On Refined TSSCPP Conjectures

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1 TSSCPP

G.E. Andrews and W.H. Burge, “Determinant Identities”, Pacific J. Math. **158** (1993), 1–14.

Theorem 1.1. (Andrews-Burge)

$$M_n(x,y) = \det\left(\binom{i+j+x}{2i-j} + \binom{i+j+y}{2i-j}\right)_{0\le i,j\le n-1}$$
$$= \prod_{k=0}^{n-1} \Delta_{2k}(x+y),$$

where $\Delta_0(u) = 2$ and for $j > 0$

$$\Delta_{2j}(u) = \frac{(u+2j+2)_j(\frac{1}{2}u+2j+\frac{3}{2})_{j-1}}{(j)_j(\frac{1}{2}u+j+\frac{3}{2})_{j-1}}$$

with

$$(A)_j = A(A+1)\cdots(A+j-1).$$

Corollary 1.2. (Andrews)

$$\det(a_{ij})_{0\le i,j\le n-1} = A_n^2,$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } i = j > 0, \\ \sum_{s=2i-j+1}^{2j-i} \binom{i+j}{s} & \text{if } i < j, \\ -a_{ji} & \text{if } i > j, \end{cases}$$

and

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Example.

$$A_1 = 1, \quad A_2 = 2, \quad A_3 = 7, \quad A_4 = 42, \quad A_5 = 429.$$

If A be a matrix with n rows, we denote by $d_n(A)$ the sum of all minors of size n from A.

Theorem 1.3. Let

$$P_n = \left(\binom{i}{j-i}\right)_{0\le i\le n-1,\, 0\le j\le 2n-2}.$$

Then

$$d_n(P_n) = \mathrm{Pf}\,(a_{ij})_{0\le i,j\le n-1}.$$

Example.

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 \end{pmatrix} \qquad (a_{ij}) = \begin{pmatrix} 1 & 2 & 4 & 8 \\ -2 & 0 & 7 & 16 \\ -4 & -7 & 0 & 25 \\ -8 & -16 & -25 & 0 \end{pmatrix}$$

Proof of Corollary 1.2. We define several new matrices:

$$w_n = \left(\binom{i+j+1}{2j-i} + \binom{i+j}{2j-i-1}\right)_{0\le i,j\le n-1},$$
$$u(n) = (\delta_{ij} - 2\delta_{i,j+1})_{0\le i,j\le n-1},$$
$$v_n = \left(\frac{(3i+1)(3j+1)(3j-3i)}{(i+j)(i+j+1)(i+j+2)}\binom{i+j+2}{2j-i+1}\right)_{0\le i,j\le n-1},$$

(where we define the $(0,0)$-th entry of $v(n)$ to be 1),

$$u_1(n) = u(n) + (2\delta_{(i-1)^2+j,0})_{0\le i,j\le n-1},$$
$$st_n = (a_{ij})_{0\le i,j\le n-1}.$$

Then elementary algebra reveals

$$u(n)w_n = v_n,$$
$$u_1(n)st_n{}^t u_1(n) = v_n.$$

If we expand $M_{n+1}(-2,-1)$ along the top row, we find

$$M_{n+1}(-2,-1) = 2\det\left(\binom{i+j}{2i-j+1} + \binom{i+j+1}{2i-j+1}\right)_{0\le i,j\le n-1}$$
$$= 2\det\left(w_n\right).$$

Since the determinant of u and u_1 are each 1, it follows

$$\det(a_{ij})_{0\le i,j\le n-1} = \det(st_n) = \det(v_n)$$
$$= \det(w_n) = \frac{1}{2}M_{n+1}(-2,-1)$$
$$= \prod_{k=1}^{n}\Delta_{2k}(-3) = A_n^2$$

because

$$\Delta_{2k}(-3) = \left(\frac{(3k-1)!(k-1)!}{(2k-2)!(2k-1)!}\right)^2. \qquad \square$$

Let $A_n(k)$ denote the number defined by

$$A_n(k) = \frac{\binom{n+k-1}{n-1}\binom{2n-k-2}{n-1}}{\binom{2n-2}{n-1}} A_n$$

Then it satisfies

$$\sum_{k=1}^{n} A_n(k) = A_n.$$

and

$$\frac{A_n(k+1)}{A_n(k)} = \frac{(n-k)(n+k-1)}{k(2n-k-1)}$$

for $0 < k < n$.

Let $F_n(t) = \sum_{k=1}^{n} A_n(k) t^{k-1}$.

Example. $A_n(k)$ for $1 \le k \le n \le 6$.

					1					
				1		1				
			2		3		2			
		7		14		14		7		
	42		105		135		105		42	
429		1287		2002		2002		1287		429

Example.

$$F_1(t) = 1$$

$$F_2(t) = 1 + t$$

$$F_3(t) = 2 + 3t + 2t^2$$

$$F_4(t) = 7(1 + t)(1 + t + t^2)$$

$$F_5(t) = 42 + 105t + 135t^2 + 105t^3 + 42t^4$$

$$F_6(t) = 429 + 1287x + 2002x^2 + 2002x^3 + 1287x^4 + 429x^5$$

Let

$$P_n(t) = \left(\begin{cases} \delta_{i,j} & \text{if } i = 0, \\ \binom{i-1}{j-i-1} + \binom{i-1}{j-i} t & \text{if } i > 0. \end{cases} \right)_{0 \le i \le n-1,\, 0 \le j \le 2n-2}.$$

Example.

$$P_4(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1+t & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 2+t & 1+2t & t \end{pmatrix}$$

Conjecture 1.4.

$$d_n\left(P_n(t)\right) = F_n(t)$$

Theorem 1.5.

$$d_n(P_n(t)) = \mathrm{Pf}\,(a_{ij}(t))_{0\le i,j\le n-1}$$

where

$$a_{ij}(t) = \begin{cases} 1 & \text{if } i=j=0, \\ 0 & \text{if } i=j>0, \\ 2^{j-1}(1+t) & \text{if } i=0,\ j>0, \\ (1+t^2)\sum_{s=2i-j+1}^{2j-i}\binom{i+j-2}{s-1} & \\ +t\left\{\sum_{s=2i-j+1}^{2j-i}\binom{i+j-2}{s-2}+\sum_{s=2i-j+1}^{2j-i}\binom{i+j-2}{s}\right\} & \text{if } 0<i<j, \\ -a_{ji} & \text{if } i>j. \end{cases}$$

Example.

$$(a_{ij}(t))_{1\le i,j\le 4}$$

$$= \begin{pmatrix} 1 & 1+t & 2(1+t) & 4(1+t) \\ -(1+t) & 0 & 2+3t+2t^2 & 4\ (1+t)^2 \\ -2(1+t) & -(2+3t+2t^2) & 0 & 7+11\,t+7\,t^2 \\ -4(1+t) & -4\ (1+t)^2 & -(7+11t+7t^2) & 0 \end{pmatrix}$$

Conjecture 1.4 is equivalent to the following conjecture:

Conjecture 1.6.

$$\det(a_{ij}(t))_{0\leq i,j\leq n-1} = F_n(t)^2$$

Let

$$w_n(t) =$$

$$\begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } i = j > 0, \\ \delta_{1,j}(1+t) & \text{if } i = 0,\ j > 0 \\ (1+t^2)\left\{\binom{i+j-2}{2i-j-1} + 2\binom{i+j-2}{2i-j}\right\} & \\ \quad + t\left\{\binom{i+j-2}{2i-j-2} + 2\binom{i+j-2}{2i-j-1} + \binom{i+j-2}{2i-j} + 2\binom{i+j-2}{2i-j+1}\right\} & \\ & \text{if } 0 < i < j, \\ -a_{ji} & \text{if } i > j. \end{cases}$$

Assume $B_n(r,s)$ satisfies the following recursion:

$$B_n(1,s) = \begin{cases} 0 & \text{if } s = 1, \\ A_{n-1}(n-s) & \text{if } 2 \le s \le n, \end{cases}$$

$$B_n(r,1) = \begin{cases} 0 & \text{if } r = 1, \\ A_{n-1}(n-r) & \text{if } 2 \le r \le n, \end{cases}$$

and

$$B_n(r+1,s+1) - B_n(r,s)$$
$$= \frac{A_{n-1}(r)\{A_n(s+1) - A_n(s)\} + A_{n-1}(s)\{A_n(r+1) - A_n(r)\}}{A_n(1)}$$

for $1 \le r, s \le n-1$.

Note that $A_n(1) = A_{n-1}$ and

$$\sum_{r=1}^{n} B_n(r,s) = A_n(s)$$

$$\sum_{s=1}^{n} B_n(r,s) = A_n(r)$$

Let

$$G_n(t,u) = \sum_{r,s=1}^{n} B_n(r,s) t^{r-1} u^{n-s}.$$

Example.

$$[B_1(r,s)]_{1\le r,s\le 1} = \begin{bmatrix} 1 \end{bmatrix}$$

$$[B_2(r,s)]_{1\le r,s\le 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[B_3(r,s)]_{1\le r,s\le 3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$[B_4(r,s)]_{1\le r,s\le 4} = \begin{bmatrix} 0 & 2 & 3 & 2 \\ 2 & 4 & 5 & 3 \\ 3 & 5 & 4 & 2 \\ 2 & 3 & 2 & 0 \end{bmatrix}$$

$$[B_5(r,s)]_{1\le r,s\le 5} = \begin{bmatrix} 0 & 7 & 14 & 14 & 7 \\ 7 & 21 & 33 & 30 & 14 \\ 14 & 33 & 41 & 33 & 14 \\ 14 & 30 & 33 & 21 & 7 \\ 7 & 14 & 14 & 7 & 0 \end{bmatrix}$$

Example.

$$G_1(t,u) = 1$$
$$G_2(t,u) = t + u$$
$$G_3(t,u) = t + u + t^2 + tu + u^2 + t^2u + tu^2$$
$$G_4(t,u) = 2t + 2u + 3t^2 + 4tu + 3u^2 + 2t^3 + 5t^2u + 5tu^2 + 2u^3 + 3t^3u + 4t^2u^2 + 3tu^3 + 2t^3u^2 + 2t^2u^3$$

Let

$Q_n(t,u)$

$$= \left(\begin{cases} \delta_{i,j} & \text{if } i=0, \\ u\binom{i-1}{j-i} + t\binom{i-1}{j-i-1} & \text{if } i=1, \\ u\binom{i-2}{j-i} + (1+tu)\binom{i-2}{j-i-1} + t\binom{i-2}{j-i-2} & \text{if } i \geq 2. \end{cases} \right)_{0\leq i\leq n-1,\, 0\leq j\leq 2n-2}.$$

Example.

$$Q_4(t,u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & t & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 1+tu & tu & 0 & 0 \\ 0 & 0 & 0 & u & 1+u+tu & 1+t+tu & tu \end{pmatrix}$$

Conjecture 1.7.

$$d_n(Q_n(t,u)) = G_n(t,u)$$

Let $n \le N$. Let A be an n by N matrix.

Define $d'_n(A)$ by

$$\sum_{\substack{1\le j_1<\cdots<j_n\le N \\ j_{2k}=j_{2k-1}+1\ (k\ge 1)}} \det\left(\Delta_{j_1,\ldots,j_n}(A)\right)$$

if n is even,

$$\sum_{\substack{1\le j_1<\cdots<j_n\le N \\ j_{2k+1}=j_{2k}+1\ (k\ge 1)}} \det\left(\Delta_{j_1,\ldots,j_n}(A)\right)$$

if n is odd.

Example.

$$d'_1(P_1) = 1$$
$$d'_2(P_2) = 1$$
$$d'_3(P_3) = 3$$

$$d_4'(P_4) = 9$$
$$d_5'(P_5) = 78$$
$$d_6'(P_6) = 676$$
$$d_7'(P_7) = 16796$$
$$d_8'(P_8) = 417316$$

Set $S(n) = 3^{-n(n-1)} Sp(4n)(n-1, n-1, n-2, n-2, ...1, 1, 0, 0)$ where $Sp(4n)(n-1, n-1, n-2, n-2, ...1, 1, 0, 0)$ stands for the dimmension of the irreducible representation of the symmplectic group with the highest weight $(n-1, n-1, n-2, n-2, ...1, 1, 0, 0)$.

Example.

$Sp(4)(0, 0) = 1$, $Sp(8)(1, 1, 0, 0) = 27$, $Sp(12)(2, 2, 1, 1, 0, 0) = 18954$.

$S(1) = 1$, $S(2) = 3$, $Sp(3) = 26$.

Conjecture 1.8. 1. If n is even, then

$$\begin{aligned} d'_n(P_n) &= S(n)^2 \\ &= A^{(1)}_{UO}(8n, \vec{1}; \zeta_6) = A_V(2n+1; \vec{1}, \vec{1}; \zeta_6)^2 \end{aligned}$$

2. If n is odd, then

$$d'_n(P_n) = S(n)S(n+1)$$

(c.f. [15])

Let

$$a'_{ij} = \binom{i+j}{2j-i-1} - \binom{i+j}{2i-j-1}$$
$$= \frac{(3j-3i)(i+j+1)}{(2i-j+1)(2j-i+1)}\binom{i+j}{2j-i}$$

and let $st'_n = (a'_{ij})_{0\le i,j\le n-1}$

(where we define the $(0,0)$-th entry of $st(n)$ to be 1),

Theorem 1.9.

$$d'_n(P_n) = \mathrm{Pf}\,(st'_n).$$

Example.

$$st'_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 9 & 6 & 1 \\ 0 & -1 & -9 & 0 & 28 & 27 \\ 0 & 0 & -6 & -28 & 0 & 90 \\ 0 & 0 & -1 & -27 & -90 & 0 \end{bmatrix}$$

$\mathrm{Pf}(st'_6) = 676.$

Define $d''_n(A)$ by

$$\sum_{\substack{1 \leq j_1 < \cdots < j_n \leq N \\ j_{2k-1} \text{ odd, } j_{2k} \text{ even } (k \geq 1)}} \det\left(\Delta_{j_1,\ldots,j_n}(A)\right)$$

Conjecture 1.10.

$$d''_n(P_n) = A_{n-1}$$

A plane partition is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the weight n.

A part of a plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is a positive entry $\pi_{ij} > 0$. The shape of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row. The shape of π is denoted by $\mathrm{sh}(\pi)$. We say that π has r rows if $r = \ell(\lambda)$. Similarly, π has s columns if $s = \ell(\lambda')$.

Example. shape $(9, 8, 4, 1)$, 4 rows, 9 columns, weight 49.

5	5	4	3	3	2	2	2	1
4	4	2	2	1	1	1	1	
2	1	1	1					
1								

Example.

Plane partition of 0: $\emptyset$

Plane partition of 1: $\boxed{1}$

Plane partition of 2:

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}$$

Plane partition of 3:

$$\begin{array}{|c|}\hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

A plane partition is said to be column-strict if it is weakly decreasing in rows and strictly decreasing in coulumns.

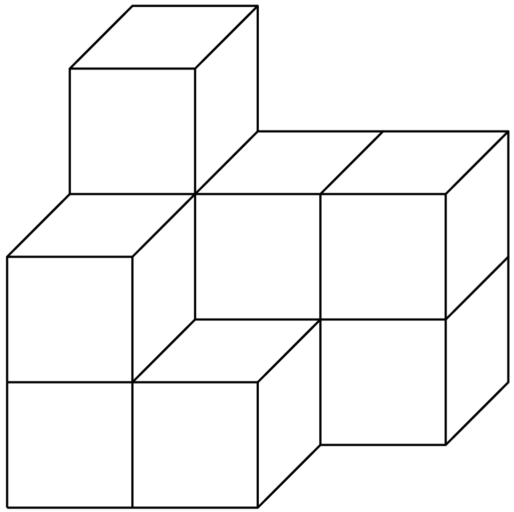
The **Ferrers graph** $F(\pi)$ of π is the set of all lattice points $(i,j,k) \in \mathbb{P}^3$ such that $k \leq \pi_{ij}$.

Example.

The Ferrers graph of

3	2	2
2	1	

is as follows:

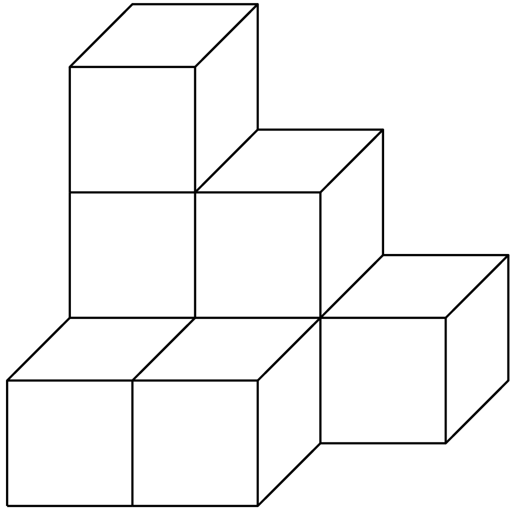


Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition with at most r rows, at most c columns, and with largest part at most t. We say that $\pi' = (\pi'_{ij})_{i,j\geq 1}$ is (r,c,t)-complementary plane partition of π if $\pi'_{ij} = t - \pi_{r-i,c-j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$.

Example. The $(3,2,3)$-complementary PP of the above PP is

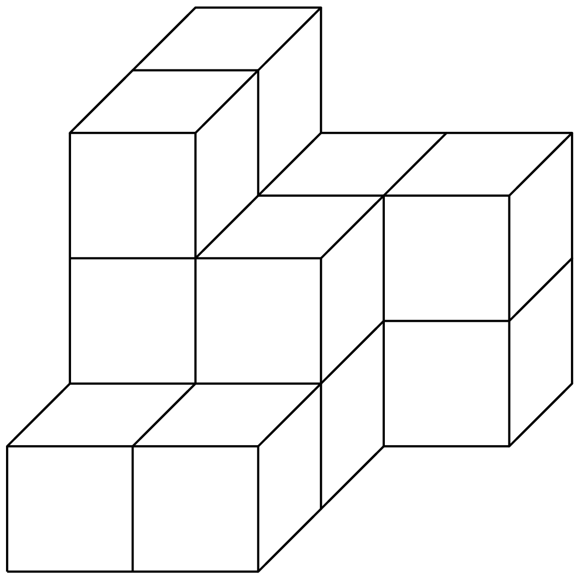
3	2	1
1	1	

and its Ferrers graph is as follows:



Let $\mathbf{P}$ denote the set of positive integers. Consider the elements of $\mathbb{P}^3$, regarded as the lattice points of $\mathbb{R}^3$ in the positive orthant. The symmetric group S_3 is acting on $\mathbb{P}^3$ as permutations of the coordinate axies. A plane partition is said to be totally symmetric if its Ferrors graph is mapped to itself under all 6 permutations in S_3.

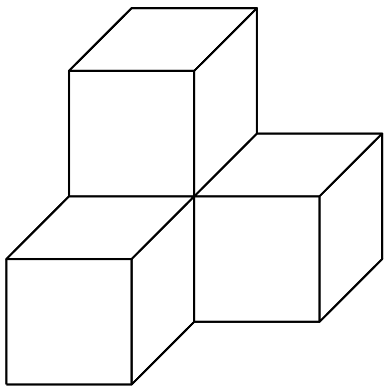
A plane partition is said to be cyclically symmetric if its Ferrors graph is mapped to itself under all 3 permutations in A_3.



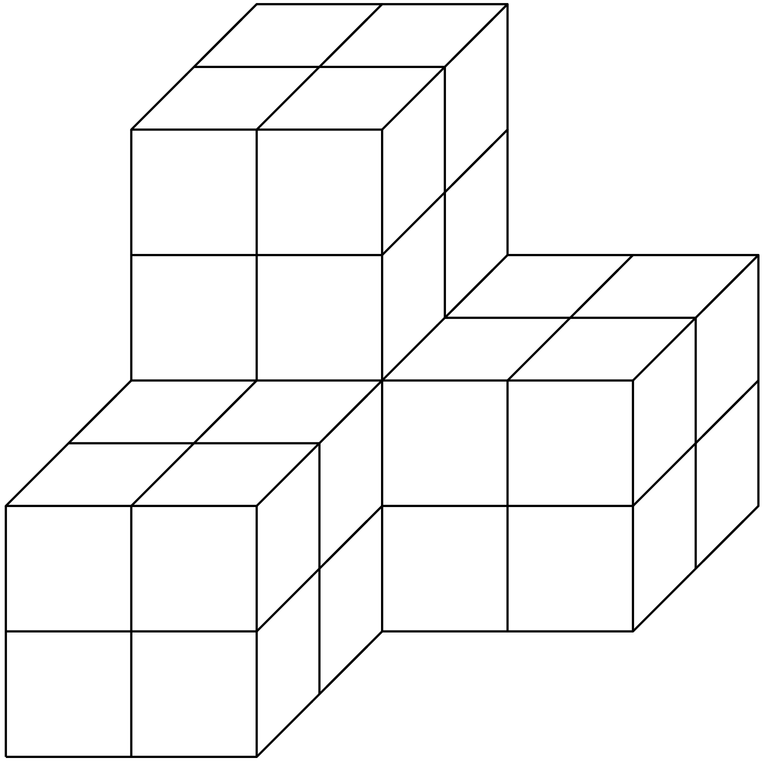
A plane partition $\pi = (\pi_{ij})_{i,j \geq 1}$ is said to be (r, c, t)-self-complementary if $\pi_{ij} = t - \pi_{r-i,c-j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$. Let $\mathcal{T}_n$ denote the set of all plane partitions which is contained in the box $X_n = [2n] \times [2n] \times [2n]$, $(2n, 2n, 2n)$-self-complementary and totally symmetric. An element of $\mathcal{T}_n$ is called a totally symmetric self-complementary plane partition (abbreviated as TSSCPP) of size n.

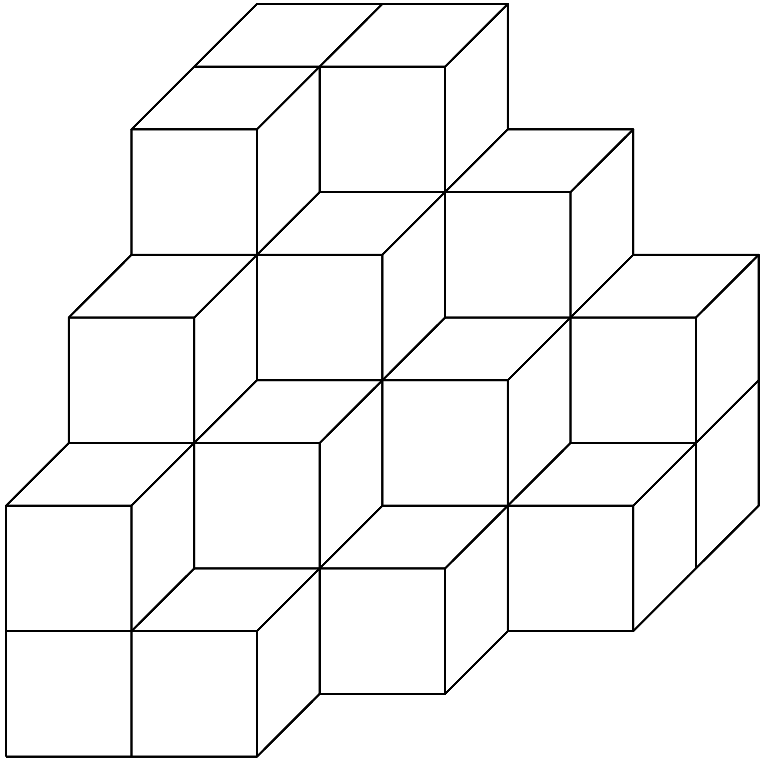
Example.

$\mathcal{T}_1$



$\mathcal{T}_2$





$\mathcal{T}_3$

6	6	6	3	3	3
6	6	6	3	3	3
6	6	6	3	3	3
3	3	3			
3	3	3			
3	3	3			

6	6	6	4	3	3
6	6	6	3	3	3
6	6	5	3	3	2
4	3	3	1		
3	3	3			
3	3	2			

6	6	6	4	3	3
6	6	6	4	3	3
6	6	4	3	2	2
4	4	3	2		
3	3	2			
3	3	2			

6	6	6	5	4	3
6	6	5	3	3	2
6	5	5	3	3	1
5	3	3	1	1	
4	3	3	1		
3	2	1			

6	6	6	5	4	3
6	6	5	4	3	2
6	5	4	3	2	1
5	4	3	2	1	
4	3	2	1		
3	2	1			

6	6	6	5	5	3
6	5	5	3	3	1
6	5	5	3	3	1
5	3	3	1	1	
5	3	3	1	1	
3	1	1			

6	6	6	5	5	3
6	5	5	4	3	1
6	5	4	3	2	1
5	4	3	2	1	
5	3	2	1	1	
3	1	1			

We can define a shifted plane partition similarly. A shifted plane partition is an array $\tau = (\tau_{ij})_{1 \le i \le j}$ of nonnegative integers such that τ has finite support and is weakly decreasing in rows and columns. The shifted shape of τ is the distinct partition μ for which τ has μ_i nonzero parts in the ith row. In [?], Mills, Robbins and Rumsey considered a class $\mathcal{B}_n$ of triangular shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$,

and they constructed a bijection between $\mathcal{T}_n$ and $\mathcal{B}_n$. In this paper we call an element of $\mathcal{B}_n$ a triangular shifted plane partition (abbreviated as TSPP) of size n.

When $n = 3$, $\mathcal{B}_3$ consists of the followng 7 elements:

$$\begin{array}{cc} 3 & 3 \\ & 3 \end{array} \quad \begin{array}{cc} 3 & 3 \\ & 2 \end{array} \quad \begin{array}{cc} 3 & 3 \\ & 1 \end{array} \quad \begin{array}{cc} 3 & 2 \\ & 2 \end{array} \quad \begin{array}{cc} 3 & 2 \\ & 1 \end{array} \quad \begin{array}{cc} 2 & 2 \\ & 2 \end{array} \quad \begin{array}{cc} 2 & 2 \\ & 1 \end{array}$$

In this talk we consider another classes of plane partitions. Let $\mathcal{P}_n$ denote the class of column-strict (ordinary) plane partitions in which each part in the jth column does not exceed $n-j$. We call an element of $\mathcal{P}_n$ a restricted column-stricted plane partition.

Example.

$\mathcal{P}_1$ consists of the following 1 PPs: $\emptyset$

$\mathcal{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \boxed{1}$$

$\mathcal{P}_3$ consists of the following 7 PPs:

$$\emptyset \qquad \boxed{1} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \boxed{2} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Let $\pi \in \mathcal{P}_n$. A part π_{ij} of π is said to be saturated if $\pi_{ij} = n - j$. A saturated part, if it exists, appears only in the first row.

Example. $n = 7$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

For $\pi \in \mathcal{P}_n$ let

$$\overline{U}_k(\pi) = \sharp\{(i,j) | \pi_{ij} = k\} + \sharp\{1 \leq i < k | \pi_{1,n-i} = i\}$$

for $1 \leq k \leq n$.

Especially,

$\overline{U}_1(\pi)$: the number of 1s in π,

$\overline{U}_n(\pi)$: the number of saturated parts in π.

Example. $n = 7$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

$\overline{U}_1(\pi) = 3$, $\overline{U}_2(\pi) = 4$, $\overline{U}_3(\pi) = 3$, $\overline{U}_4(\pi) = 3$, $\overline{U}_5(\pi) = 4$,
$\overline{U}_6(\pi) = 3$, $\overline{U}_7(\pi) = 3$.

Let $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots, x_{n-1})$ be sets of variables. Let $\overline{U}(\pi) = (\overline{U}_1(\pi), \ldots, \overline{U}_n(\pi))$ and we set $t^{\overline{U}(\pi)} = \prod_{k=1}^{n} t_k^{\overline{U}_k(\pi)}$. Similarly we write x^π for $\prod_{ij} x_{\pi_{ij}}$.

Theorem 1.11.

$$\sum_{\substack{\pi \in \mathcal{P}_n \\ \mathrm{sh}(\pi) = \lambda'}} t^{\overline{U}(\pi)} x^\pi$$

$$= \det \left(e^{(n-i)}_{\lambda_j - j + i} \left(t_1 x_1, \ldots, t_{n-i-1} x_{n-i-1}, \prod_{r=1}^{n} t_r x_{n-i} \right) \right)_{1 \le i,j \le n}$$

where $e_r^{(m)}(x)$ denote the rth elementary symmetric function in the viariables $(x_1, \ldots, x_m)$, i.e.

$$\sum_r e_r^{(m)}(x) z^r = \prod_{i=1}^{m} (1 + x_i z)$$

Corollary 1.12.

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}(\pi)} x^{\pi}$$

is the sum of the all minors of the rectangular matrix

$$\left[e^{(i)}_{j-i} \left(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, \prod_{r=1}^{n} t_r x_{n-i} \right) \right]_{\substack{0 \le i \le n-1 \\ 0 \le j \le 2n-2}}$$

of size n.

Example.

When $n = 3$, the sum of all minors of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_1 t_2 t_3 x_1 & 0 & 0 \\ 0 & 0 & 1 & t_1 x_1 + t_2 t_3 x_2 & t_1 t_2 t_3 x_1 x_2 \end{bmatrix}$$

is $1 + t_1 x_1 + t_2 t_3 x_2 + t_1 t_2 t_3 x_1 x_2 + t_1^2 t_2 t_3 x_1^2 + t_1 t_2^2 t_3^2 x_1 x_2 + t_1^2 t_2^2 t_3^2 x_1^2 x_2$.

Each term corresponds to the following PPs:

$\emptyset$	$\overline{U}_1(\pi) = 0$	$\overline{U}_2(\pi) = 0$	$\overline{U}_3(\pi) = 0$	1
$\boxed{1}$	$\overline{U}_1(\pi) = 1$	$\overline{U}_2(\pi) = 0$	$\overline{U}_3(\pi) = 0$	$t_1 x_1$
$\boxed{1}\boxed{1}$	$\overline{U}_1(\pi) = 2$	$\overline{U}_2(\pi) = 1$	$\overline{U}_3(\pi) = 1$	$t_1^2 t_2 t_3 x_1^2$
$\boxed{2}$	$\overline{U}_1(\pi) = 0$	$\overline{U}_2(\pi) = 1$	$\overline{U}_3(\pi) = 1$	$t_2 t_3 x_2$
$\boxed{2}\boxed{1}$	$\overline{U}_1(\pi) = 1$	$\overline{U}_2(\pi) = 2$	$\overline{U}_3(\pi) = 2$	$t_1 t_2^2 t_3^2 x_1 x_2$
$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\overline{U}_1(\pi) = 1$	$\overline{U}_2(\pi) = 1$	$\overline{U}_3(\pi) = 1$	$t_1 t_2 t_3 x_1 x_2$
$\begin{array}{\|c\|c\|}\hline 2 & 1\\\hline 1 & \\\cline{1-1}\end{array}$	$\overline{U}_1(\pi) = 2$	$\overline{U}_2(\pi) = 2$	$\overline{U}_3(\pi) = 2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

Corollary 1.13.

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}_k(\pi)} = d_n\left(P_n(t)\right)$$

where $d_n(A)$ stands for the sum of all minors of size n from A.

Corollary 1.14.

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}_1(\pi)} s^{\overline{U}_2(\pi)} = d_n\left(Q_n(t, s)\right)$$

Conjecture 1.6 is equivalent to the following conjecture.

Conjecture 1.15. (Refined TSSCPP conjecture)

The number of $\pi \in \mathcal{P}_n$ such that $\overline{U}_k(\pi) = r - 1$ is $A_n(r)$ for $1 \leq r \leq n$ and $1 \leq k \leq n$.

(cf. [13][14])

Conjecture 1.16. (Double refined TSSCPP conjecture)

The number of $\pi \in \mathcal{P}_n$ such that $\overline{U}_1(\pi) = r - 1$ and $\overline{U}_2(\pi) = n - s$ is $B_n(r, s)$ for $1 \leq r, s \leq n$. (cf. [14][21])

2 ASM

A alternating sign matrix (ASM) is, by definition, a matrix of 0s, 1s, and -1s in which the entries in each row or column sum to 1 and the nonzero entries in each row and column alternate in sign. The additional restriction is added that any -1s in a row or column must have a "outside" it (i.e., all -1s are "bordered" by $+1$s),

Let $\mathcal{A}_n$ denote the set of all ASMs of size n.

Example.

$\mathcal{A}_1$: $\begin{bmatrix} 1 \end{bmatrix}$

$\mathcal{A}_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\mathcal{A}_3$:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix}$$

Theorem 2.1. (Zeilberger, Kuperberg) [Alternating sign matrix conjecture]

The number of the ASMs of size n is A_n.

D.P. Robbins, "Symmetry Classes of Alternating Sign Matrices", `arXiv:math.CO/0008045`.

The 8-element group of symmetries of square acts on square matrices. For any subgroup of the group we may consider the subset of matrices invariant under elements of the subgroup. There are 8 conjugacy classes of these subgroups giving rise to 8 symmetry classes of matrices.

1. no conditions (ASM)
2. $a_{ij} = a_{i,n-1-j}$ vertical symmetric (VSASM)
3. $a_{ij} = a_{n-1-i,n-1-j}$ half turn symmetric (HTSASM)
4. $a_{ij} = a_{ji}$ diagonal symmetric (DSASM)
5. $a_{ij} = a_{j,n-1-i}$ quarter-turn symmetric (QTSASM)
6. $a_{ij} = a_{i,n-1-j} = a_{n-1-i,j}$ veriticallly and horizontally symmetric (VHSASM)
7. $a_{ij} = a_{ji} = a_{n-1-j,n-1-i}$ both diagonals (DASASM)

8. $a_{ij} = a_{ji} = a_{i,n-1-j}$ all symmetries (TSASM)

Let μ be a non-negative integer and define

$$Z_n(x,y,\mu) = \det(\delta_{ij} + z_{ij})_{0 \le i,j \le n-1}$$

where

$$z_{ij} = \sum_{t,k=0}^{n-1} \binom{i+\mu}{t} \binom{k}{t} \binom{j-k+\mu-1}{j-k} x^{k-t}$$

for $0 \le i \le n-2$, $0 \le j \le n-1$ and

$$z_{n-1,j} = \sum_{t,k,l=0}^{n-1} \binom{n-2+\mu-l}{t-l} \binom{k}{t} \binom{j-k+\mu-1}{j-k} x^{k-t} y^{l+1}$$

for $0 \le j \le n-1$.

Let

$$T_n(x,\mu)=\det\left(\sum_{t=0}^{2n-2}\binom{i+\mu}{t-i}\binom{j}{2j-t}x^{2j-t}\right)_{0\le i,j\le n-1}.$$

Let

$$Y(i,t,\mu)=\binom{i+\mu}{2i+1+\mu-t}+\binom{i+1+\mu}{2i+1+\mu-t}$$

and define

$$R_n(x,\mu)=\det\left(\sum_{t=0}^{2n-1}Y(i,t,\mu)Y(j,t,0)x^{2j+1-t}\right)_{0\le i,j\le n-1}.$$

Let

$$f(i,j)=\sum_{0\le k<l}\begin{vmatrix}\binom{x+i-1}{k-i-1}+\binom{x+i-1}{k-i}t & \binom{x+i-1}{l-i-1}+\binom{x+i-1}{l-i}t\\ \binom{y+j-1}{k-j-1}+\binom{y+j-1}{k-j}t & \binom{y+j-1}{l-j-1}+\binom{y+j-1}{l-j}t\end{vmatrix}.$$

Then

$$\begin{aligned}f(i,j)=&\sum_{k\ge x+2i-j}\Bigg[(1+t^2)\binom{x+y+i+j-2}{k-1}\\&\qquad+t\left\{\binom{x+y+i+j-2}{k-2}+\binom{x+y+i+j-2}{k}\right\}\Bigg]\\&+\sum_{k\ge y+2j-i}\Bigg[(1+t^2)\binom{x+y+i+j-2}{k-1}\\&\qquad+t\left\{\binom{x+y+i+j-2}{k-2}+\binom{x+y+i+j-2}{k}\right\}\Bigg].\end{aligned}$$

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