j)}(x;q,t)\right)_{1\le i,j\le \ell(\lambda)},$$

where $Q_\lambda(x;q,t)$ stands for the Macdonald polynomial corresponding to the partition λ, and $Q_{(r)}(x;q,t)$ is the one corresponding to the one row partition (r) (See [8], IV, sec.4).

Corollary

Let

$$Z(x;q,t) = \sum_{\lambda} \omega(\boldsymbol{\lambda}) T_{\boldsymbol{\lambda}}(x;q,t),$$

Here the sum runs over all partitions $\boldsymbol{\lambda}$.

Then we have

$$
\begin{aligned}
\log Z(x;q,t) \\
&= \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n) \frac{1-t^{2n}}{1-q^{2n}} p_{2n} \\
&- \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n \frac{(1-t^{2n})^2}{(1-q^{2n})^2} p_{2n}^2 \\
&\in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].
\end{aligned}
$$

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