Several refined conjectures on TSSCPP and ASM

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Contents of this talk

References

- Mills-Robbins-Rumsey, “Self-complementary totally symmetric plane partitions” J. Combin. Theory Ser. A, **42** (1986), 277 – 292.
- Masao Ishikawa, “Refined enumerations of Totally Symmetric Self-Complementary Plane Partitions”, in preparation.

Partitions

1. Ordinary parttions
2. Strict partitions

Partitions

A partition of a positive integer n is a finite nonincreasing sequence of positive integers λ_1, λ_2,. . . , λ_r such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition, and n is called the weight of the partition, denoted by $|\lambda|$. Many times the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be denoted by λ, and we shall write $\lambda \vdash n$ to denote "λ is a partition of n". The number of (non-zero) parts is the length, denoted by $\ell(\lambda)$.

Example

The empty sequence $\emptyset$ forms the only partition of zero.

$n = 1$: (1);

$n = 2$: (2), (1^2);

$n = 3$: (3), (21), (1^3);

$n = 4$: (4), (31), (2^2), (21^2), (1^4);

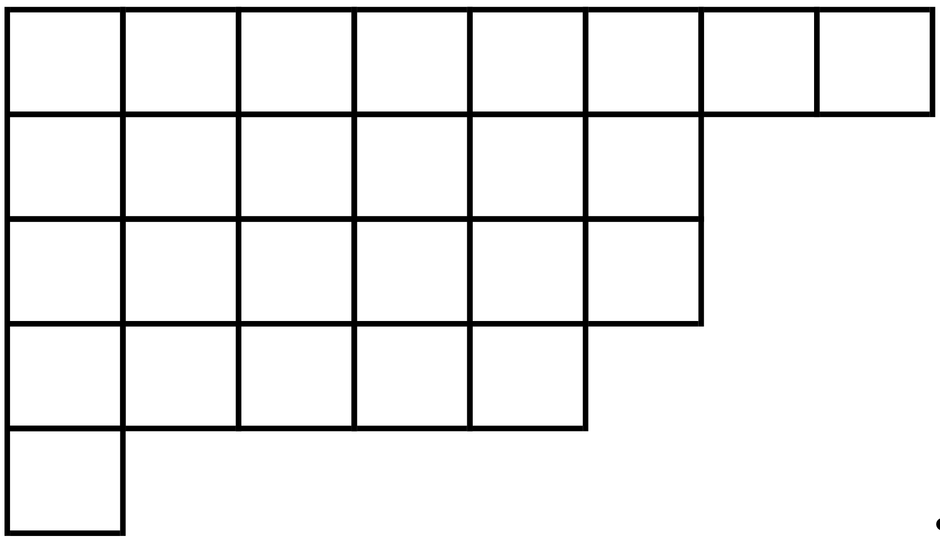
$n = 5$: (5), (41), (32), (31^2), (2^21), (21^3), (1^5);

Young Diagram

To each partition λ is associated its graphical representation (Young diagram) $\mathcal{D}_\lambda$, which formally is the set of points with integral coordinates (i, j) in the plane such that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, then $(i, j) \in \mathcal{D}_\lambda$ if and only if $1 \leq j \leq \lambda_i$. We sometimes identify the Ferres graph $\mathcal{D}_\lambda$ with the partition λ and use the same symbol λ to express its Young diagram.

Example

The Young diagram of the partition $(8,6,6,5,1)$ is



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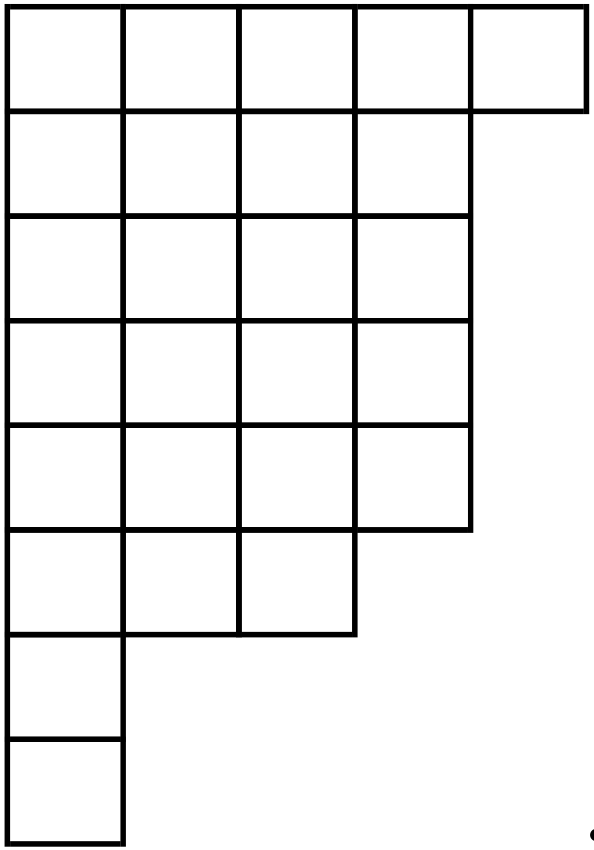
Conjugate

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition, we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_r)$ by choosing λ'_i as the number of parts of λ that are $\geq i$. The partition λ' is called the conjugate of λ.

Example

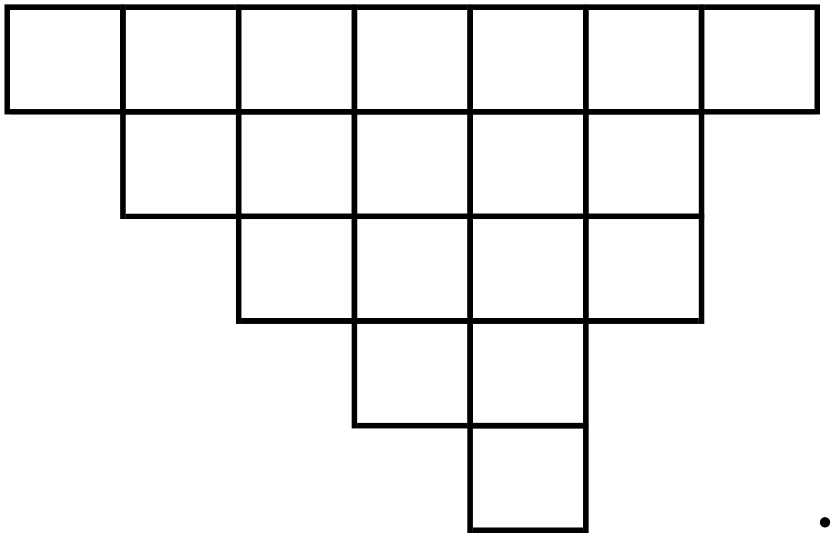
The conjugate of the partition (86^251) is (54^431^2)



Strict Partitions

A partition μ all of whose parts are distinct (have multiplicity 1) is called a strict partition. For a strict partition $\mu = (\mu_1, > \mu_2 > \cdots > \mu_r)$, the shifted diagram $\mathcal{S}_\mu$ is obtained from the Young diagram of μ by moving the ith row $(i-1)$ squares to the right, for each $i > 1$.

If $\mu = (7, 5, 4, 2, 1)$ then $\mathcal{S}_\mu$ is



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Symmetric functions

1. Complete symmetric functions
2. Elmentary symmetric functions
3. Schur functions
   (a) Ratio of determinants
   (b) Tableaux
   (c) Jacobi-Trudi formula
   (d) Bender-Knuth involution

Complete symmetric functions

Let $x = (x_1, x_2, \ldots)$ be countably many variables. For a positive integer l, we write the rth complete symmetric function in n variables $x_1, \ldots, x_n$ by $h_r^{(n)}(x) = h_r^{(n)}(x_1, \ldots, x_n)$, i.e. we have

$$\sum_{r=0}^{\infty} h_r^{(n)}(x) y^r = \prod_{i=1}^{l} (1 - x_i y)^{-1}.$$

Example

$$h_2^{(3)}(x) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3.$$

Elementary symmetric functions

For a positive integer n, we write the rth elementary symmetric function in n variables $x_1, \ldots, x_n$ by $e_r^{(n)}(x) = e_r^{(n)}(x_1, \ldots, x_n)$, i.e. we have

$$\sum_{r=0}^{\infty} e_r^{(n)}(x) y^r = \prod_{i=1}^{n} (1 + x_i y).$$

Example

$$e_2^{(3)}(x) = x_1 x_2 + x_1 x_3 + x_2 x_3.$$

The Schur functions

For a positive integer n and a partition λ such that $\ell(\lambda) \le n$, let

$$s_\lambda^{(n)}(x) = \frac{\det(x_i^{\lambda_j+n-j})_{1\le i,j\le n}}{\det(x_i^{n-j})_{1\le i,j\le n}}.$$

$s_\lambda^{(n)}(x)$ is called the Schur function corresponding to λ.

Tableaux

Given a partition λ, A tableaux T of shape λ is a filling of the diagram with numbers $1, \ldots, n$ whereas the numbers must strictly increase down each column and weakly from left to right along each row.

Schur functions

The Schur function $s_\lambda^{(n)}(x)$ is

$$s_\lambda^{(n)}(x) = \sum_T x^T,$$

where the sum runs over all tableaux of shape λ.

Here $x^T = x_1^{\sharp 1\text{s in } T} x_2^{\sharp 2\text{s in } T} \cdots x_n^{\sharp n\text{s in } T}$

Example

A Tableau T of shape (5441).

1	1	1	2	2
2	2	3	4	
3	3	4	5	
5				

The weight of T is $x_1^3 x_2^4 x_3^3 x_4^2 x_5^2$.

Example

When $\lambda = (2,2)$ and $X = (x_1, x_2, x_3, x_4)$,

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 4&4\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&3\\\hline 4&4\\\hline\end{array}$$

$$\begin{aligned} s_\lambda^{(4)}(X) = {} & x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2 + 2x_1x_2x_3x_4 \\ & + x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_2^2x_1x_3 + x_2^2x_1x_4 + x_2^2x_3x_4 \\ & + x_3^2x_1x_2 + x_3^2x_1x_4 + x_3^2x_2x_4 + x_4^2x_1x_2 + x_4^2x_1x_3 + x_4^2x_2x_3 \end{aligned}$$

Jacobi-Trudi formula

For a positive integer n and a partition λ, we have

$$\begin{aligned} s_\lambda^{(n)}(x) &= \det(h^{(n)}_{\lambda_i+j-i})_{1\le i,j\le \ell(\lambda)} \\ &= \det(e^{(n)}_{\lambda'_i+j-i})_{1\le i,j\le \ell(\lambda')}. \end{aligned}$$

Bender-Knuth involution

A classical method to prove that a Schur function is symmetric is to define involutions s_i on tableaux which swaps the number of i's and $(i-1)$'s, for each i. This is well-known as the Bender-Knuth involution.

Swapping rule s_r

Consider the parts of T equal to $r-1$ or r. Since T is column-strict, some columns of T will contain neither $r-1$ nor r, while some others will contain one $r-1$ and one r. These columns we ignore. The remaining parts equal to $r-1$ or r occur once in each column. Assume row i has a certain number k of $r-1$'s followed by a certain number l of r's. In row i, convert the k $r-1$'s and l r's to l $r-1$'s and k r's.

Swapping rule s_r

For example, the three consecutive rows $i-1$, i and $i+1$ of c could look as follows.

$i-1$	$\vdots$		$\vdots$							$r-1$	$\dots$	$r-$
i	$r-1$	$\dots$	$r-1$	$r-1$	$\dots$	$r-1$	r	$\dots$	r	r	$\dots$	
$i+1$	r	$\dots$	r									

Example

A Tableau T

1	1	1	2	2
2	2	3	4	
3	3	4	5	
5				

of shape (5441) is mapped to

1	1	1	1	2
2	2	3	4	
3	3	4	5	
5				

by the swap s_2.

Alternating Sign Matrices and Symmetries

1. Alternating sign matrices
2. Half-turn
3. Vertical flip
4. Monotone triangles

Alternating sign matrices

An alternating sign matrices is a square matrix which satisfies:

(i) all entries are 1, -1, or 0,

(ii) every row and column has sum 1,

(iii) in every row and column the nonzero entries alternate in sign.

Examples

All permutation matrices are alternating sign matrices. For 1×1 and 2×2 matrices these are only alternating sign matrices. There are exactly seven 3×3 alternating sign matrices, six permutation matrices and the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Symmetries

1. no symmetry
2. $a_{ij} = a_{n-1-i,n-1-j}$ half turn
3. $a_{ij} = a_{i,n-1-j}$ vertical axis

Example

3×3 alternating matrices $A_3(t) = 2 + 3t + 2t^2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Double distribution

3×3 alternating matrices

$$(B_3(k,l))_{1\le k,l\le 3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$B_n(k,l)$ is the number of $n \times n$ alternating sign matrices which has a 1 in the kth column of the top row and has a 1 in the lth column of the bottom row.

Example

Half-turn symmetric 3×3 alternating matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$A_3^{\mathsf{HTS}}(t) = 1 + t + t^2$.

Example

Vertical symmetric 3×3 alternating matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$A_3^{\mathsf{VS}}(t) = 1.$

Monotone triangles

A monotone triangle of size n is, by definition, a triangular array of positive integers

$$\begin{array}{cccc} & & & m_{n,n} \\ & & m_{n-1,n-1} & m_{n-1,n} \\ & \iddots & \vdots & \vdots \\ m_{1,1} & \cdots & m_{1,n-1} & m_{1,n} \end{array}$$

subject to the constraints that

(M1) $m_{ij} < m_{i,j+1}$ whenever both sides are defined,

(M2) $m_{ij} \ge m_{i+1,j}$ whenever both sides are defined,

(M3) $m_{ij} \le m_{i+1,j+1}$ whenever both sides are defined,

(M4) the bottom row $(m_{1,1}, m_{1,2}, \dots, m_{1,n})$ is $(1, 2, \dots, n)$.

Let $\mathcal{M}_n$ denote the set of monotone triangles of size n.

Example

$\mathcal{M}_3$ consists of the following seven elements.

$$\begin{array}{ccc} & & 1\\ & 1 & 2\\ 1 & 2 & 3\end{array}\qquad\begin{array}{ccc} & & 2\\ & 1 & 2\\ 1 & 2 & 3\end{array}\qquad\begin{array}{ccc} & & 1\\ & 1 & 3\\ 1 & 2 & 3\end{array}\qquad\begin{array}{ccc} & & 2\\ & 1 & 3\\ 1 & 2 & 3\end{array}$$

$$\begin{array}{ccc} & & 3\\ & 1 & 3\\ 1 & 2 & 3\end{array}\qquad\begin{array}{ccc} & & 2\\ & 2 & 3\\ 1 & 2 & 3\end{array}\qquad\begin{array}{ccc} & & 3\\ & 2 & 3\\ 1 & 2 & 3\end{array}$$

Certain Numbers

1. A_n : ASM numbers
2. $A_{n,r}$, $A_n(t)$: the refined ASM numbers.
3. $B_n(k,l)$: the doubly refined ASM numbers.
4. A_n^{HTS}, $A_n^{\mathsf{HTS}}(t)$: the number of half-turn symmetric ASMs.
5. A_n^{VS}, $A_n^{\mathsf{VS}}(t)$: the number of ASMs invariant under the vertical flip.

A_n

Let A_n denote the number defined by

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

This number is famous for the number of alternating sign matrices.

$A_{n,r}$

Let n be a positive number and let $1 \leq r \leq n$. Set $A_{n,r}$ to be the number

$$A_{n,r} = \frac{\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}}{\binom{2n-2}{n-1}} A_{n-1} = \frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}} A_n.$$

Then the number $A_{n,r}$ satisfies the recurrence $A_{n,1} = A_{n-1}$ and

$$\frac{A_{n,r+1}}{A_{n,r}} = \frac{(n-r)(n+r-1)}{k(2n-r-1)}.$$

We also define the polynomial $A_n(t) = \sum_{r=1}^{n} A_{n,r} t^{r-1}$. For instance, the first few terms are $A_1(t) = 1$, $A_2(t) = 1 + t$, $A_3(t) = 2 + 3t + 2t^2$, $A_4(t) = 7 + 14t + 14t^2 + 7t^3$.

$B_n(k,l)$

Let n be a positive integer and let $B_n(k,l)$, $1 \leq k,l \leq n$, denote the number which satisfies the initial condition

$$B_n(k,1) = B_n(1,k) = \begin{cases} 0 & \text{if } k = 1 \\ A_{n-1,n-k} & \text{if } 2 \leq k \leq n \end{cases}$$

and the recurrence equation

$$B_n(k+1,l+1) - B_n(k,l) = \frac{A_{n-1,k}(A_{n,l+1} - A_{n,l}) + A_{n-1,l}(A_{n,k+1} - A_{n,k})}{A_{n,1}}$$

for $1 \leq k,l \leq n-1$.

Example

This recurrence equation satisfied by $B_n(k,l)$ has been introduced by Stroganov to describe the double distribution of the positions of the 1's in the top row and the bottom row of an alternating sign matrix.

$$(B_4(k,l))_{1\le k,l\le 4} = \begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 4 & 5 & 3 \\ 3 & 5 & 4 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

A_n^{HTS}

Let A_n^{HTS} be the number defined by

$$A_{2n}^{\mathsf{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{\{(n+i)!\}^2}$$

and

$$A_{2n+1}^{\mathsf{HTS}} = \frac{n!(3n)!}{\{(2n)!\}^2} \cdot A_{2n}^{\mathsf{HTS}}.$$

The first few terms are 1, 2, 3, 10, 25, 140, 588. This is the number of half-turn symmetric alternating sign matrices.

$\widetilde{A}_n^{\mathsf{HTS}}(t)$

We also define the polynomial $\widetilde{A}_n^{\mathsf{HTS}}(t)$ by

$$\frac{\widetilde{A}_{2n}^{\mathsf{HTS}}(t)}{\widetilde{A}_{2n}^{\mathsf{HTS}}} = \frac{(3n-2)(2n-1)!}{(n-1)!(3n-1)!}$$

$$\sum_{r=0}^{n} \frac{\{n(n-1)-nr+r^2\}(n+r-2)!(2n-r-2)!}{r!(n-r)!} t^r$$

where $\widetilde{A}_{2n}^{\mathsf{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{(3i+1)!(n+i)!}$. For instance, the first few terms are $\widetilde{A}_2^{\mathsf{HTS}}(t) = 1+t$, $\widetilde{A}_4^{\mathsf{HTS}}(t) = 2+t+2t^2$,
$\widetilde{A}_6^{\mathsf{HTS}}(t) = 5+5t+5t^2+5t^3$ and
$\widetilde{A}_8^{\mathsf{HTS}}(t) = 20+30t+32t^2+30t^3+20t^4$.

$A_{2n}^{\mathsf{HTS}}(t)$

Let

$$A_{2n}^{\mathsf{HTS}}(t) = \widetilde{A}_{2n}^{\mathsf{HTS}}(t) A_n(t),$$

and

$$A_{2n+1}^{\mathsf{HTS}}(t) = \frac{1}{3}\left\{ A_{n+1}(t)\widetilde{A}_{2n}^{\mathsf{HTS}}(t) + A_n(t)\widetilde{A}_{2n+2}^{\mathsf{HTS}}(t) \right\}.$$

The first few terms are $A_2^{\mathsf{HTS}}(t) = 1 + t$, $A_3^{\mathsf{HTS}}(t) = 1 + t + t^2$, $A_4^{\mathsf{HTS}}(t) = 2 + 3t + 3t^2 + 2t^3$, $A_5^{\mathsf{HTS}}(t) = 3 + 6t + 7t^2 + 6t^3 + 3t^4$. Let $A_{n,r}^{\mathsf{HTS}}$ denote the coefficient of t^r in $A_n^{\mathsf{HTS}}(t)$.

A^{VS}_{2n+1}

Let A^{VS}_{2n+1} be the number defined by

$$A^{\mathsf{VS}}_{2n+1} = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}$$

and let $A^{\mathsf{VS}}_{2n+1,r}$ be the number given by

$$A^{\mathsf{VS}}_{2n+1,r} = \frac{A^{\mathsf{VS}}_{2n-1}}{(4n-2)!} \sum_{k=1}^{r} (-1)^{r+k} \frac{(2n+k-2)!(4n-k-1)!}{(k-1)!(2n-k)!}.$$

This number A^{VS}_{2n+1} is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$. For example, the first few terms of A^{VS}_{2n+1} is 1, 3, 26, 646 and 45885.

$A^{\mathsf{VS}}_{2n+1}(t)$

We also define the polynomial $A^{\mathsf{VS}}_{2n+1}(t)$ by

$$A^{\mathsf{VS}}_{2n+1}(t) = \sum_{r=1}^{2n} A^{\mathsf{VS}}_{2n+1,r} t^{r-1}.$$

For instance, the first few terms are $A^{\mathsf{VS}}_{3}(t) = 1$, $A^{\mathsf{VS}}_{5}(t) = 1 + t + t^2$, $A^{\mathsf{VS}}_{7}(t) = 3 + 6t + 8t^2 + 6t^3 + 3t^4$ and $A^{\mathsf{VS}}_{9}(t) = 26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$.

Plane Partitions

1. Plane parttions
2. Shifted plane partitions
3. Domino plane partitions

Plane Partitions

A plane partition is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the weight n.

A part of a plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is a positive entry $\pi_{ij} > 0$. The shape of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row. The shape of π is denoted by $\mathrm{sh}(\pi)$. We say that π has r rows if $r = \ell(\lambda)$. Similarly, π has s columns if $s = \ell(\lambda')$.

Example

The following is a plane aprtition of shape $(9, 8, 4, 1)$, 4 rows, 9 columns, weight 49.

5	5	4	3	3	2	2	2	1
4	4	2	2	1	1	1	1	
2	1	1	1					
1								

Example

Plane partition of 0: $\emptyset$

Plane partition of 1: $\begin{array}{|c|}\hline 1\\\hline\end{array}$

Plane partition of 2:

$$\begin{array}{|c|}\hline \mathbf{2}\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{1}\\\hline\end{array} \qquad \begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{1}\\\hline\end{array}$$

Plane partition of 3:

$$\begin{array}{|c|}\hline \mathbf{3}\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline \mathbf{1} & \mathbf{1} & \mathbf{1}\\\hline\end{array} \qquad \begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{1}\\\hline \mathbf{1}\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline \mathbf{2} & \mathbf{1}\\\hline\end{array} \qquad \begin{array}{|c|}\hline \mathbf{2}\\\hline \mathbf{1}\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{1}\\\hline \mathbf{1} \\\cline{1-1}\end{array}$$

Column-strict plane partitions

A plane partition is said to be **column-strict** if it is weakly decreasing in rows and strictly decreasing in coulumns.

Example

5	5	4	3	3	3	1
4	4	2	2	1	1	
3	2	1	1			
1	1					

is a column-stric plane partition.

Ferrers graph

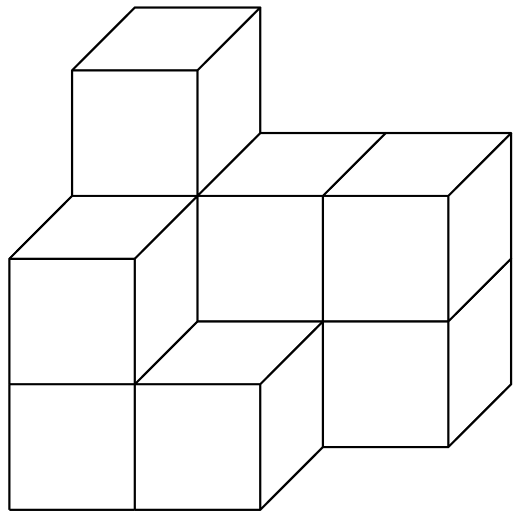
The Ferrers graph $F(\pi)$ of π is the set of all lattice points $(i, j, k) \in \mathbb{P}^3$ such that $k \leq \pi_{ij}$.

Example

The Ferrers graph of

3	2	2
2	1	

is as follows:



Shifted plane partitions

We can define a shifted plane partition similarly. A shifted plane partition is an array $\tau = (\tau_{ij})_{1 \le i \le j}$ of nonnegative integers such that τ has finite support and is weakly decreasing in rows and columns. The shifted shape of τ is the distinct partition μ for which τ has μ_i nonzero parts in the ith row.

Example

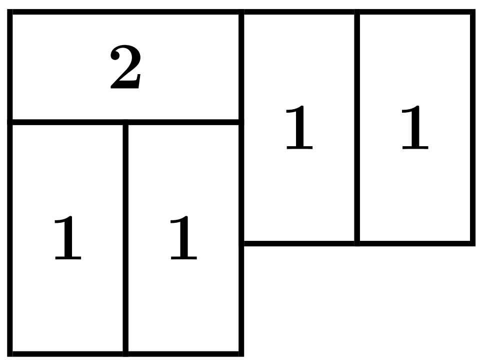
4	4	3	3	2	1	1
	4	3	2	1	1	
		2	2	1	1	
			1			

.

Domino plane partitions

Let λ be a partition. A domino plane partition of shape λ is a tiling of this shape by means of dominoes (2×1 or 1×2 rectangles), where each domino is numbered by a positive integer and those intergers are weakly decreasing in rows and columns. The integers in the dominoes are called parts. A domino plane partition is said to be column-strict if it is strictly decreasing in columns.

Example



Symmetries

1. Self-complementary plane parttions
2. Totally symmetric plane parttions
3. Cyclically cymmetric plane parttions

Complementary

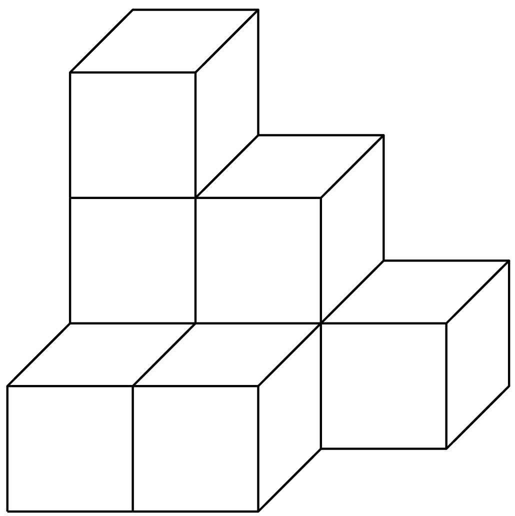
Let $\pi = (\pi_{ij})_{i,j\ge 1}$ be a plane partition with at most r rows, at most c columns, and with largest part at most t. We say that $\pi' = (\pi'_{ij})_{i,j\ge 1}$ is (r, c, t)-complementary plane partition of π if $\pi'_{ij} = t - \pi_{r+1-i,c+1-j}$ for all $1 \le i \le r$ and $1 \le j \le c$.

Example

The $(3, 2, 3)$-complementary PP of the above PP is

$\mathbf{3}$	$\mathbf{2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	

and its Ferrers graph is as follows:



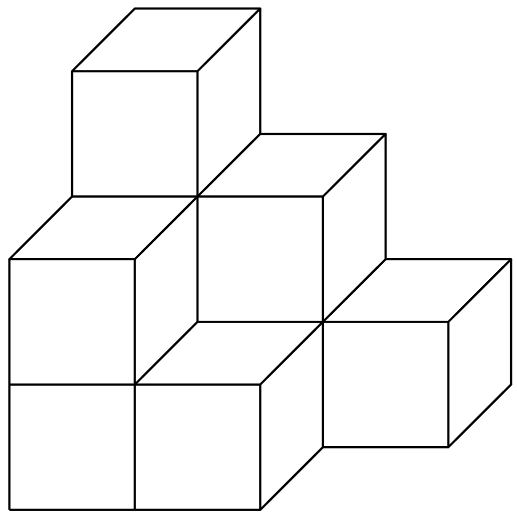
Self-complementary plane partitions

A plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is said to be (r, c, t)-self-complementary if $\pi_{ij} = t - \pi_{r+1-i,c+1-j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$.

Example

3	2	1
2	1	

is a $(3,2,3)$-self-complementary plane partition and its Ferrers graph is as follows:



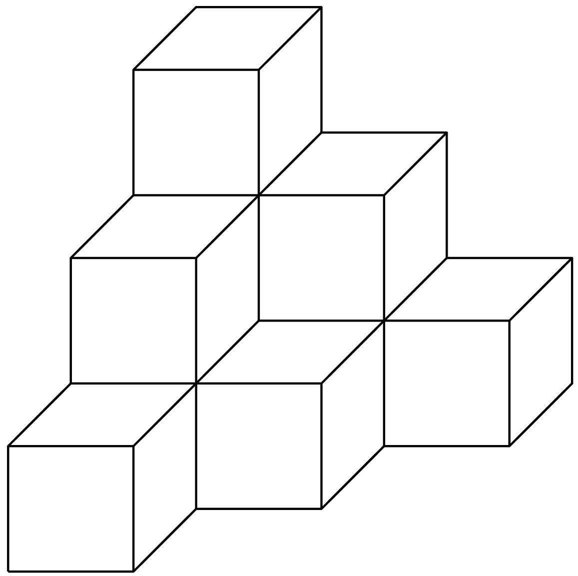
Totally symmetric plane partitions

Let $\mathbf{P}$ denote the set of positive integers. Consider the elements of $\mathbb{P}^3$, regarded as the lattice points of $\mathbb{R}^3$ in the positive orthant. The symmetric group S_3 is acting on $\mathbb{P}^3$ as permutations of the coordinate axies. A plane partition is said to be totally symmetric if its Ferrors graph is mapped to itself under all 6 permutations in S_3.

Example

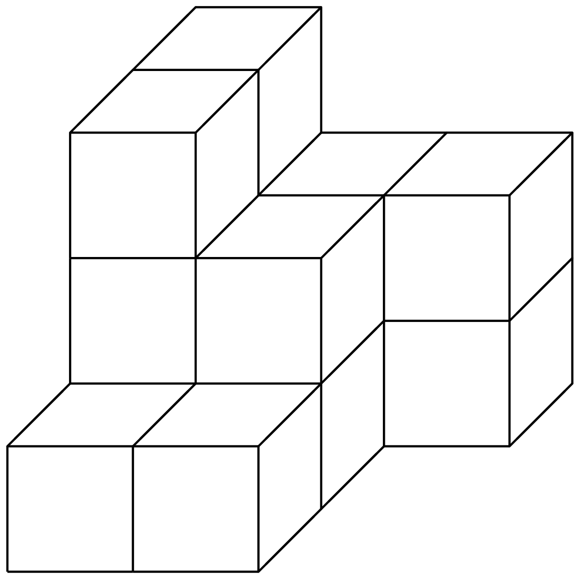
3	2	1
2	1	
1		

is a totally symmetric plane partition and its Ferrers graph is as follows:



Cyclically symmetric plane partitions

A plane partition is said to be cyclically symmetric if its Ferrors graph is mapped to itself under all 3 permutations in A_3.



is cyclically symmetric, but not totally symmetric.

Certain Classes of Plane Partitions

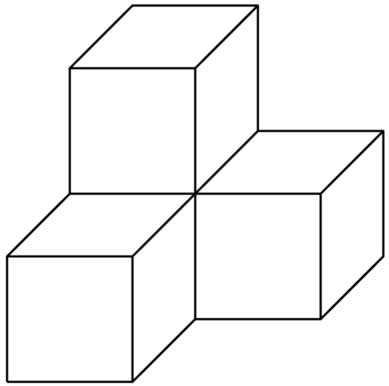
1. Totally symmetric self-complementary plane parttions
2. Triangular shifted plane partitions
3. Restricted column-stricted plane partitions
4. Restricted column-stricted domino plane partitions

Totally symmetric self-complementary plane partitions

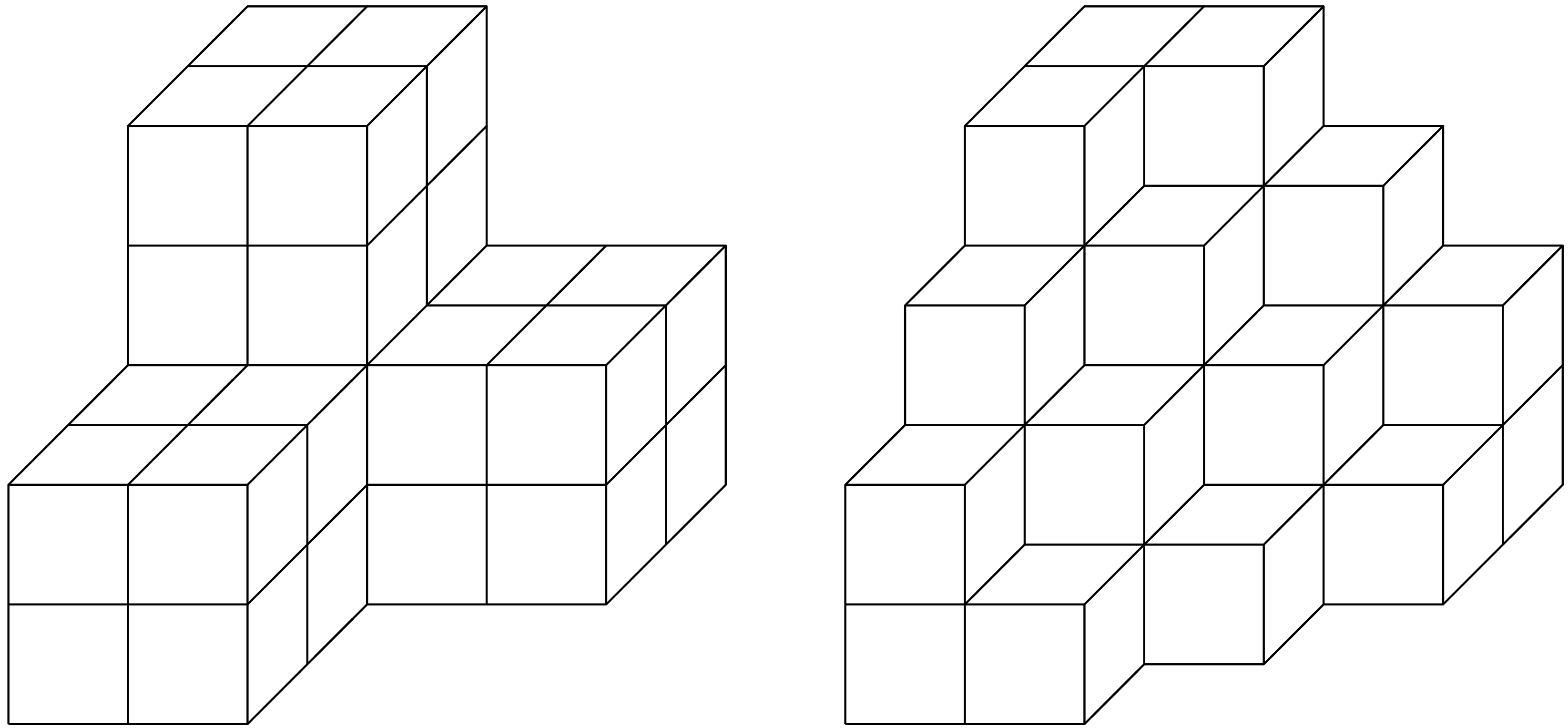
Let $\mathcal{T}_n$ denote the set of all plane partitions which is contained in the box $X_n = [2n] \times [2n] \times [2n]$, $(2n, 2n, 2n)$-self-complementary and totally symmetric. An element of $\mathcal{T}_n$ is called a totally symmetric self-complementary plane partition (abbreviated as TSSCPP) of size n.

Example

$\mathcal{T}_1$



$\mathcal{T}_2$



$\mathcal{T}_3$

6	6	6	3	3	3
6	6	6	3	3	3
6	6	6	3	3	3
3	3	3			
3	3	3			
3	3	3			

6	6	6	4	3	3
6	6	6	3	3	3
6	6	5	3	3	2
4	3	3	1		
3	3	3			
3	3	2			

6	6	6	4	3	3
6	6	6	4	3	3
6	6	4	3	2	2
4	4	3	2		
3	3	2			
3	3	2			

6	6	6	5	4	3
6	6	5	3	3	2
6	5	5	3	3	1
5	3	3	1	1	
4	3	3	1		
3	2	1			

6	6	6	5	4	3
6	6	5	4	3	2
6	5	4	3	2	1
5	4	3	2	1	
4	3	2	1		
3	2	1			

6	6	6	5	5	3
6	5	5	3	3	1
6	5	5	3	3	1
5	3	3	1	1	
5	3	3	1	1	
3	1	1			

6	6	6	5	5	3
6	5	5	4	3	1
6	5	4	3	2	1
5	4	3	2	1	
5	3	2	1	1	
3	1	1			

Triangular shifted plane partitions

Mills, Robbins and Rumsey considered a class $\mathcal{B}_n$ of triangular shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \ldots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$,

and they constructed a bijection between $\mathcal{T}_n$ and $\mathcal{B}_n$. In this paper we call an element of $\mathcal{B}_n$ a triangular shifted plane partition (abbreviated as TSPP) of size n.

Example

$\mathcal{B}_1$ consists of the following 1 PPs: $\emptyset$

$\mathcal{B}_2$ consists of the following 2 PPs:

$$\boxed{2} \qquad \boxed{1}$$

$\mathcal{B}_3$ consists of the followng 7 elements:

$$\begin{array}{cc} \boxed{3} & \boxed{3} \\ & \boxed{3} \end{array} \quad \begin{array}{cc} \boxed{3} & \boxed{3} \\ & \boxed{2} \end{array} \quad \begin{array}{cc} \boxed{3} & \boxed{3} \\ & \boxed{1} \end{array} \quad \begin{array}{cc} \boxed{3} & \boxed{2} \\ & \boxed{2} \end{array} \quad \begin{array}{cc} \boxed{3} & \boxed{2} \\ & \boxed{1} \end{array} \quad \begin{array}{cc} \boxed{2} & \boxed{2} \\ & \boxed{2} \end{array} \quad \begin{array}{cc} \boxed{2} & \boxed{2} \\ & \boxed{1} \end{array}$$

A statistics

In this talk, for $b = (b_{ij})_{1 \le i \le j \le n-1} \in \mathcal{B}_n$, we set $b_{i,n} = n - i$ for all i and $b_{0,j} = n$ for all j by convention.

Definition (Mills, Robbins and Rumsey)

For a $b = (b_{ij})_{1 \le i \le j \le n-1}$ in $\mathcal{B}_n$ and integers $r = 1, \dots, n$, let

$$U_r(b) = \sum_{t=1}^{n-r} (b_{t,t+r-1} - b_{t,t+r}) + \sum_{t=n-r+1}^{n-1} \{b_{t,n-1} > n - t\}.$$

Here $\{\dots\}$ has value 1 when the statement "$\dots$" is true and 0 otherwise. for $1 \le k \le n$,

Example

$n = 7.$

7	7	7	7	7	7
	6	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

$U_1(b) = 3, \quad U_2(b) = 1, \quad U_3(b) = 3, \quad U_4(b) = 2, \quad U_5(b) = 2,$
$U_6(b) = 3, \quad U_7(b) = 3.$

Example

$\mathcal{B}_3$ consists of the followng 7 elements:

$$
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
$$

$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

Flip

Mills, Robbins and Rumsey defined the notion of flip.

Let $b = (b_{ij})_{1 \le i \le j \le n-1}$ be an element of $\mathcal{B}_n$ and let $1 \le i < j \le n-1$ so that b_{ij} is a part of b off the main diagonal. Then the flip of the part b_{ij} is the operation of replacing b_{ij} by b'_{ij} where

$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$

When the part is in the main diagonal, the flip of a part b_{ii} is the operation replacing b_{ii} by b'_{ii} where

$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

Involution

Let $1 \leq r \leq n$ and $b = (b_{ij})_{1 \leq i \leq j \leq n-1} \in \mathcal{B}_n$. Define an opration

$$\pi_r : \mathcal{B}_n \longrightarrow \mathcal{B}_n$$
$$b \longmapsto \pi_r(b)$$

where $\pi_r(b)$ is the result of flipping all the $b_{i,i+r-1}$, $1 \leq i \leq n+m-r$. Since none of these parts of b are neighbors, the result is indpendent of the order in which the flips are applied, and this operation π_r is evidently an involution, i.e. $\pi_r^2 = id$.

Example

The seven elements of $\mathcal{B}_3$

$$
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&3\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}
$$

is mapped to

$$
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&3\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}
$$

by π_1, respectively.

An involution corresponding to the half-turn

Mills, Robbins and Rumsey defined an involution ρ of $\mathcal{B}_n$ by

$$\rho = \pi_2 \pi_4 \cdots$$

where the product is over all π_i with i even and $\leq n$, and presented a conjecture that this involution ρ corresponds to the half turn of an alternating matrix.

Example

The seven elements of $\mathcal{B}_3$

$$
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
$$

is mapped to

$$
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
$$

by ρ, respectively. So the three elements remains invariant under ρ.

An involution corresponding to the vertical flip

Mills, Robbins and Rumsey defined an involution γ of $\mathcal{B}_n$ by

$$\gamma = \pi_1 \pi_3 \cdots$$

where the product is over all π_i with i odd and $\leq n$, and presented a conjecture that this involution γ corresponds to the vertical flip of an alternating matrix.

Example

The seven elements of $\mathcal{B}_3$

$$
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&3\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}
$$

is mapped to

$$
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&3\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}
$$

by γ, respectively. So one element is invariant under γ.

Conjectures and Progress

Mills-Robbins-Rumsey, "Self-complementary totally symmetric plane partitions" J. Combin. Theory Ser. A, **42** (1986), 277 – 292.

The conjectures by Mills-Robbins-Rumsey

1. Conjecture 2 : the refined TSSPP conjecture.
2. Conjecture 3 : the doubly refined TSSCPP conjecture.
3. Conjecture 4 : HTS refined TSSCPP conjecture.
4. Conjecture 6 : VS refined TSSCPP conjecture.
5. Conjecture 7, 7' : MT refined TSSCPP conjecture.

The TSSCPP conjecture

Theorem (Andrews)

The number of totally symmetric self-complementary plane partition of size n is equal to A_n.

Definition

If A be a matrix with n rows, we denote by $d_n(A)$ the sum of all minors of size n from A.

The number of the TSSCPPs

Theorem

Let

$$P_n = \left(\binom{i}{j-i} \right)_{0 \le i \le n-1,\, 0 \le j \le 2n-2}.$$

Then the number of TSSCPPs of size n is equal to $d_n(P_n)$.

Example

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 \end{pmatrix}$$

The refined TSSCPP conjecture

<u>Conjecture</u> (MRR, Conjecture 2)

Let $1 \leq k \leq n$ and $1 \leq r \leq n$. Then the number of elements b of $\mathcal{B}_n$ such that $U_r(b) = k - 1$ would be $A_{n,k}$. Namely, $\sum_{b \in \mathcal{B}_n} t^{U_r(b)} = A_n(t)$ would hold.

Theorem

Let

$$P_n(t) = \left(\begin{cases} \delta_{i,j} & \text{if } i = 0, \\ \binom{i-1}{j-i-1} + \binom{i-1}{j-i} t & \text{if } i > 0. \end{cases} \right)_{0 \le i \le n-1,\, 0 \le j \le 2n-2}.$$

The polynomial $\sum_{b \in \mathcal{B}_n} t^{U_r(b)}$ is equal to $d_n(P_n(t))$.

Example

$$P_4(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1+t & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 2+t & 1+2t & t \end{pmatrix}$$

Theorem

The polynomial $\sum_{b\in\mathcal{B}_n} t^{U_r(b)}$ is given by the Pfaffian $\operatorname{Pf}(a_{ij}(t))_{1\le i,j\le n}$ if n is even, and $\operatorname{Pf}(a_{ij}(t))_{0\le i,j\le n}$ if n is odd. Here $a_{0j} = (1+t)\delta_{0,j-1}$ and

$$
\begin{aligned}
a_{ij}(t) = (1+t^2)\Bigg\{ & 2\binom{i+j-3}{2i-j} + 3\binom{i+j-3}{2i-j-1} \\
& - 3\binom{i+j-3}{2i-j-2} - 2\binom{i+j-3}{2i-j-3}\Bigg\} \\
& + t\Bigg\{2\binom{i+j-3}{2i-j+1} + 3\binom{i+j-3}{2i-j} - \binom{i+j-3}{2i-j-1} \\
& + \binom{i+j-3}{2i-j-2} - 3\binom{i+j-3}{2i-j-3} - 2\binom{i+j-3}{2i-j-4}\Bigg\}
\end{aligned}
$$

when $0 < i < j$.

The doubly refined TSSCPP conjecture

Conjecture (MRR, Conjecture 3)

Let $n \geq 2$ and $1 \leq k, l \leq n$ be integers. Then the number of elements b of $\mathcal{B}_n$ such that $U_1(b) = k - 1$ and $U_2(b) = n - l$ would be $B_n(k, l)$.

Theorem

Let n be a positive integer and let $2 \le r \le n$.

Let

$$Q_n(t,u)$$

$$= \left(\begin{cases} \delta_{i,j} & \text{if } i = 0, \\ u\binom{i-1}{j-i} + t\binom{i-1}{j-i-1} & \text{if } i = 1, \\ u\binom{i-2}{j-i} + (1+tu)\binom{i-2}{j-i-1} + t\binom{i-2}{j-i-2} & \text{if } i \ge 2. \end{cases} \right)_{0 \le i \le n-1,\, 0 \le j \le 2n-2}.$$

The polynomial $\sum_{b \in \mathcal{B}_n} t^{U_1(b)} u^{U_r(b)}$ is equal to $d_n(Q_n(t,u))$.

Example

$$Q_4(t,u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & t & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 1+tu & t & 0 & 0 \\ 0 & 0 & 0 & u & 1+u+tu & 1+t+tu & t \end{pmatrix}$$

The refined HTS TSSCPP conjecture

Conjecture (MRR, Conjecture 4)

Let $n \geq 2$ and r, $0 \leq r < n$ be integers. Then the number of elements of $\mathcal{B}_n$ with $\rho(b) = b$ and $U_1(b) = r$ would be $A_{n,r}^{\mathsf{HTS}}$. Namely, $\sum_{\substack{b \in \mathcal{B}_n \\ \rho(b)=b}} t^{U_1(b)} = A_n^{\mathsf{HTS}}(t)$ would hold.

The refined VS TSSCPP conjecture

Conjecture (MRR, Conjecture 6)

Let $n \geq 3$ an odd integer and r, $0 \leq r < n$ be an integer. Then the number of elements of $\mathcal{B}_n$ with $\gamma(b) = b$ and $U_2(b) = r$ would be $A^{\mathsf{VS}}_{n,r}$. Namely, $\sum_{\substack{b \in \mathcal{B}_n \\ \gamma(b) = b}} t^{U_2(b)} = A^{\mathsf{VS}}_n(t)$ would hold.

Theorem

Let $n \geq 3$ an odd integer.

The polynomial $\sum_{\substack{b \in \mathcal{B}_n \\ \gamma(b)=b}} t^{U_2(b)}$ is given by the determinant $\det(c_{ij}(t))_{0 \leq i,j \leq n-1}$, where $c_{00} = 1$, $c_{0j} = \binom{j}{2j} + \binom{j}{2j+1}t$ when $j \geq 1$, $c_{i0} = \binom{i}{-i+1} + \binom{i}{-i}t$ when $i \geq 1$, and

$$c_{ij}(t) = \binom{i+j-1}{2j-i} + \left\{ \binom{i+j-1}{2j-i-1} + \binom{i+j-1}{2j-i+1} \right\} t + \binom{i+j-1}{2j-i} t^2$$

when $i, j \geq 1$.

Theorem

Let $n \geq 3$ an odd integer.

Then the number of elements of $\mathcal{B}_n$ with $\gamma(b) = b$ is equal to A_n^{VS}.

The refined MT TSSCPP conjecture

For $k = 0, \ldots, n-1$, let $\mathcal{M}_n^k$ be the set of monotone triangles with all entries m_{ij} in the first $n-k$ columns equal to their minimum values $j-i+1$. For example, $\mathcal{M}_3^0$ is composed of one element, $\mathcal{M}_3^1$ is composed of five elements, and $\mathcal{M}_3^2 = \mathcal{M}_3$.

For $k = 0, \ldots, n-1$, let $\mathcal{B}_n^k$ be the subset of those b in $\mathcal{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximal values n.

Conjecture (MRR, Conjecture 7)

For $n \geq 2$ and $k = 0, \ldots, n-1$, the cardinality of $\mathcal{B}_n^k$ is equal to the cardinality of $\mathcal{M}_n^k$.

The MT TSSCPPs

Theorem

Let $n \geq 2$ and $k = 1, \ldots, n-1$. Let

$$P_n^k = \left(\binom{i}{j-i} \right)_{0 \leq i \leq n-1,\, 0 \leq j \leq n+k-1}.$$

Then the cardinality of $\mathcal{B}_n^k$ is equal to $d_n(P_n^k)$.

Example

$$P_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Restricted column-stricted plane partitions

Let $\mathcal{P}_n$ denote the class of column-strict (ordinary) plane partitions in which each part in the jth column does not exceed $n-j$. We call an element of $\mathcal{P}_n$ a restricted column-stricted plane partition.

Example

$\mathcal{P}_1$ consists of the following 1 PPs: $\emptyset$

$\mathcal{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \begin{array}{|c|}\hline 1\\\hline\end{array}$$

$\mathcal{P}_3$ consists of the following 7 PPs:

$$\emptyset \qquad \begin{array}{|c|}\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline 1 \\\cline{1-1}\end{array}$$

Saturated parts

Let $\pi \in \mathcal{P}_n$. A part π_{ij} of π is said to be saturated if $\pi_{ij} = n - j$. A saturated part, if it exists, appears only in the first row.

Example

$n = 7$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Definition

Let $c = (c_{ij})_{1 \le i \le n, 1 \le j \le n}$ be a RCSPP in $\mathcal{P}_n$ and let k be a positive integer. Let $c_{\ge k}$ denote the plane partition formed by the parts $\ge k$. Let

$$\theta_i(c_{\ge k}) = \sharp\{l : c_{i,l} \ge k\}$$

denote the length of the ith row of $c_{\ge k}$, i.e. the rightmost column containing a letter $\ge k$ in the ith row of c.

A bijection

Theorem

Let $n \geq 1$ be nonnegative integers and $c = (c_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ be a RCSPP in $\mathcal{P}_{n,m}$. Associate to the array $c = (c_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ the array $b = (b_{ij})_{1 \leq i \leq j \leq n-1}$ defined by

$$n - b_{ij} = \theta_{n-j}(c_{\geq 1-i+j})$$

with $1 \leq i \leq j \leq n-1$. Then b is in $\mathcal{B}_n$, and this mapping φ_n, which associate to a RCSPP c the TSPP $b = \varphi_n(c)$, is a bijection of $\mathcal{P}_n$ onto $\mathcal{B}_n$.

A statistics

Definition

For $\pi \in \mathcal{P}_n$ let

$$\overline{U}_k(\pi) = \sharp\{(i,j)|\pi_{ij} = k\} + \sharp\{1 \leq i < k|\pi_{1,n-i} = i\}$$

for $1 \leq k \leq n$, i.e. $\overline{U}_k(\pi)$ is the number of parts equal to k plus the number of saturated parts less than k.

Especially,

$\overline{U}_1(\pi)$: the number of 1s in π,

$\overline{U}_n(\pi)$: the number of saturated parts in π.

Example $n = 7.$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

$$\overline{U}_1(\pi) = 3, \quad \overline{U}_2(\pi) = 5, \quad \overline{U}_3(\pi) = 3, \quad \overline{U}_4(\pi) = 4,$$
$$\overline{U}_5(\pi) = 4, \quad \overline{U}_6(\pi) = 3, \quad \overline{U}_7(\pi) = 3.$$

The statistics

Theorem

Let $n \geq 1$ be nonnegative integers and let $c \in \mathcal{P}_n$. Then

$$\overline{U}_r(c) = n - 1 - U_r(\varphi_n(c))$$

A deformed Bender-Knuth involution

Now we define a Bender-Knuth type involution $\widetilde{\pi}_r : \mathcal{P}_n \longrightarrow \mathcal{P}_n$. Let $2 \leq r \leq n$ and $c \in \mathcal{P}_n$. Consider the parts of c equal to r or $r-1$. Since c is column-strict, some columns of c will contain neither r nor $r-1$, while some others will contain one r and one $r-1$. These columns we ignore. We also ignore an $r-1$ in column $n-r+1$, i.e. we ignore a saturated part which is equal to $r-1$ because a saturated $r-1$ can't be changed to r. The remaining parts equal to r or $r-1$ occur once in each column. Assume row i has a certain number k of r's followed by a certain number l of $r-1$'s. Note that we don't count an $r-1$ if it is saturated so that a saturated $r-1$ always remains untouched. In row i, convert the k r's and l $r-1$'s to l r's and k $r-1$'s.

Involution $\widetilde{\pi}_r$

Define an operation $\widetilde{\pi}_r : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by $c \mapsto \widetilde{\pi}_r(c)$ where $\widetilde{\pi}_r(c)$ is the result of swapping r's and $r-1$'s in row i of c by this deformed rule for $1 \le i \le n-r$. We call the involution $\widetilde{\pi}_r$, $1 \le r \le n$, the deformed Bender-Knuth involution (abbreviated to the DBK involution).

$i-1$	$\vdots$		$\vdots$							r
i	r	$\ldots$	r	r	$\ldots$	r	$r-1$	$\ldots$	$r-1$	$r-1$
$i+1$	$r-1$	$\ldots$	$r-1$							

Example

$\mathcal{P}_3$ consists of the following 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline 1 \\\cline{1-1}\end{array}$$

and mapped to

$$\emptyset \qquad \begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1\\\hline 1 \\\cline{1-1}\end{array}$$

by $\widetilde{\pi}_2$, respectively.

Example

$\mathcal{P}_3$ consists of the following 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and mapped to

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \emptyset \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array}$$

by $\widetilde{\pi}_1$, respectively.

Proposition

Let $n \geq 1$ be non-negative integers. Let $2 \leq r \leq n$ and let c in $\mathcal{P}_n$. Then

$$\overline{U}_r\left(\widetilde{\pi}_r(c)\right) = \overline{U}_{r-1}\left(c\right)$$

and

$$\overline{U}_r\left(c\right) = \overline{U}_{r-1}\left(\widetilde{\pi}_r(c)\right).$$

Theorem

Let $n \geq 1$ be non-negative integers and let $1 \leq r \leq n$.

Then we have

$$\pi_r\left(\varphi_n\left(c\right)\right) = \varphi_n\left(\widetilde{\pi}_r(c)\right).$$

A HT involution

Define an involution $\widetilde{\gamma} : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by

$$\widetilde{\rho} = \widetilde{\pi}_2 \widetilde{\pi}_4 \widetilde{\pi}_6 \cdots$$

where the product is over all $\widetilde{\pi}_i$ with i even and $\leq n$.

Let $\mathcal{P}_n^{\widetilde{\rho}}$ denote the set of elements of $\mathcal{P}_n$ which is invariant under $\widetilde{\rho}$.

Example

There are 1 elements of $\mathcal{P}_1$ that is invariant under $\widetilde{\rho}$.

$$\emptyset$$

There are 2 elements of $\mathcal{P}_2$ that is invariant under $\widetilde{\rho}$.

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

There are 3 elements of $\mathcal{P}_3$ that is invariant under $\widetilde{\rho}$.

$$\emptyset \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & \color{red}{1} \\ \hline 1 & \\ \cline{1-1} \end{array}$$

There are 10 elements of $\mathcal{P}_4$ that is invariant under $\widetilde{\rho}$.

There are 25 elements of $\mathcal{P}_5$ that is invariant under $\widetilde{\rho}$.

A vertical flip involution

Define an involution $\widetilde{\gamma} : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by

$$\widetilde{\gamma} = \widetilde{\pi}_1 \widetilde{\pi}_3 \widetilde{\pi}_5 \cdots$$

where the product is over all $\widetilde{\pi}_i$ with i odd and $\leq n$.

Let $\mathcal{P}_n^{\widetilde{\gamma}}$ denote the set of elements of $\mathcal{P}_n$ which is invariant under $\widetilde{\gamma}$.

$\mathcal{P}_n^{\widetilde{\gamma}}$ is empty unless n is odd.

Example

There are 1 element of $\mathcal{P}_3$ which is invariant under $\widetilde{\gamma}$.

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

There are 3 element of $\mathcal{P}_5$ which is invariant under $\widetilde{\gamma}$.

$$\emptyset \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

There are 26 element of $\mathcal{P}_7$

Restricted column-stricted domino plane partitions

Let $\mathcal{P}_{2n+1}^{\mathsf{VS}}$ be the set of domino plane partitions c which satisfies

- (F1) the shape of c is even;
- (F2) c is column-strict;
- (F3) each part in the jth column does not exceed $\lfloor (2n+2-j)/2 \rfloor$.

We call an element of $\mathcal{P}_{2n+1}^{\mathsf{VS}}$ a restricted column-strict domino plane partition (abbreviated to RCSDPP). The condition (F3) can be restated as follows; if $c \in \mathcal{P}_{2n+1}^{\mathsf{VS}}$, then all the parts in the 1st and 2nd row of c are $\leq n-1$, all the parts in the 3rd and 4th row of c are $\leq n-2$, and so on.

Example

For example, if n=5, then $\mathcal{P}_5^{\mathsf{VS}}$ is composed of the following three elements.

$\emptyset$

We also let $\overline{U}_1(c)$ denote the number of 1's in c for $c \in \mathcal{P}_{2n+1}^{\mathsf{VS}}$. From the above example, we have $\sum_{c\in\mathcal{P}_5^{\mathsf{VS}}} t^{\overline{U}_1(c)} = 1 + t + t^2$. The reader can easily check that there are 26 elements in $\mathcal{P}_7^{\mathsf{VS}}$ and $\sum_{c\in\mathcal{P}_7^{\mathsf{VS}}} t^{\overline{U}_1(c)} = 3 + 6t + 8t^2 + 6t^3 + 3t^4$.

A bijection

Theorem

There is a bijection between RCSPPs $\mathcal{P}_{2n+1}$ invariant under $\widetilde{\gamma}$ and RCSDPPs $\mathcal{P}_{2n+1}^{\mathsf{VS}}$. By this bijection $\overline{U}_2$ of $\mathcal{P}_{2n+1}$ corresponds to $\overline{U}_1$ of $\mathcal{P}_{2n+1}^{\mathsf{VS}}$.

Another restricted column-stricted domino plane partitions

Let $\mathcal{P}^{\mathsf{HTS}}_{2n+1}$ be the set of domino plane partitions c which satisfies

(F1') the conjugate of the shape of c is even;

(F2) c is column-strict;

(F3) each part in the jth column does not exceed $\lfloor (2n+2-j)/2 \rfloor$.

The condition (F3) can be restated as follows; if $c \in \mathcal{P}^{\mathsf{HTS}}_{2n+1}$, then all the parts in the 1st and 2nd row of c are $\le n-1$, all the parts in the 3rd and 4th row of c are $\le n-2$, and so on.

Another bijection

There is a strong evidence that the following conjecture holds.

Conjecture

There would be a bijection between RCSPPs $\mathcal{P}_{2n+1}$ invariant under $\widetilde{\rho}$ and $\mathcal{P}^{\mathsf{HTS}}_{2n+1}$. By this bijection $\overline{U}_1$ of $\mathcal{P}_n$ corresponds to $\overline{U}_1$ of $\mathcal{P}^{\mathsf{HTS}}_{2n+1}$.

Carré-Leclerc bijection

Proposition

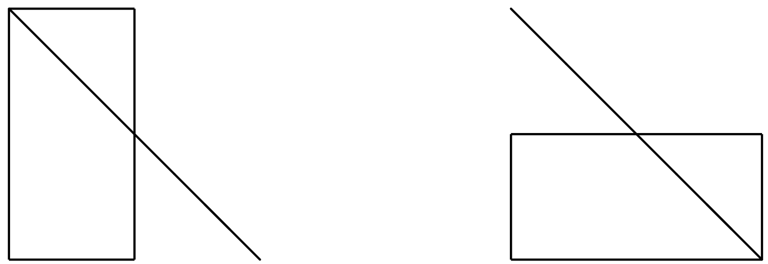
Carré-Leclerc defined a bijection between a domino plane partition T and a pair of plane partitions (T^0, T^1). By this bijection,

1. the shape of T is even if and only if the shape T^0 is obtained by removing a vertical strip from the shape of T^1;
2. the conjugate of the shape of T is even if and only if the shape T^1 is obtained by removing a horizontal strip from the shape of T^0,

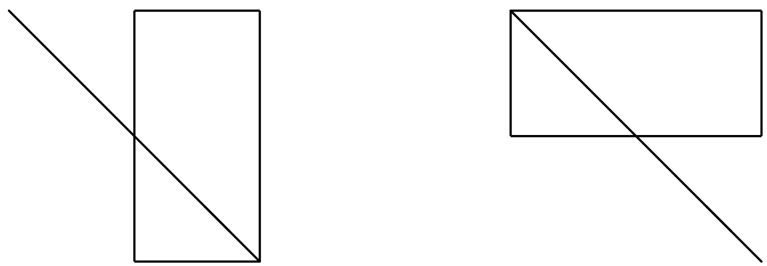
C. Carré and B. Leclerc, “Splitting the Square of a Schur Function into its Symmetric and Antisymmetric Parts”, J. Algebraic Combin. **4** (1995), 201 – 231.

Color rule

Color 0:

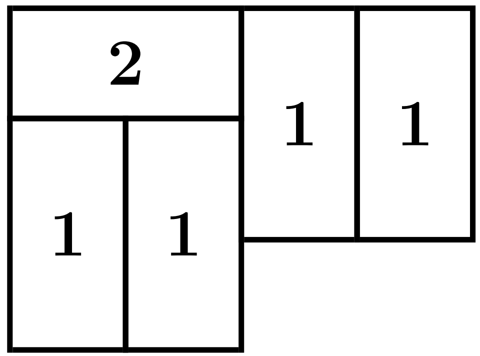


Color 1:

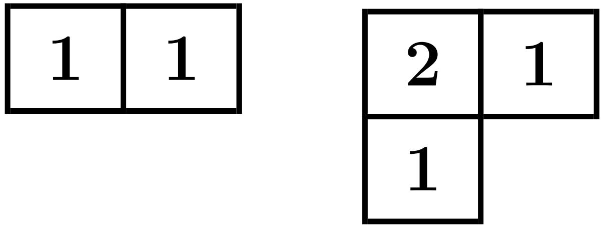


Example

The domino plane partition



correspond to the following pair of plane partitions:



Paired restriced column-stricted plane partitions

Let $\mathcal{Q}_n^{\mathsf{VS}}$ be the set of pairs (c^0, c^1) of plane partitions which satisfies

(G1) $c^0, c^2 \in \mathcal{P}_n$;

(G2) The shape of c^0 is obtained by removing a vertical strip from the shape of c^1.

We call an element of $\mathcal{Q}_n^{\mathsf{VS}}$ a paired restricted column-strict plane partition (abbreviated to PRCSPP).

<u>Theorem</u>

There is a bijection between RCSPPs $\mathcal{P}_n$ invariant under $\widetilde{\gamma}$ and PRCSPPs $\mathcal{Q}_n^{\mathsf{VS}}$.

Example

$\mathcal{P}_{\mathbf{5}}^{\mathsf{VS}}$ is composed of the following three elements

$$\emptyset\;, \qquad \boxed{1}\;, \qquad \boxed{1}\boxed{1}\;,$$

which corresponds to

$$(\emptyset, \emptyset), \qquad \left(\emptyset, \boxed{1}\right), \qquad \left(\boxed{1}, \boxed{1}\right),$$

respectively.

$\mathcal{P}_n^k$

Definition

For $k = 0, \ldots, n-1$, let $\mathcal{P}_n^k$ denote the subset of those $c = (c_{ij})$ in $\mathcal{P}_n$ which has at most k rows.

Example

$\mathcal{P}_3$ consists of the following seven plane partitions.

$\emptyset$	1	$1\ \mathbf{1}$	$\mathbf{2}$	$\mathbf{2}\ \mathbf{1}$	$\mathbf{2}$	$\mathbf{2}\ \mathbf{1}$
					1	1

There are only one element, i.e. $\emptyset$, of $\mathcal{P}_3$ with no row, five elements of $\mathcal{P}_3$ with with at most one row, and seven elements of $\mathcal{P}_3$ with at most two rows.

Bijection

Theorem

Let $n \geq 1$ be nonnegative integers. Let $0 \leq k \leq n-1$. By the bijection φ_n defined above, the subset $\mathcal{B}_n^k$ of $\mathcal{B}_n$ is in one-to-one correspondence with the subset $\mathcal{P}_n^k$ of $\mathcal{P}_n$.

Let $t=(t_1,\ldots,t_n)$ and $x=(x_1,\ldots,x_{n-1})$ be sets of variables. Let $\overline{U}(\pi)=(\overline{U}_1(\pi),\ldots,\overline{U}_n(\pi))$ and we set $t^{\overline{U}(\pi)}=\prod_{k=1}^{n} t_k^{\overline{U}_k(\pi)}$. Similarly we write x^π for $\prod_{ij} x_{\pi_{ij}}$.

Theorem 0.1.

$$\sum_{\substack{\pi\in\mathcal{P}_n\\ \mathrm{sh}(\pi)=\lambda'}} t^{\overline{U}(\pi)}x^\pi$$

$$=\det\left(e^{(n-i)}_{\lambda_j-j+i}\left(t_1x_1,\ldots,t_{n-i-1}x_{n-i-1},\prod_{r=1}^{n} t_r x_{n-i}\right)\right)_{1\le i,j\le n}$$

where $e_r^{(m)}(x)$ denote the rth elementary symmetric function in the viariables $(x_1,\ldots,x_m)$, i.e.

$$\sum_r e_r^{(m)}(x)z^r=\prod_{i=1}^{m}(1+x_iz)$$

Corollary 0.2.

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}(\pi)} x^{\pi}$$

is the sum of the all minors of the rectangular matrix

$$\left[e^{(i)}_{j-i} \left(t_1 x_1, \ldots, t_{n-i-1} x_{n-i-1}, \prod_{r=1}^{n} t_r x_{n-i} \right) \right]_{\substack{0 \le i \le n-1 \\ 0 \le j \le 2n-2}}$$

of size n.

Example.

When $n = 3$, the sum of all minors of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_1 t_2 t_3 x_1 & 0 & 0 \\ 0 & 0 & 1 & t_1 x_1 + t_2 t_3 x_2 & t_1 t_2 t_3 x_1 x_2 \end{bmatrix}$$

is $1 + t_1 x_1 + t_2 t_3 x_2 + t_1 t_2 t_3 x_1 x_2 + t_1^2 t_2 t_3 x_1^2 + t_1 t_2^2 t_3^2 x_1 x_2 + t_1^2 t_2^2 t_3^2 x_1^2 x_2$.

Each term corresponds to the following PPs:

$\emptyset$	$\overline{U}_1(\pi) = 0$	$\overline{U}_2(\pi) = 0$	$\overline{U}_3(\pi) = 0$	1
$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\overline{U}_1(\pi) = 1$	$\overline{U}_2(\pi) = 0$	$\overline{U}_3(\pi) = 0$	$t_1 x_1$
$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$	$\overline{U}_1(\pi) = 2$	$\overline{U}_2(\pi) = 1$	$\overline{U}_3(\pi) = 1$	$t_1^2 t_2 t_3 x_1^2$
$\begin{array}{\|c\|}\hline 2 \\ \hline\end{array}$	$\overline{U}_1(\pi) = 0$	$\overline{U}_2(\pi) = 1$	$\overline{U}_3(\pi) = 1$	$t_2 t_3 x_2$
$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline\end{array}$	$\overline{U}_1(\pi) = 1$	$\overline{U}_2(\pi) = 2$	$\overline{U}_3(\pi) = 2$	$t_1 t_2^2 t_3^2 x_1 x_2$
$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline\end{array}$	$\overline{U}_1(\pi) = 1$	$\overline{U}_2(\pi) = 1$	$\overline{U}_3(\pi) = 1$	$t_1 t_2 t_3 x_1 x_2$
$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$	$\overline{U}_1(\pi) = 2$	$\overline{U}_2(\pi) = 2$	$\overline{U}_3(\pi) = 2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

Corollary 0.3.

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}_k(\pi)} = d_n\left(P_n(t)\right)$$

where $d_n(A)$ stands for the sum of all minors of size n from A.

Corollary 0.4.

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}_1(\pi)} s^{\overline{U}_2(\pi)} = d_n\left(Q_n(t, s)\right)$$

Conjecture ?? is equivalent to the following conjecture.

Conjecture 0.5. (Refined TSSCPP conjecture)

The number of $\pi \in \mathcal{P}_n$ such that $\overline{U}_k(\pi) = r - 1$ is $A_n(r)$ for $1 \leq r \leq n$ and $1 \leq k \leq n$.

(cf. [13][14])

Conjecture 0.6. (Double refined TSSCPP conjecture)

The number of $\pi \in \mathcal{P}_n$ such that $\overline{U}_1(\pi) = r - 1$ and $\overline{U}_2(\pi) = n - s$ is $B_n(r, s)$ for $1 \leq r, s \leq n$. (cf. [14][21])

1 ASM

A alternating sign matrix (ASM) is, by definition, a matrix of 0s, 1s, and -1s in which the entries in each row or column sum to 1 and the nonzero entries in each row and column alternate in sign. The additional restriction is added that any -1s in a row or column must have a "outside" it (i.e., all -1s are "bordered" by $+1$s),

Let $\mathcal{A}_n$ denote the set of all ASMs of size n.

Example.

$\mathcal{A}_1$: $\begin{bmatrix} 1 \end{bmatrix}$

$\mathcal{A}_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\mathcal{A}_3$:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix}$$

Theorem 1.1. (Zeilberger, Kuperberg) [Alternating sign matrix conjecture]

The number of the ASMs of size n is A_n.

D.P. Robbins, "Symmetry Classes of Alternating Sign Matrices", `arXiv:math.CO/0008045`.

The 8-element group of symmetries of square acts on square matrices. For any subgroup of the group we may consider the subset of matrices invariant under elements of the subgroup. There are 8 conjugacy classes of these subgroups giving rise to 8 symmetry classes of matrices.

1. no conditions (ASM)
2. $a_{ij} = a_{i,n-1-j}$ vertical symmetric (VSASM)
3. $a_{ij} = a_{n-1-i,n-1-j}$ half turn symmetric (HTSASM)
4. $a_{ij} = a_{ji}$ diagonal symmetric (DSASM)
5. $a_{ij} = a_{j,n-1-i}$ quarter-turn symmetric (QTSASM)
6. $a_{ij} = a_{i,n-1-j} = a_{n-1-i,j}$ veriticallly and horizontally symmetric (VHSASM)
7. $a_{ij} = a_{ji} = a_{n-1-j,n-1-i}$ both diagonals (DASASM)

8. $a_{ij} = a_{ji} = a_{i,n-1-j}$ all symmetries (TSASM)

Let μ be a non-negative integer and define

$$Z_n(x, y, \mu) = \det(\delta_{ij} + z_{ij})_{0 \le i,j \le n-1}$$

where

$$z_{ij} = \sum_{t,k=0}^{n-1} \binom{i+\mu}{t} \binom{k}{t} \binom{j-k+\mu-1}{j-k} x^{k-t}$$

for $0 \le i \le n-2$, $0 \le j \le n-1$ and

$$z_{n-1,j} = \sum_{t,k,l=0}^{n-1} \binom{n-2+\mu-l}{t-l} \binom{k}{t} \binom{j-k+\mu-1}{j-k} x^{k-t} y^{l+1}$$

for $0 \le j \le n-1$.

Let

$$T_n(x,\mu)=\det\left(\sum_{t=0}^{2n-2}\binom{i+\mu}{t-i}\binom{j}{2j-t}x^{2j-t}\right)_{0\le i,j\le n-1}.$$

Let

$$Y(i,t,\mu)=\binom{i+\mu}{2i+1+\mu-t}+\binom{i+1+\mu}{2i+1+\mu-t}$$

and define

$$R_n(x,\mu)=\det\left(\sum_{t=0}^{2n-1}Y(i,t,\mu)Y(j,t,0)x^{2j+1-t}\right)_{0\le i,j\le n-1}.$$

Let

$$f(i,j)=\sum_{0\le k<l}\begin{vmatrix}\binom{x+i-1}{k-i-1}+\binom{x+i-1}{k-i}t & \binom{x+i-1}{l-i-1}+\binom{x+i-1}{l-i}t\\ \binom{y+j-1}{k-j-1}+\binom{y+j-1}{k-j}t & \binom{y+j-1}{l-j-1}+\binom{y+j-1}{l-j}t\end{vmatrix}.$$

Then

$$\begin{aligned}f(i,j)=\sum_{k\ge x+2i-j}&\left[(1+t^2)\binom{x+y+i+j-2}{k-1}\right.\\&\left.+t\left\{\binom{x+y+i+j-2}{k-2}+\binom{x+y+i+j-2}{k}\right\}\right]\\+\sum_{k\ge y+2j-i}&\left[(1+t^2)\binom{x+y+i+j-2}{k-1}\right.\\&\left.+t\left\{\binom{x+y+i+j-2}{k-2}+\binom{x+y+i+j-2}{k}\right\}\right].\end{aligned}$$

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