Multisoliton solutions of the vector nonlinear Schrödinger equation (Kulish–Sklyanin model) and the vector mKdV equation

Takayuki Tsuchida

December 21, 2015

Abstract

There exist two natural vector generalizations of the completely integrable nonlinear Schrödinger (NLS) equation in 1 + 1 dimensions: the well-known Manakov model and the lesser-known Kulish–Sklyanin model. In this paper, we propose a binary Darboux (or Zakharov–Shabat dressing) transformation that can be directly applied to the Kulish–Sklyanin model. By deriving a simple closed expression for iterations of the binary Darboux transformation, we obtain an explicit formula for the $N$-soliton solution of the Kulish–Sklyanin model under vanishing boundary conditions. Because the third-order symmetry of the vector NLS equation can be reduced to a vector generalization of the modified KdV (mKdV) equation, we can also obtain multisoliton (or multi-breather) solutions of the vector mKdV equation in closed form.
1 Introduction

The cubic nonlinear Schrödinger (NLS) equation [1, 2]

\[ i \frac{\partial q}{\partial t} + q_{xx} + 2\sigma|q|^2q = 0, \quad \sigma = +1 \text{ or } -1, \]  

(1.1)

is a representative integrable system in 1 + 1 dimensions. The case \( \sigma = +1 \) and the case \( \sigma = -1 \) correspond to the self-focusing and self-defocusing NLS equation, respectively. The NLS equation can be generalized to a single vector equation involving the standard scalar product \( \langle \cdot, \cdot \rangle \) in two distinct ways while preserving the integrability [3]; that is, the Manakov model [4]

\[ i q_t + q_{xx} + 2\langle q, q^* \rangle q = 0 \]  

(1.2)

and the Kulish–Sklyanin model [5]

\[ i q_t + q_{xx} + 4\langle q, q^* \rangle q - 2\langle q, q \rangle q^* = 0. \]  

(1.3)

Here, \( q \) is a vector dependent variable and the asterisk denotes the complex conjugation. For brevity, we write down only the self-focusing case here, but it is straightforward to extend these models to the self-defocusing or a mixed focusing-defocusing case [6–11]. Note that these models often appear in some disguised forms; any invertible linear transformation can be applied to the vector \( q \), which mixes its components. The Kulish–Sklyanin model (1.3) can be reduced to the Manakov model (1.2) by setting \( \langle q, q \rangle = 0 \), up to a trivial rescaling; this can be realized by restricting the components of \( q \) as, e.g.,

\[ q = (q_1, \pm iq_1, q_3, \pm iq_3, \ldots, q_{2m-1}, \pm iq_{2m-1}). \]

This simple observation demonstrates that the explicit formula for the \( N \)-soliton solution of the Kulish–Sklyanin model (1.3) and the vector soliton interactions thereof are highly nontrivial and more complicated than those for the Manakov model (1.2) reported in [4, 12–14].

Clearly, the Manakov model (1.2) is obtained from the (generally rectangular) matrix generalization of the scalar NLS equation, i.e., the matrix NLS equation [15]:

\[ iQ_t + Q_{xx} + 2QQ^\dagger Q = O, \]  

(1.4)

as a special case. Here, the dagger denotes the Hermitian conjugation and the symbol \( O \) is used to stress that this is a matrix equation. In contrast, the Kulish–Sklyanin model (1.3) is obtained from the matrix NLS equation (1.4) through the nontrivial reduction

\[ Q = q_1 I + \sum_{j=1}^{2m-1} q_{j+1} e_j. \]
Here, $I$ is the identity matrix; $\{e_1, e_2, \ldots, e_{2m-1}\}$ are skew-Hermitian ($e_j^\dagger = -e_j$) matrices that form generators of the Clifford algebra, i.e., they satisfy the anticommutation relations:

$$\{e_j, e_k\}_+ := e_je_k + e_ke_j = -2\delta_{jk}I, \quad (1.5)$$

where $\delta_{jk}$ is the Kronecker delta. We require that $\{I, e_1, e_2, \ldots, e_{2m-1}\}$ are linearly independent.

Because the ancestor model, the matrix NLS equation (1.4), can be solved using the inverse scattering method and the $N$-soliton solution can be written down explicitly, it is straightforward to obtain the $N$-soliton solution of the Kulish–Sklyanin model (1.3) through the reduction. However, the obtained expression is “non-classical” in the sense that it involves the generators of the Clifford algebra $\{e_1, e_2, \ldots, e_{2m-1}\}$ explicitly in a rather complicated manner; it is a highly nontrivial task to translate such a “non-classical” expression into a more user-friendly “classical” expression not involving $\{e_1, e_2, \ldots, e_{2m-1}\}$, using the anticommutation relations (1.5). Indeed, this can be achieved for the one- and two-soliton solutions, but not for the general $N$-soliton solution in practice.

The main objective of this paper is to derive a simple closed expression for the general $N$-soliton solution of the Kulish–Sklyanin model (1.3) without recourse to the $N$-soliton solution of the matrix NLS equation (1.4). To this end, we consider a nonstandard Lax representation for the Kulish–Sklyanin model (1.3) [16], which does not involve the generators of the Clifford algebra $\{e_1, e_2, \ldots, e_{2m-1}\}$, and apply a binary Darboux (or Zakharov–Shabat dressing) transformation [17–21]. A peculiar structure of the binary Darboux transformation allows us to express an arbitrary number of its iterations in simple explicit form. Thus, by applying the $N$-fold binary Darboux transformation to the trivial zero solution, we obtain the bright $N$-soliton solution of the Kulish–Sklyanin model (1.3) in closed form. Actually, the binary Darboux transformation can be applied to all the isospectral flows that belong to the same integrable hierarchy as the Kulish–Sklyanin model (1.3). Among the higher flows of this integrable hierarchy, the third-order flow is particularly interesting because it simplifies to a vector analog of the modified KdV (mKdV) equation [22, 23]:

$$q_y + q_{xxx} + 6\langle q, q \rangle q_x = 0 \quad (1.6)$$

under the reduction $q = q^*$. Thus, with a minor tune-up of the multifold binary Darboux transformation, we can obtain multisoliton solutions, multibreather solutions and their mixtures of the vector mKdV equation (1.6).

This paper is organized as follows. In section 2, we summarize two different Lax representations for the Kulish–Sklyanin model (1.3) and make some
remarks on its soliton solutions. In section 3, we propose the binary Darboux transformation and apply its $N$-fold version to the Kulish–Sklyanin model (1.3) to obtain its general $N$-soliton solution in simple explicit form. We also discuss how to obtain exact solutions such as the $N$-soliton solution of the vector mKdV equation (1.6); the obtained $N$-soliton formula is different from the multisoliton formula proposed by Iwao and Hirota [24] using the Hirota bilinear method [25], and our formula has its own advantages. Section 4 is devoted to concluding remarks.

2 Lax representations

We start with the matrix generalization of the nonreduced NLS system [26,27] proposed by Zakharov and Shabat as early as 1974 [15]:

$$\begin{cases}
    iQ_t + Q_{xx} - 2QRQ = O, \\
    iR_t - R_{xx} + 2RQR = O.
\end{cases} \tag{2.1}$$

Here, $Q$ and $R$ are $l_1 \times l_2$ and $l_2 \times l_1$ (generally rectangular) matrices. Some relevant information and references on the matrix NLS system (2.1) can be found in [28].

The Lax representation [29] for the matrix NLS system (2.1) is given by the following overdetermined linear system [30,31]:

$$\begin{bmatrix}
    \Psi_1 \\
    \Psi_2
\end{bmatrix}_x = \begin{bmatrix}
    -i\zeta I_{l_1} & Q \\
    R & i\zeta I_{l_2}
\end{bmatrix} \begin{bmatrix}
    \Psi_1 \\
    \Psi_2
\end{bmatrix}, \tag{2.2}
$$

$$\begin{bmatrix}
    \Psi_1 \\
    \Psi_2
\end{bmatrix}_t = \begin{bmatrix}
    -2i\zeta^2 I_{l_1} - iQR & 2\zeta Q + iQ_x \\
    2\zeta R - iR_x & 2i\zeta^2 I_{l_2} + iRQ
\end{bmatrix} \begin{bmatrix}
    \Psi_1 \\
    \Psi_2
\end{bmatrix}. \tag{2.3}
$$

Here, $\zeta$ is a spectral parameter independent of $x$ and $t$, and $I_{l_1}$ and $I_{l_2}$ are the $l_1 \times l_1$ and $l_2 \times l_2$ identity matrices, respectively; in this paper, we usually consider the $l_1 = l_2$ case and omit the index of the identity matrix.

The matrix NLS system (2.1) is a positive flow in the integrable hierarchy associated with the spectral problem (2.2). The next higher flow in the integrable hierarchy is a matrix analog [30, 31] of the nonreduced complex mKdV equation [26,27,32], i.e.

$$\begin{cases}
    Q_y + Q_{xxx} - 3Q_xRQ - 3QRRQ_x = O, \\
    R_y + R_{xxx} - 3R_xQR - 3RQR_x = O.
\end{cases} \tag{2.4}$$

To reduce the matrix NLS system (2.1) to the Kulish–Sklyanin model (1.3) or, more generally, the matrix NLS hierarchy to the Kulish–Sklyanin hierarchy, we introduce $2^{m-1} \times 2^{m-1}$ skew-Hermitian matrices $\{e_1, e_2, \ldots, e_{2m-1}\}$
that satisfy the anticommutation relations (1.5). Then, we set

\[ Q = q_1 I + \sum_{j=1}^{2m-1} q_{j+1} e_j, \quad R = r_1 I - \sum_{j=1}^{2m-1} r_{j+1} e_j. \quad (2.5) \]

The matrices \( \{ I, e_1, e_2, \ldots, e_{2m-1} \} \) are assumed to be linearly independent. Lax representations involving the generators of the Clifford algebra (or quaternions in the \( m = 2 \) case) can be traced back to the references [5,33,34].

As a natural extension of the complex conjugate, we define “Clifford conjugate” denoted as \( \hat{} \), which acts on the linear span of \( \{ I, e_1, e_2, \ldots, e_{2m} \} \) to reverse the sign of the coefficients of \( \{ e_1, e_2, \ldots, e_{2m} \} \). For instance,

\[ \hat{Q} = q_1 I - \sum_{j=1}^{2m-1} q_{j+1} e_j, \quad \hat{R} = r_1 I + \sum_{j=1}^{2m-1} r_{j+1} e_j. \]

Note that \( \hat{Q} = Q \). Because of the anticommutation relations (1.5), we have useful relations such as

\[ Q \hat{Q} = \hat{Q} Q = \langle q, q \rangle I, \]
\[ QR + \hat{R} Q = \hat{Q} \hat{R} + R Q = 2 \langle q, r \rangle I, \]
\[ QRQ = \left( QR + \hat{R} \hat{Q} \right) Q - \hat{R} \hat{Q} Q = 2 \langle q, r \rangle Q - \langle q, q \rangle \hat{R}, \quad (2.6) \]
\[ RQR = R \left( QR + \hat{R} \hat{Q} \right) - \hat{R} \hat{Q} Q = 2 \langle q, r \rangle R - \langle r, r \rangle \hat{Q}, \quad (2.7) \]
\[ (I - \mu QR) \left( I - \mu \hat{R} \hat{Q} \right) = \left( 1 - 2 \mu \langle q, r \rangle + \mu^2 \langle q, q \rangle \langle r, r \rangle \right) I. \quad (2.8) \]

Here, \( q = (q_1, q_2, \ldots, q_{2m}) \) and \( r = (r_1, r_2, \ldots, r_{2m}) \) are 2m-component row vectors; \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product, e.g., \( \langle q, r \rangle = \sum_{j=1}^{2m} q_j r_j \), etc.

Owing to (2.6) and (2.7), the reduction (2.5) simplifies the matrix NLS system (2.1) to the nonreduced Kulish–Sklyanin model:

\[
\begin{cases}
    i q_t + q_{xx} - 4 \langle q, r \rangle q + 2 \langle q, q \rangle r = 0, \\
    i r_t - r_{xx} + 4 \langle q, r \rangle r - 2 \langle r, r \rangle q = 0.
\end{cases} \quad (2.9)
\]

Note that \( \langle q, r \rangle \) and \( q_j r_k - q_k r_j \) are conserved densities for (2.9). By further imposing the general complex conjugation reduction

\[ r_j = \sigma_j q_j^*, \quad \sigma_j = \pm 1, \quad j = 1, 2, \ldots, 2m, \]

\[ i q_t + q_{xx} - 4 \langle q, r \rangle q + 2 \langle q, q \rangle r = 0, \]
\[ i r_t - r_{xx} + 4 \langle q, r \rangle r - 2 \langle r, r \rangle q = 0. \]
we obtain the Kulish–Sklyanin model with a mixed focusing-defocusing non-linearity:

\[ i q_t + q_{xx} - 4 \langle q, q^\ast \Sigma \rangle q + 2 \langle q, q \rangle q^\ast \Sigma = 0. \]  

(2.10)

Here, \( \Sigma := \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{2m}) \) is a diagonal matrix with each entry \( \sigma_j \) equal to +1 or −1. In the following, we mainly consider the Kulish–Sklyanin model in the self-focusing case:

\[ i q_t + q_{xx} + 4 \langle q, q^\ast \rangle q - 2 \langle q, q \rangle q^\ast = 0. \]  

(2.11)

The third-order symmetry of the nonreduced Kulish–Sklyanin model (2.9) is obtained by imposing the reduction (2.5) on the matrix complex mKdV system (2.4) and noting the identities \( Q_x R Q + Q R Q_x = (Q R Q)_x - Q R_x Q, \)
\( R_x Q R + R Q R_x = (R Q R)_x - R Q_x R \) in view of (2.6) and (2.7); by further setting \( r = -q \) (and thus \( R = -Q \) in (2.5)), (2.4) reduces to the vector mKdV equation [22,23]:

\[ q_y + q_{xxx} + 6 \langle q, q \rangle q_x = 0. \]  

(2.12)

The matrix NLS hierarchy can be solved using the inverse scattering method based on the spectral problem (2.2), so the exact solutions such as the \( N \)-soliton solution of the matrix NLS system (2.1), as well as the third-order symmetry (2.4), can be obtained explicitly in closed form. Thus, the exact solutions of the Kulish–Sklyanin model (2.11), as well as the vector mKdV equation (2.12), can also be obtained by imposing the corresponding reduction conditions on the scattering data involved in the solution. However, this approach is useful only if the number of components or solitons is small enough. Indeed, the obtained formula for the \( N \)-soliton solution of the \( 2m \)-component Kulish–Sklyanin model (2.11) involves the inverse of an \( N \times N \) block matrix, where each block is a \( 2^m \times 2^m \) matrix taking values in the linear span of \( \{ I, e_1, e_2, \ldots, e_{2m-1} \} \). The formula is too bulky and not a mathematically tractable object for \( 2m > 4 \) and \( N > 2 \).

In the four-component case \((2m = 4)\), the reduction (2.5) is no longer a restriction. Indeed, one can employ \( 2 \times 2 \) Pauli’s matrices multiplied by the imaginary unit i as a matrix representation for \( \{ e_1, e_2, e_3 \} \):

\[
e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.
\]

These matrices together with the identity matrix form a basis, i.e., any \( 2 \times 2 \) complex matrix can be expressed as a linear combination of \( \{ I, e_1, e_2, e_3 \} \); thus, (2.5) is merely a linear transformation mixing the elements in the \( 2 \times 2 \)
matrices $Q$ and $R$. In the self-focusing case, this linear transformation reads

$$Q = \begin{bmatrix} q_1 + iq_4 & iq_2 + q_3 \\ iq_2 - q_3 & q_1 - iq_4 \end{bmatrix}, \quad R = -Q^\dagger = \begin{bmatrix} -q_1^* + iq_4^* & i q_2^* + q_3^* \\ i q_2^* - q_3^* & -q_1^* - iq_4^* \end{bmatrix},$$

where $Q$ satisfies the matrix NLS equation (1.4). Clearly, the $N$-soliton solution of the Kulish–Sklyanin model (2.11) for a four-component vector $\mathbf{q}$ can be directly obtained from the $N$-soliton solution of the matrix NLS equation (1.4) for a $2 \times 2$ matrix $Q$ by applying this linear transformation.

The Kulish–Sklyanin model (2.11) for a three-component vector $\mathbf{q}$ is obtained by setting one component, say $q_3$, in the four-component case as identically zero. The reduction $q_3 = 0$ corresponds to the restriction of $Q$ to a symmetric matrix [35]; the corresponding reduction of the $N$-soliton solution from the four-component case to the three-component case is straightforward [36,37].

It is clear by setting $q_2 = q_3 = 0$ in the above representation that the Kulish–Sklyanin model (2.11) in the two-component case, say $\mathbf{q} = (q_1, q_4)$ can be decoupled into two scalar NLS equations in the variables $q_1 \pm i q_4$ [38]. Thus, any solution of the two-component Kulish–Sklyanin model can be written as a linear combination of two solutions of the scalar NLS equation; in this sense, the two-component case is trivial and less interesting. The rank-1 one-soliton solution in the two-component case is

$$\mathbf{q}(x, t) = 2\eta \text{sech} \left[2\eta (x + 4\xi t) + \alpha\right] e^{-2i\xi x - 4i(\xi^2 - \eta^2) t + i\varphi} \left( \frac{1}{2}, \pm \frac{1}{2} \right),$$

and the rank-2 one-soliton solution is

$$\mathbf{q}(x, t) = 2\eta \text{sech} \left[2\eta (x + 4\xi t) + \alpha_1\right] e^{-2i\xi x - 4i(\xi^2 - \eta^2) t + i\varphi_1} \left( \frac{1}{2}, -\frac{i}{2} \right) + 2\eta \text{sech} \left[2\eta (x + 4\xi t) + \alpha_2\right] e^{-2i\xi x - 4i(\xi^2 - \eta^2) t + i\varphi_2} \left( \frac{1}{2}, \frac{i}{2} \right).$$

Here, $\eta > 0$ and the other parameters are real constants. This implies that the rank-1 one-soliton solution in the general component case is

$$\mathbf{q}(x, t) = 2\eta \text{sech} \left[2\eta (x + 4\xi t) + \alpha\right] e^{-2i\xi x - 4i(\xi^2 - \eta^2) t + i\varphi} \langle \mathbf{u}, \mathbf{u} \rangle \mathbf{u}$$

(2.13)

where $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and $\langle \mathbf{u}, \mathbf{u}^* \rangle = \frac{1}{2}$, and the rank-2 one-soliton solution is

$$\mathbf{q}(x, t) = 2\eta \text{sech} \left[2\eta (x + 4\xi t) + \alpha_1\right] e^{-2i\xi x - 4i(\xi^2 - \eta^2) t + i\varphi} \langle \mathbf{u}, \mathbf{u} \rangle \mathbf{u} + 2\eta \text{sech} \left[2\eta (x + 4\xi t) + \alpha_2\right] e^{-2i\xi x - 4i(\xi^2 - \eta^2) t + i\varphi} \langle \mathbf{u}, \mathbf{u}^* \rangle \mathbf{u}^*$$

(2.14)
where $\langle u, u \rangle = 0$ and $\langle u, u^* \rangle = \frac{1}{2}$.

In the two-component case, the Kulish–Sklyanin model with a mixed focusing-defocusing nonlinearity is more interesting than the model with a simple focusing (or defocusing) nonlinearity. Indeed, (2.10) with $q = (q_1, q_4)$ and $\Sigma = \text{diag}(-1, 1)$ [39]:

\[
\begin{align*}
&i q_{1,t} + q_{1,xx} + 2 \left( |q_1|^2 - 2 |q_4|^2 \right) q_1 - 2 q_1^* q_1^* = 0, \\
i q_{4,t} + q_{4,xx} + 2 \left( 2 |q_1|^2 - |q_4|^2 \right) q_4 + 2 q_1^* q_4^* = 0,
\end{align*}
\]

is obtained from the matrix NLS system (2.1) through the reduction $Q = \left[ q_1 + i q_4 \quad 0 \\
0 \quad q_1 - i q_4 \right]$, $R = \left[ -q_1^* - i q_4^* \quad 0 \\
0 \quad -q_1^* + i q_4^* \right]$. Thus, the two-component Kulish–Sklyanin model with a mixed focusing-defocusing nonlinearity (2.15) is equivalent to the nonreduced scalar NLS system [26,27]:

\[
\begin{align*}
i q_t + q_{xx} - 2 q^2 r &= 0, \\
i r_t - r_{xx} + 2 r^2 q &= 0,
\end{align*}
\]

through the linear change of variables $q = q_1 + i q_4, r = -q_1^* - i q_4^*$ (or $q = q_1 - i q_4, r = -q_1^* + i q_4^*$).

In this paper, we aim to obtain a compact and tractable expression for the $N$-soliton solution of the Kulish–Sklyanin model (2.11), which is valid for an arbitrary number of components and does not involve the generators of the Clifford algebra. To derive such a “classical” expression, we first rewrite the spectral problem (2.2) under the reduction (2.5) to a more convenient form. We consider a linear eigenfunction the first component of which is an invertible matrix; then, the spectral problem (2.2) can be rewritten in terms of $P := \Psi_2 \Psi_1^{-1}$ as a matrix Riccati equation (see [40–42] for the scalar case and [43,44] for the vector case):

\[
P_x = R + 2 i \zeta P - PQP.
\]

Thus, under the reduction (2.5) and appropriate boundary conditions, we can confine $P$ to the linear span of $\{ I, e_1, e_2, \ldots, e_{2m-1} \}$. By setting

\[
P = p_1 I - \sum_{j=1}^{2m-1} p_{j+1} e_j, \quad p = (p_1, p_2, \ldots, p_{2m})
\]

and noting the relation (2.6), we can simplify (2.16) to a vector Riccati equation:

\[
p_x = r + 2 i \zeta p - 2 \langle p, q \rangle p + \langle p, p \rangle q.
\]
We introduce the scalar denominator $f$ and the vector numerator $g$ as

$$p = \frac{g}{f},$$

and set

$$\langle g, g \rangle = fh.$$  \hspace{1cm} (2.18b)

Noting the freedom to multiply $f$ and $g$ by any common factor, we can linearize the vector Riccati equation (2.17) as

$$\begin{pmatrix} f \\ g^T \\ h \end{pmatrix}_x = \begin{pmatrix} -2i\zeta & 2q & 0 \\ r^T & 0 & q^T \\ 0 & 2r & 2i\zeta \end{pmatrix} \begin{pmatrix} f \\ g^T \\ h \end{pmatrix},$$

where the superscript $^T$ denotes the matrix transpose. This spectral problem is the spatial part of a nonstandard Lax representation for the nonreduced Kulish–Sklyanin model (2.9) [16]; this kind of nonstandard spectral problem first appeared in [7, 27] through the investigation of the squared eigenfunctions associated with the scalar NLS hierarchy and a certain vector generalization was studied in [45, 46]. The corresponding time part of the Lax representation can, in principle, be derived from (2.3) in an analogous manner, but it is easier to obtain the temporal Lax matrix from the compatibility condition as a truncated power series in the spectral parameter $\zeta$ [7, 26, 27]. For later convenience, we rescale $q$, $r$, and $g^T$ by a factor of $1/\sqrt{2}$ and set $2\zeta =: \lambda$ to reformulate the nonstandard Lax representation in a more symmetric and concise form.

**Proposition 2.1.** The nonreduced Kulish–Sklyanin model with an arbitrary number of vector components:

$$\begin{align*}
\{ \begin{align*}
&i q_t + q_{xx} - 2\langle q, r \rangle q + \langle q, q \rangle r = 0, \\
&i r_t - r_{xx} + 2\langle q, r \rangle r - \langle r, r \rangle q = 0,
\end{align*} \} \quad (2.19)
\end{align*}$$

is equivalent to the compatibility condition for the overdetermined linear system [16]:

$$\begin{align*}
\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_x &= \begin{pmatrix} -i\lambda & q & 0 \\ r^T & 0 & q^T \\ 0 & r & i\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \\
\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_t &= \begin{pmatrix} -i\lambda^2 - i\langle q, r \rangle & \lambda q + iq_x & 0 \\ \lambda r^T - ir^T_x & 0 & \lambda q^T + iq^T_x \\ 0 & \lambda r - ir_x & i\lambda^2 + i\langle q, r \rangle \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.
\end{align*} \quad (2.20, 2.21)$$
Here, $\lambda$ is a constant spectral parameter, $\mathbf{q}$ and $\mathbf{r}$ are row vectors and $\psi_2$ is a column vector.

By rewriting the spectral problem (2.20) as the adjoint problem

$$\begin{bmatrix} \psi_3 - \psi_2^T \psi_1 \end{bmatrix}_x = - \begin{bmatrix} \psi_3 - \psi_2^T \psi_1 \end{bmatrix} \begin{bmatrix} -i\lambda & \mathbf{q} & 0 \\ \mathbf{r}^T & \mathbf{O} & \mathbf{q}^T \\ 0 & \mathbf{r} & i\lambda \end{bmatrix},$$

(2.22)

or noting the identity

$$\begin{bmatrix} \psi_3 - \psi_2^T \psi_1 \end{bmatrix}_x = - \begin{bmatrix} \psi_3 - \psi_2^T \psi_1 \end{bmatrix} \begin{bmatrix} -i\lambda & \mathbf{q} & 0 \\ \mathbf{r}^T & \mathbf{O} & \mathbf{q}^T \\ 0 & \mathbf{r} & i\lambda \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = 0,$$

and similar for $t$-differentiation, we notice that the quantity $2\psi_1\psi_3 - \langle \psi_2, \psi_2 \rangle$ is a constant. In fact, the derivation from the standard Lax representation, (2.2) and (2.3), through the reduction (2.5) implies that we only need to consider linear eigenfunctions satisfying the condition $2\psi_1\psi_3 = \langle \psi_2, \psi_2 \rangle$ (cf. (2.18b)). In this paper, we use the notation $\langle \cdot, \cdot \rangle$ to denote the scalar product of two row vectors as well as column vectors.

**Proposition 2.2.** The third-order symmetry of the nonreduced Kulish–Sklyanin model (2.19) reads [47, 48]

$$\begin{aligned}
q_y + q_{xxx} - 3\langle q_x, r \rangle q - 3\langle q, r \rangle q_x + 3\langle q, q_x \rangle r &= 0, \\
r_y + r_{xxx} - 3\langle q, r_x \rangle r - 3\langle q, r \rangle r_x + 3\langle r, r_x \rangle q &= 0.
\end{aligned}$$

(2.23)

The reduction $r = -q$ simplifies (2.23) to the vector mKdV equation [22, 23]:

$$q_y + q_{xxx} + 3\langle q, q \rangle q_x = 0,$$

(2.24)

which is obtained as the compatibility condition for the overdetermined linear system:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}_x = \begin{bmatrix} -i\lambda & \mathbf{q} & 0 \\ -\mathbf{q}^T & \mathbf{O} & \mathbf{q}^T \\ 0 & -\mathbf{q} & i\lambda \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix},$$

(2.25)

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}_y = \begin{bmatrix} -i\lambda^3 + i\lambda\langle q, q \rangle & \lambda^2 q + i\lambda q_x - \alpha & 0 \\ -\lambda^2 q^T + i\lambda q_x^T + \alpha^T & -2\lambda^2 q + 2\lambda q^T q_x & \lambda^2 q^T + i\lambda q_x - \alpha^T \\ 0 & -\lambda^2 q + i\lambda q_x + \alpha & i\lambda^3 - i\lambda\langle q, q \rangle \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix},$$

(2.26)

with $\alpha := q_{xx} + \langle q, q \rangle q$. 

10
3 Darboux transformations and multisoliton solutions

3.1 Darboux transformations

We propose the binary Darboux (or Zakharov–Shabat dressing) transformation \([17–21]\) that can be applied to the spectral problem (2.20) associated with the Kulish–Sklyanin hierarchy. This can be obtained by considering how the binary Darboux transformation for the spectral problem (2.2) acts on \(P = \Psi_2 \Psi_1^{-1}\) under the reduction (2.5), confining the result to the linear span of \(\{I, e_1, e_2, \ldots, e_{2m-1}\}\) and then linearizing the discrete vector equation through the transformation (2.18).

Let \(\Lambda\) be the block anti-diagonal matrix:

\[
\Lambda := \begin{bmatrix} 1 & -I & 1 \\ 1 & & \end{bmatrix}, \quad \Lambda^T = \Lambda, \quad \Lambda^2 = I,
\]

and denote a column-vector eigenfunction of the spectral problem (2.20) at \(\lambda = \mu\) and its matrix transpose (i.e., row vector) as

\[
|\mu\rangle := \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}_{\lambda=\mu}, \quad \langle \mu | := \begin{bmatrix} \psi_1^T & \psi_2^T & \psi_3^T \end{bmatrix}_{\lambda=\mu},
\]

which satisfy the condition

\[
\langle \mu | \Lambda | \mu \rangle = 2\psi_1\psi_3 - \langle \psi_2, \psi_2 \rangle_{\lambda=\mu} = 0.
\]

In the same manner, we introduce a column-vector eigenfunction \(|\nu\rangle\) of the spectral problem (2.20) at \(\lambda = \nu\) and its matrix transpose \(\langle \nu |\).

Proposition 3.1. The spectral problem (2.20) is form-invariant under the action of the binary Darboux transformation defined as

\[
\begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{bmatrix} \propto \left\{ I + \left( \frac{\nu - \mu}{\lambda - \nu} \right) \frac{|\mu\rangle \langle \nu | \Lambda | \mu \rangle}{\langle \nu | \Lambda | \nu \rangle} + \left( \frac{\mu - \nu}{\lambda - \mu} \right) \frac{|\nu\rangle \langle \mu | \Lambda | \nu \rangle}{\langle \mu | \Lambda | \mu \rangle} \right\} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}, \quad (3.1)
\]

up to an overall constant, where \(\langle \mu | \Lambda | \mu \rangle = \langle \nu | \Lambda | \nu \rangle = 0\) and the transformed potentials \(\tilde{q}\) and \(\tilde{r}\) are given by

\[
\tilde{q} = q + i (\mu - \nu) \frac{(|\mu\rangle \langle \nu | - |\nu\rangle \langle \mu |)}{\langle \nu | \Lambda | \mu \rangle}, \quad (3.2a)
\]

\[
\tilde{r} = r + i (\nu - \mu) \frac{(|\nu\rangle \langle \mu | - |\mu\rangle \langle \nu |)}{\langle \nu | \Lambda | \mu \rangle}. \quad (3.2b)
\]
Here, the subscripts $12$ and $32$ denote the $(1, 2)$ and $(3, 2)$ sub-matrices (row vectors in this case) in the $3 \times 3$ block matrix.

In (3.1),

$$|\mu\rangle \langle \nu| \Lambda |\mu\rangle, \quad |\nu\rangle \langle \mu| \Lambda |\nu\rangle$$

provide linear eigenfunctions of the transformed spectral problem at $\lambda = \nu$ and $\lambda = \mu$, respectively; for a suitable choice of $|\mu\rangle$ and $|\nu\rangle$, these correspond to bound states generated by the binary Darboux transformation. Note that overall factors of $|\mu\rangle$ and $|\nu\rangle$ play no role in the definition of the binary Darboux transformation.

If $|\mu\rangle$ and $|\nu\rangle$ satisfy not only the spectral problem (2.20) but also the isospectral evolution equation (2.21) at $\lambda = \mu$ and $\lambda = \nu$, respectively, the binary Darboux transformation (3.1) preserves the Lax representation, (2.20) and (2.21), form-invariant with the potentials transformed as $q \rightarrow q$ and $r \rightarrow r$. This is also true for other flows of the integrable hierarchy. Thus, (3.2) can be used to generate a new nontrivial solution of the Kulish–Sklyanin hierarchy from its trivial solution.

Similar results on the Darboux transformations have been obtained by Mikhailov and coworkers (see, in particular, the pioneering paper [18] and the recent papers [49, 50]).

The Darboux matrix defined in (3.1):

$$D_{\mu,\nu} = I + \left( \frac{\nu - \mu}{\lambda - \nu} \right) |\mu\rangle \langle \nu| \Lambda + \left( \frac{\mu - \nu}{\lambda - \mu} \right) |\nu\rangle \langle \mu| \Lambda$$

has the important invariance property:

$$D_{\mu,\nu}^T \Lambda D_{\mu,\nu} = \Lambda,$$

which implies $\det D_{\mu,\nu} = 1$ and

$$D_{\mu,\nu}^{-1} = \Lambda D_{\mu,\nu}^T \Lambda.$$

Thus, the constant quantity $2\psi_1 \psi_3 - \langle \psi_2, \psi_2 \rangle$ for any linear eigenfunction is invariant under the binary Darboux transformation, i.e.

$$2\tilde{\psi}_1 \tilde{\psi}_3 - \langle \tilde{\psi}_2, \tilde{\psi}_2 \rangle = 2\psi_1 \psi_3 - \langle \psi_2, \psi_2 \rangle.$$

In particular, if we start from a linear eigenfunction satisfying the condition $2\psi_1 \psi_3 = \langle \psi_2, \psi_2 \rangle$, any linear eigenfunction generated by iterations of the binary Darboux transformation also satisfies the same condition.
We can consider an arbitrary number of iterations of the binary Darboux transformation (3.1) with different values of \( \mu \) and \( \nu \) in each step. For instance, the twofold binary Darboux transformation can be represented by the Darboux matrix:

\[
\bar{D}_{\mu_2, \nu_2} D_{\mu_1, \nu_1} = \left\{ I + \left( \frac{\nu_2 - \mu_2}{\lambda - \nu_2} \right) \langle \bar{\mu}_2 | \Lambda | \bar{\nu}_2 \rangle + \left( \frac{\mu_2 - \nu_2}{\lambda - \mu_2} \right) \langle \bar{\nu}_2 | \Lambda | \bar{\mu}_2 \rangle \right\} \times \left\{ I + \left( \frac{\nu_1 - \mu_1}{\lambda - \nu_1} \right) \langle \mu_1 | \Lambda | \mu_1 \rangle + \left( \frac{\mu_1 - \nu_1}{\lambda - \mu_1} \right) \langle \nu_1 | \Lambda | \nu_1 \rangle \right\},
\]

where

\[
|\bar{\mu}_2 \rangle = |\mu_2 \rangle + \left( \frac{\nu_1 - \mu_1}{\mu_2 - \nu_1} \right) \langle \nu_1 | \Lambda | \mu_2 \rangle |\mu_1 \rangle + \left( \frac{\mu_1 - \nu_1}{\mu_2 - \mu_1} \right) \langle \mu_1 | \Lambda | \mu_1 \rangle |\nu_1 \rangle,
\]

\[
|\bar{\nu}_2 \rangle = |\nu_2 \rangle + \left( \frac{\nu_1 - \mu_1}{\nu_2 - \nu_1} \right) \langle \nu_2 | \Lambda | \nu_2 \rangle |\mu_1 \rangle + \left( \frac{\mu_1 - \nu_1}{\nu_2 - \nu_1} \right) \langle \mu_2 | \Lambda | \nu_1 \rangle |\nu_1 \rangle,
\]

and \( \langle \mu_j | \Lambda | \mu_j \rangle = \langle \nu_j | \Lambda | \nu_j \rangle = 0 \) (\( j = 1, 2 \)).

Noting that a multifold binary Darboux transformation can be defined as the order-independent composition of binary Darboux transformations, we can assume that the \( N \)-fold binary Darboux transformation takes the following form (cf. [18]):

\[
D_{\lambda_1, \lambda_2, \ldots, \lambda_{2N}} = I + \sum_{k=1}^{2N} \frac{1}{\lambda - \lambda_k} \left( \sum_{j=1}^{2N} g_{jk} |\lambda_j \rangle \right) \langle \lambda_k | \Lambda. \tag{3.3}
\]

Here, \( \{ \lambda_1, \lambda_2, \ldots, \lambda_{2N} \} \) are pairwise distinct constants, \( g_{jk} \) is a scalar function to be determined, \( |\lambda_j \rangle \) is a nonzero column-vector eigenfunction of the spectral problem (2.20) at \( \lambda = \lambda_j \), and \( |\lambda_j \rangle \) and its matrix transpose (i.e., row vector) \( \langle \lambda_j | \) satisfy the condition \( \langle \lambda_j | \Lambda | \lambda_j \rangle = 0 \). Then, substituting (3.3) into the invariance property:

\[
D_{\lambda_1, \lambda_2, \ldots, \lambda_{2N}}^T \Lambda D_{\lambda_1, \lambda_2, \ldots, \lambda_{2N}} = \Lambda,
\]

and noting that this is an identity in \( \lambda \), we obtain the relations:

\[
g_{kk} = 0, \quad k = 1, 2, \ldots, 2N,
\]

\[
(\lambda_k - \lambda_j) g_{jk} + \sum_{i=1}^{2N} \sum_{l=1}^{2N} g_{ik} g_{lj} \langle \lambda_i | \Lambda | \lambda_l \rangle = 0, \quad j, k = 1, 2, \ldots, 2N. \tag{3.4}
\]

Thus, \( g_{jk} + g_{kj} = 0 \) and (3.4) can be written as a \( 2N \times 2N \) matrix equation:

\[
GA - AG - GLG = O,
\]

13
which is equivalent to
\[ AG^{-1} - G^{-1}A = L. \tag{3.5} \]

Here, \( A \) is a diagonal matrix, \( G \) is a skew-symmetric matrix and \( L \) is a symmetric matrix, defined as
\[
A := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2N}), \quad G := (g_{jk})_{j,k=1,2,\ldots,2N}, \quad L := (\langle \lambda_i | A | \lambda_i \rangle)_{i=1,2,\ldots,2N}.
\]

By solving the linear equation (3.5) for \( G^{-1} \), we find that off-diagonal entries of the skew-symmetric matrix \( G^{-1} \) are given by
\[ (G^{-1})_{jk} = \frac{\langle \lambda_j | A | \lambda_k \rangle}{\lambda_j - \lambda_k}, \quad j \neq k. \tag{3.6} \]

In view of (2.20) at \( \lambda = \lambda_k \) and (2.22) at \( \lambda = \lambda_j \), we can compute the \( x \)-derivative of \( G^{-1} \) as
\[
\partial_x (G^{-1})_{jk} = -(G^{-1}G_xG^{-1})_{jk} = \langle \lambda_j | A \text{diag}(1,0,\ldots,0,-i) | \lambda_k \rangle. \tag{3.7}
\]

With the aid of (3.4) and (3.7), we can prove a multifold generalization of Proposition 3.1 by a direct calculation.

**Proposition 3.2.** The spectral problem (2.20) is form-invariant under the action of the \( N \)-fold binary Darboux transformation defined as
\[
\begin{bmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 \\
\tilde{\psi}_3
\end{bmatrix}
\propto \begin{bmatrix}
I + \sum_{k=1}^{2N} \frac{1}{\lambda - \lambda_k} \left( \sum_{j=1}^{2N} g_{jk} \langle \lambda_j | \Lambda | \lambda_j \rangle \right) \langle \lambda_k | \Lambda \rangle
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{bmatrix},
\]
up to an overall constant, where \( |\lambda_j\rangle \) is a linear eigenfunction of the original spectral problem (2.20) at \( \lambda = \lambda_j \) satisfying \( \langle \lambda_j | A | \lambda_j \rangle = 0 \) and \( g_{jk} \) is the \((j,k)\) element of the inverse of the skew-symmetric matrix \( G^{-1} \) determined by (3.6).

The transformed potentials \( \tilde{q} \) and \( \tilde{r} \) are given by
\[
\tilde{q} = q - i \sum_{1 \leq j < k \leq 2N} g_{jk} (|\lambda_j\rangle \langle \lambda_k| - |\lambda_k\rangle \langle \lambda_j|)_{12},
\]
\[
\tilde{r} = r + i \sum_{1 \leq j < k \leq 2N} g_{jk} (|\lambda_j\rangle \langle \lambda_k| - |\lambda_k\rangle \langle \lambda_j|)_{32},
\]
where the subscripts \((1,2)\) and \((3,2)\) sub-matrices (row vectors in this case) in the \(3 \times 3\) block matrix.

Note that overall factors of \( |\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_{2N}\rangle \) are irrelevant to the definition of the \( N \)-fold binary Darboux transformation.
3.2 Multisoliton solutions

We first notice that the spectral problem (2.20) under the complex conjugation reduction \( r = -q^* \) has the following symmetry property (a kind of involution): if

\[
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{bmatrix}
\]

is a linear eigenfunction at \( \lambda = \mu \), then

\[
\begin{bmatrix}
\psi_3^* \\
-\psi_2^* \\
\psi_1^*
\end{bmatrix}
= \Lambda
\begin{bmatrix}
\psi_1^* \\
\psi_2^* \\
\psi_3^*
\end{bmatrix}
\]

is a linear eigenfunction at \( \lambda = \mu^* \). By applying Proposition 3.1 using these two linear eigenfunctions as \( |\mu\rangle \) and \( |\nu\rangle \), we obtain new potentials \( \tilde{q} \) and \( \tilde{r} \), which also satisfy the same relation \( \tilde{r} = -\tilde{q}^* \).

To obtain the bright \( N \)-soliton solution of the Kulish–Sklyanin model ((2.19) under the reduction \( r = -q^* \)):

\[
iq_t + q_{xx} + 2\langle q, q^* \rangle q - \langle q, q \rangle q^* = 0,
\]

we start with the trivial zero solution \( q = r = 0 \) and apply Proposition 3.2. In view of the above symmetry property, we consider a set of \( 2N \) eigenvalues \( \{\lambda_1, \lambda_2, \ldots, \lambda_{2N}\} \) that consist of \( N \) complex conjugate pairs. The ordering of the \( 2N \) eigenvalues is irrelevant to the definition of the \( N \)-fold binary Darboux transformation, so it can be altered depending on one’s preference; in this paper, we number the \( 2N \) eigenvalues as

\[
\lambda_{N+j} = \lambda_j^*, \quad j = 1, 2, \ldots, N,
\]

and choose a column-vector eigenfunction \( |\lambda_j\rangle \) of the linear problem (2.20) and (2.21) at \( \lambda = \lambda_j \) as

\[
|\lambda_j\rangle = \begin{bmatrix} e^{-i\lambda_j x - i\lambda_j^2 t} \\ \frac{1}{2} \langle c_j, c_j^\dagger \rangle e^{i\lambda_j x + i\lambda_j^2 t} \end{bmatrix} \propto \begin{bmatrix} 1 \\ \frac{1}{2} \langle c_j, c_j^\dagger \rangle e^{i\lambda_j x + i\lambda_j^2 t} \end{bmatrix}, \quad j = 1, 2, \ldots, N,
\]

and

\[
|\lambda_{N+j}\rangle = \begin{bmatrix} \frac{1}{2} \langle c_j^*, c_j^* \rangle e^{-i\lambda_j^* x - i\lambda_j^* t} \\ e^{i\lambda_j^* x + i\lambda_j^* t} \end{bmatrix} \propto \begin{bmatrix} \frac{1}{2} \langle c_j^*, c_j^* \rangle e^{-2i\lambda_j^* x - 2i\lambda_j^* t} \\ -c_j^\dagger e^{-i\lambda_j^* x - i\lambda_j^* t} \end{bmatrix}, \quad j = 1, 2, \ldots, N,
\]
where \( c_j \) is a constant row vector. Note that these linear eigenfunctions indeed satisfy the condition \( \langle \lambda_j \mid A \mid \lambda_j \rangle = 0, \ j = 1, 2, \ldots, 2N \).

Recalling that overall factors of \( |\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_{2N}\rangle \) are irrelevant in the \( N \)-fold binary Darboux transformation, we can rescale these eigenfunctions as in (3.9) and translate the skew-symmetric matrix \( G^{-1} \) determined by (3.6) into a slightly simpler skew-symmetric matrix:

\[
G^{-1} \rightarrow \begin{bmatrix} U & V \\ -V^T & W \end{bmatrix}, \quad U^T = -U, \quad W^T = -W,
\]

where the entries of the \( N \times N \) matrices \( U := (u_{jk})_{j,k=1,2,\ldots,N}; \ V := (v_{jk})_{j,k=1,2,\ldots,N} \) and \( W := (w_{jk})_{j,k=1,2,\ldots,N} \) are defined as

\[
u_{jk} := \frac{\frac{1}{2} \langle c_j, c_k \rangle e^{2i\lambda_j x + 2i\lambda_k^2 t} + \frac{1}{2} \langle c_k, c_j \rangle e^{2i\lambda_k x + 2i\lambda_j^2 t} - \langle c_j, c_k \rangle e^{i(\lambda_j + \lambda_k)x + i(\lambda_j^2 + \lambda_k^2)t}}{\lambda_j - \lambda_k}, \quad j < k, \tag{3.10a}
\]

\[
v_{jk} := \frac{1 + \langle c_j, c_j^* \rangle e^{i(\lambda_j - \lambda_k^*)x + i(\lambda_j^2 - \lambda_k^2)t} + \frac{1}{4} \langle c_j, c_j \rangle \langle c_k, c_k^* \rangle e^{2i(\lambda_j - \lambda_k^*)x + 2i(\lambda_j^2 - \lambda_k^2)t}}{\lambda_j - \lambda_k^*}, \tag{3.10b}
\]

\[
w_{jk} := \frac{\frac{1}{2} \langle c_j^*, c_k \rangle e^{-2i\lambda_j^* x - 2i\lambda_j^2 t} + \frac{1}{2} \langle c_k^*, c_k \rangle e^{-2i\lambda_k^* x - 2i\lambda_k^2 t} - \langle c_j^*, c_k \rangle e^{-i(\lambda_j^* + \lambda_k^*)x - i(\lambda_j^2 + \lambda_k^2)t}}{\lambda_j^* - \lambda_k^*}, \quad j < k.
\]

Note that \( u_{jj} = w_{jj} = 0 \) and \( u_{jk} \) and \( w_{jk} \) for \( j > k \) are given by \(-u_{kj}\) and \(-w_{kj}\), respectively. Moreover, we have \( w_{jk} = u_{jk}^* \) and \( v_{jk} = -v_{kj} \), so \( W = U^* \) and \( V^T = -V \).

Now, by applying Proposition 3.2, we obtain

\[
\tilde{q} = -i \sum_{1 \leq j < k \leq N} \begin{bmatrix} U & V \\ V^* & U^* \end{bmatrix}_{j,k}^{-1} \left( c_k e^{i\lambda_k x + i\lambda_k^2 t} - c_j e^{i\lambda_j x + i\lambda_j^2 t} \right)
\]

\[
- i \sum_{1 \leq j, k \leq N} \begin{bmatrix} U & V \\ V^* & U^* \end{bmatrix}_{j,N+k}^{-1} \left( -c_k^* e^{-i\lambda_k^* x - i\lambda_k^2 t} - \frac{1}{2} \langle c_k^*, c_j^* \rangle c_j e^{i(\lambda_j^* - 2\lambda_k^*)x + i(\lambda_j^2 - 2\lambda_k^2)t} \right)
\]

\[
- i \sum_{1 \leq j, k \leq N} \begin{bmatrix} U & V \\ V^* & U^* \end{bmatrix}_{N+j,N+k}^{-1} \left( -\frac{1}{2} \langle c_j^*, c_j \rangle c_k e^{-i(2\lambda_j^* + \lambda_k^*)x - i(2\lambda_j^2 + \lambda_k^2)t} + \frac{1}{2} \langle c_j^*, c_k \rangle c_j^* e^{-i(\lambda_j^* + 2\lambda_k^*)x - i(\lambda_j^2 + 2\lambda_k^2)t} \right), \tag{3.11}
\]

16
and \( \tilde{r} = -\tilde{q}^* \), so the complex conjugation reduction is realized. It only remains to compute the entries of the inverse matrix in (3.11). While the inverse of a general square matrix is given in terms of the determinant and cofactors [51], the inverse of a skew-symmetric matrix can be expressed more simply in terms of the Pfaffian and cofactors [25]. The Pfaffian [25, 51] is a square root of the determinant of a skew-symmetric matrix of even dimension (see https://en.wikipedia.org/wiki/Pfaffian). For example, the inverse of a \( 4 \times 4 \) skew-symmetric matrix is given as

\[
\begin{bmatrix}
0 & d_{12} & d_{13} & d_{14} \\
-d_{12} & 0 & d_{23} & d_{24} \\
-d_{13} & -d_{23} & 0 & d_{34} \\
-d_{14} & -d_{24} & -d_{34} & 0 \\
\end{bmatrix}^{-1} = \frac{1}{d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23}} \begin{bmatrix}
0 & -d_{34} & d_{24} & -d_{23} \\
d_{34} & 0 & -d_{14} & d_{13} \\
d_{24} & d_{14} & 0 & -d_{12} \\
d_{23} & -d_{13} & d_{12} & 0 \\
\end{bmatrix}.
\]

Following the notation and definition in Hirota’s book [25], we write the Pfaffian of the \( 2N \times 2N \) skew-symmetric matrix

\[
\begin{bmatrix}
U & V \\
V^* & U^* \\
\end{bmatrix}, \quad U^T = -U, \quad V^\dagger = -V
\]

as

\[
(1, 2, \ldots, 2N)
\]

and denote cofactors as

\[
\Gamma(j, k) = (-1)^{j+k-1} (1, 2, \ldots, j-1, j+1, \ldots, \tilde{k}, k+1, \ldots, 2N), \quad 1 \leq j < k \leq 2N.
\]

Then, we have

\[
\begin{bmatrix}
U & V \\
V^* & U^* \\
\end{bmatrix}^{-1} = \frac{1}{(1, 2, \ldots, 2N)} \begin{bmatrix}
0 & -\Gamma(1, 2) & -\Gamma(1, 3) & \cdots & -\Gamma(1, 2N) \\
\Gamma(1, 2) & 0 & -\Gamma(2, 3) & \cdots & -\Gamma(2, 2N) \\
\Gamma(1, 3) & \Gamma(2, 3) & 0 & \cdots & -\Gamma(3, 2N) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Gamma(1, 2N) & \Gamma(2, 2N) & \Gamma(3, 2N) & \cdots & 0
\end{bmatrix}.
\]

Using this formula in (3.11) and omitting the tilde of \( \tilde{q} \), we arrive at the main result of this paper.

**Proposition 3.3.** The bright \( N \)-soliton solution of the self-focusing Kulish–
Sklyanin model (3.8) is given by

\[
q = \frac{i}{(1, 2, \ldots, 2N)} \left\{ \sum_{1 \leq j < k \leq N} \Gamma(j, k) \left( c_k e^{i\lambda_k x + i\lambda_k^2 t} - c_j e^{i\lambda_j x + i\lambda_j^2 t} \right) + \sum_{1 \leq j, k \leq N} \Gamma(j, N + k) \left( -c_k^* e^{-i\lambda_k x - i\lambda_k^2 t} - \frac{1}{2} \langle c_k^*, c_k^* \rangle c_j e^{i(\lambda_j - 2\lambda_k^2)x + i(\lambda_j^2 - 2\lambda_k^2)t} \right) + \sum_{1 \leq j < k \leq N} \Gamma(N + j, N + k) \left( -\frac{1}{2} \langle c_j^*, c_j^* \rangle c_k^* e^{-i(2\lambda_j^2 + \lambda_k^2)x - i(2\lambda_j^2 + \lambda_k^2)t} \right) + \frac{1}{2} \langle c_k^*, c_k^* \rangle c_j^* e^{-i(\lambda_j + 2\lambda_k^2)x - i(\lambda_j^2 + 2\lambda_k^2)t} \right) \right\},
\]

where the Pfaffian (1, 2, \ldots, 2N) and the cofactors \( \Gamma(j, k) \) are defined from the skew-symmetric matrix in (3.12) with the entries of \( U \) and \( V \) given by (3.10).

By extending the definition of the cofactors in (3.13) as \( \Gamma(k, j) = -\Gamma(j, k) \) and \( \Gamma(j, j) = 0 \) [25], we can rewrite (3.14) more concisely as

\[
q = -\frac{i}{(1, 2, \ldots, 2N)} \left\{ \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \Gamma(j, k) + \Gamma(j, N + k) \frac{1}{2} \langle c_k^*, c_k^* \rangle e^{-2i\lambda_k^2 x - 2i\lambda_k^2 t} \right) c_j e^{i\lambda_j x + i\lambda_j^2 t} + \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \Gamma(j, N + k) + \Gamma(N + j, N + k) \frac{1}{2} \langle c_j^*, c_j^* \rangle e^{-2i\lambda_j^2 x - 2i\lambda_j^2 t} \right) c_k^* e^{-i\lambda_k x - i\lambda_k^2 t} \right\}
\]

\[
= -\frac{i}{(1, 2, \ldots, 2N)} \left\{ \sum_{j=1}^{N} (1, 2, \ldots, j - 1, \beta, j + 1, \ldots, 2N) c_j e^{i\lambda_j x + i\lambda_j^2 t} - \sum_{k=1}^{N} (1, 2, \ldots, N + k - 1, \beta, N + k + 1, \ldots, 2N) c_k^* e^{-i\lambda_k x - i\lambda_k^2 t} \right\},
\]

with \((\beta, k) = 1, (\beta, N + k) = \frac{1}{2} \langle c_k^*, c_k^* \rangle e^{-2i\lambda_k^2 x - 2i\lambda_k^2 t} \) for \( k = 1, 2, \ldots, N \).

By setting \( N = 1 \), we obtain the one-soliton solution of the Kulish–Sklyanin model (3.8) as

\[
q = \frac{-i(\lambda_1 - \lambda_1^*) \left( c_1 e^{-i\lambda_1 x - i\lambda_1^2 t} + \frac{1}{2} \langle c_1^*, c_1^* \rangle c_1 e^{i(\lambda_1 - 2\lambda_1^2)x + i(\lambda_1^2 - 2\lambda_1^2)t} \right) + 1 + \langle c_1, c_1 \rangle e^{i(\lambda_1 - \lambda_1^*)x + i(\lambda_1^2 - \lambda_1^2)t} + \frac{1}{2} \langle c_1, c_1 \rangle (c_1^*, c_1^*) e^{2i(\lambda_1 - \lambda_1^*)x + 2i(\lambda_1^2 - \lambda_1^2)t} \right)}{1 + \langle c_1, c_1 \rangle e^{i(\lambda_1 - \lambda_1^*)x + i(\lambda_1^2 - \lambda_1^2)t} + \frac{1}{2} \langle c_1, c_1 \rangle (c_1^*, c_1^*) e^{2i(\lambda_1 - \lambda_1^*)x + 2i(\lambda_1^2 - \lambda_1^2)t}}.
\]

The case \( \langle c_1, c_1 \rangle = 0 \) and the case \( \langle c_1, c_1 \rangle \neq 0 \) correspond to the rank-1 one-soliton solution (2.13) and the rank-2 one-soliton solution (2.14) respectively, up to a rescaling of \( q \). The one- and two-soliton solutions of the
Kulish–Sklyanin model (3.8) (up to a linear transformation mixing the components) have been studied in detail in [28, 36, 37, 52]. Note that the $N$-soliton solution (3.14) is a linear combination of the $2N$ constant vectors $c_1, \ldots, c_N, c_1^*, \ldots, c_N^*$, so it is mathematically redundant to consider the case where the number of the components of $\mathbf{q}$ is more than $2N$.

Let us move on to the solutions of the vector mKdV equation (2.24). We first notice that the spectral problem (2.25), i.e., (2.20) under the reduction $\mathbf{r} = -\mathbf{q}$, has the following symmetry property (a kind of involution): if

$$
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{bmatrix}
$$

is a linear eigenfunction at $\lambda = \mu$, then

$$
\begin{bmatrix}
\psi_3 \\
-\psi_2 \\
\psi_1
\end{bmatrix} = \Lambda \begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{bmatrix}
$$

is a linear eigenfunction at $\lambda = -\mu$. By applying Proposition 3.1 using these two linear eigenfunctions as $|\mu\rangle$ and $|\nu\rangle$, we obtain new potentials $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$, which also satisfy the same relation $\tilde{\mathbf{r}} = -\tilde{\mathbf{q}}$.

To obtain the multisoliton (or multi-breather) solutions of the vector mKdV equation (2.24), we start with the trivial zero solution $\mathbf{q} = \mathbf{r} = 0$ in the spectral problem (2.20) and apply Proposition 3.2. In view of the above symmetry property, we consider the case where the $2N$ eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2N}\}$ occur in plus-minus pairs. The ordering of the $2N$ eigenvalues is irrelevant to the definition of the $N$-fold binary Darboux transformation and can be altered; in this paper, we number the $2N$ eigenvalues as $\lambda_{N+j} = -\lambda_j$, $j = 1, 2, \ldots, N$, and choose a column-vector eigenfunction $|\lambda_j\rangle$ of the linear problem (2.25) and (2.26) at $\lambda = \lambda_j$ as

$$
|\lambda_j\rangle = \begin{bmatrix}
e^{-i\lambda_j x - i\lambda_j^3 y} \\
\frac{1}{2} \langle \mathbf{c}_j, \mathbf{c}_j \rangle e^{i\lambda_j x + i\lambda_j^3 y}
\end{bmatrix} \propto \begin{bmatrix}1 \\
\frac{1}{2} \langle \mathbf{c}_j, \mathbf{c}_j \rangle \end{bmatrix} e^{i\lambda_j x + i\lambda_j^3 y} e^{2i\lambda_j x + 2i\lambda_j^3 y}, \quad j = 1, 2, \ldots, N,
$$

and

$$
|\lambda_{N+j}\rangle = \Lambda |\lambda_j\rangle \propto \begin{bmatrix}
\frac{1}{2} \langle \mathbf{c}_j, \mathbf{c}_j \rangle e^{2i\lambda_j x + 2i\lambda_j^3 y} \\
-\mathbf{c}_j^T e^{i\lambda_j x + i\lambda_j^3 y}
\end{bmatrix}, \quad j = 1, 2, \ldots, N,
$$

(3.15a)

and

(3.15b)
where \(c_j\) is a constant row vector. Note that these linear eigenfunctions indeed satisfy the condition \(\langle \lambda_j \mid A \mid \lambda_j \rangle = 0, \quad j = 1, 2, \ldots, 2N\).

Recalling that overall factors of \(\mid \lambda_1 \rangle, \mid \lambda_2 \rangle, \ldots, \mid \lambda_{2N} \rangle\) are irrelevant in the \(N\)-fold binary Darboux transformation, we can rescale these eigenfunctions as in (3.15) and translate the skew-symmetric matrix \(G^{-1}\) determined by (3.6) into a slightly simpler skew-symmetric matrix:

\[
G^{-1} \rightarrow \begin{bmatrix} U & V \\ -V^T & -U \end{bmatrix}, \quad U^T = -U,
\]

where the entries of the \(N \times N\) matrices \(U := (u_{jk})_{j,k = 1, 2, \ldots, N}\) and \(V := (v_{jk})_{j,k = 1, 2, \ldots, N}\) are defined as

\[
u_{jk} := \frac{1}{2} \left( c_j c_j^* e^{i \lambda_j x + 2i \lambda_j^3 y} + \frac{1}{2} \left( c_k c_k^* e^{i \lambda_k x + 2i \lambda_k^3 y} - \langle c_j, c_k \rangle e^{i (\lambda_j + \lambda_k) x + i (\lambda_j^3 + \lambda_k^3)} \right) \right), \quad j < k, \tag{3.16a}
\]

\[
u_{jk} := \frac{1}{2} \langle c_j, c_k \rangle e^{i (\lambda_j + \lambda_k) x + i (\lambda_j^3 + \lambda_k^3)} + \frac{1}{4} \langle c_j, c_j \rangle \langle c_k, c_k \rangle e^{2i (\lambda_j + \lambda_k) x + 2i (\lambda_j^3 + \lambda_k^3)} \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k}, \tag{3.16b}
\]

Note that \(v_{jk} = v_{kj}\), so \(V^T = V\).

Now, by applying Proposition 3.2, we obtain

\[
\bar{q} = -i \sum_{1 \leq j < k \leq N} \begin{bmatrix} U & V \\ -V & -U \end{bmatrix}_{jk}^{-1} \left( c_k e^{i \lambda_k x + i \lambda_k^3 y} - c_j e^{i \lambda_j x + i \lambda_j^3 y} \right) - i \sum_{1 \leq j, k \leq N} \begin{bmatrix} U & V \\ -V & -U \end{bmatrix}_{j,N+k}^{-1} \left( -c_k e^{i \lambda_k x + i \lambda_k^3 y} - \frac{1}{2} \langle c_k, c_k \rangle c_j e^{i (\lambda_j + 2\lambda_k) x + i (\lambda_j^3 + 2\lambda_k^3)} \right) \tag{3.17}
\]

and \(\bar{r} = -\bar{q}\), so the required reduction is indeed realized. Because

\[
\begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{bmatrix} U & V \\ -V & -U \end{bmatrix}^{-1} + \begin{bmatrix} U & V \\ -V & -U \end{bmatrix}^{-1} \begin{bmatrix} O & I \\ I & O \end{bmatrix} = O,
\]

20
the inverse matrix should take the form:

\[
\begin{bmatrix}
U & V \\
-V & -U
\end{bmatrix}^{-1} = \begin{bmatrix}
X & Y \\
-Y & -X
\end{bmatrix}, \quad X^T = -X, \quad Y^T = Y,
\]

which is skew-symmetric. Thus, we can rewrite (3.17) as

\[
q = -i \sum_{1 \leq j < k \leq N} \left( \begin{bmatrix} U & V \\ -V & -U \end{bmatrix} \right)^{-1}_{jk} \left( -c_j e^{i \lambda_j x + i \lambda_k^3 y} + \frac{1}{2} \langle c_j, c_k \rangle c_k e^{i (2 \lambda_j + \lambda_k) x + i (2 \lambda_j^3 + \lambda_k^3) y} \\
c_k e^{i \lambda_k x + i \lambda_k^3 y} - \frac{1}{2} \langle c_k, c_j \rangle c_j e^{i (2 \lambda_j + \lambda_k) x + i (2 \lambda_j^3 + \lambda_k^3) y} \right)
\]

\[
- i \sum_{1 \leq j < k \leq N} \left( \begin{bmatrix} U & V \\ -V & -U \end{bmatrix} \right)^{-1}_{j,N+k} \left( -c_j e^{i \lambda_j x + i \lambda_j^3 y} - \frac{1}{2} \langle c_j, c_j \rangle c_k e^{i (2 \lambda_j + \lambda_k) x + i (2 \lambda_j^3 + \lambda_k^3) y} \\
c_k e^{i \lambda_k x + i \lambda_k^3 y} - \frac{1}{2} \langle c_k, c_k \rangle c_j e^{i (2 \lambda_j + \lambda_k) x + i (2 \lambda_j^3 + \lambda_k^3) y} \right)
\]

\[
- i \sum_{j=1}^{N} \left( \begin{bmatrix} U & V \\ -V & -U \end{bmatrix} \right)^{-1}_{j,N+j} \left( -c_j e^{i \lambda_j x + i \lambda_j^3 y} - \frac{1}{2} \langle c_j, c_j \rangle c_j e^{3i \lambda_j x + 3i \lambda_j^3 y} \right),
\]

(3.18)

where the tilde of \( \tilde{q} \) is omitted. This is a fairly general complex-valued solution of the vector mKdV equation (2.24). To turn it into real-valued solutions, we first set

\[
\lambda_j = i \eta_j, \quad j = 1, 2, \ldots, N,
\]

and rewrite the solution (3.18) as follows.

**Proposition 3.4.** An \( N \)-soliton solution of the vector mKdV equation (2.24) is given by

\[
q = \frac{1}{(1, 2, \ldots, 2N)} \left\{ \sum_{1 \leq j < k \leq N} \Gamma(j, k) \left( c_j e^{-\eta_j x + \eta_j^3 y} - \frac{1}{2} \langle c_j, c_j \rangle c_k e^{-(2 \eta_j + \eta_k) x + (2 \eta_j^3 + \eta_k^3) y} \right) \\
- c_k e^{-\eta_k x + \eta_k^3 y} + \frac{1}{2} \langle c_k, c_k \rangle c_j e^{-(2 \eta_j + \eta_k) x + (2 \eta_j^3 + \eta_k^3) y} \right) \\
+ \sum_{1 \leq j < k \leq N} \Gamma(j, N+k) \left( c_j e^{-\eta_j x + \eta_j^3 y} + \frac{1}{2} \langle c_j, c_j \rangle c_k e^{-(2 \eta_j + \eta_k) x + (2 \eta_j^3 + \eta_k^3) y} \right) \\
+ c_k e^{-\eta_k x + \eta_k^3 y} + \frac{1}{2} \langle c_k, c_k \rangle c_j e^{-(2 \eta_j + \eta_k) x + (2 \eta_j^3 + \eta_k^3) y} \right) \\
+ \sum_{j=1}^{N} \Gamma(j, N+j) \left( c_j e^{-\eta_j x + \eta_j^3 y} + \frac{1}{2} \langle c_j, c_j \rangle c_j e^{-3 \eta_j x + 3 \eta_j^3 y} \right) \right\},
\]

(3.20)
where \((1, 2, \ldots, 2N)\) is the Pfaffian (a square root of the determinant) of the skew-symmetric matrix with the entries

\[
(j, k) = -(N+j, N+k) = \frac{(c_j e^{-\eta_j x + \eta_j^3 y} - c_k e^{-\eta_k x + \eta_k^3 y})}{2(\eta_j - \eta_k)}, \quad 1 \leq j < k \leq N,
\]

\[
(j, N+k) = 1 + \frac{1}{2}(c_j, c_k) e^{-2(\eta_j + \eta_k)x + 2(\eta_j^3 + \eta_k^3)y}, \quad 1 \leq j, k \leq N,
\]

and the cofactors \(\Gamma(j, k)\) for \(1 \leq j < k \leq 2N\) are defined as in (3.13).

Note that (3.20) is of the form:

\[
q = \frac{\sum_{i=1}^{N} G_i c_i e^{-\eta_i x + \eta_i^3 y}}{F},
\]

where \(F\) and \(G_1, \ldots, G_N\) are polynomials in \(\langle c_j e^{-\eta_j x + \eta_j^3 y}, c_k e^{-\eta_k x + \eta_k^3 y} \rangle\) for \(1 \leq j \leq k \leq N\); this provides a real-valued \(N\)-soliton solution if \(\eta_1, \ldots, \eta_N\) are positive and \(c_1, \ldots, c_N\) are real. By setting \(N = 1\), we obtain

\[
q = 2\eta_1 \frac{c_1 e^{-\eta_1 x + \eta_1^3 y}}{1 + \frac{1}{2}(c_1, c_1) e^{-2\eta_1 x + 2\eta_1^3 y}},
\]

which is the straightforward vector analog of the one-soliton solution of the scalar \(mKdV\) equation, i.e., the scalar \(mKdV\) soliton with a coefficient unit vector \(c_1 / \sqrt{\langle c_1, c_1 \rangle}\). Incidentally, the scalar \(mKdV\) equation was first solved by R. Hirota (J. Phys. Soc. Jpn.), M. Wadati (J. Phys. Soc. Jpn.) and S. Tanaka (Publ. RIMS & Proc. Japan Acad.) almost independently in 1972.

The solution (3.20) in the case of real soliton parameters provides a nontrivial vector generalization of the \(N\)-soliton solution of the scalar \(mKdV\) equation involving \(N\) polarization vectors:

\[
\frac{c_1}{\sqrt{\langle c_1, c_1 \rangle}}, \quad \frac{c_2}{\sqrt{\langle c_2, c_2 \rangle}}, \quad \cdots, \quad \frac{c_N}{\sqrt{\langle c_N, c_N \rangle}}.
\]

We can consider a generalization of the vector \(mKdV\) equation (2.24) as considered by Iwao and Hirota [24]:

\[
q_y + q_{xxx} + 3 \langle q B, q \rangle q_x = 0.
\]

(3.22)
Here, $B = (b_{jk})$ is a constant square matrix, which can be assumed to be symmetric ($b_{jk} = b_{kj}$) without loss of generality. Then, an $N$-soliton solution of this generalized vector mKdV equation (3.22) is given by (3.20) with the involved scalar products generalized as $\langle c_j, c_k \rangle \rightarrow \langle c_j B, c_k \rangle$, $1 \leq j, k \leq N$.

This formula generalizes the multisoliton formula proposed by Iwao and Hirota [24] using the Hirota bilinear method [25] (also see the relevant results in [53–55]) and appears to be more efficient.

In fact, (3.20) with positive $c_1, \ldots, c_N$ and real $c_1, \ldots, c_N$ is only a special $N$-soliton solution of the vector mKdV equation (2.24), which does not exhibit any oscillating behavior in each component of the vector variable $q$. In particular, it cannot reproduce the one-soliton solution of the complex mKdV equation [7, 27, 32], involving a complex carrier wave, which is equivalent to the two-component vector mKdV equation ((2.24) with a real two-component vector $q$), up to a linear transformation.

To obtain the general real-valued multisoliton solutions of the vector mKdV equation (2.24), we require that \{1; 2; \ldots; M\} in (3.19) and (3.20), as well as the corresponding linear eigenfunctions in (3.15a), are either real or occur in complex conjugate pairs [49, 56]. That is, up to a re-ordering, we assume

(a) $\eta_{M+j} = \eta_j^*$(Re$\eta_j > 0$), $c_{M+j} = c_j^*$, $j = 1, 2, \ldots, M$,

(b) $\eta_j > 0$, $c_j^* = c_j$, $j = 2M + 1, \ldots, 2M + L (= N)$.

Thus, the original $2N$ eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2N}\}$ occur in (a) plus-minus and complex-conjugate quartets or (b) plus-minus pairs. Under these conditions, we can easily show from Proposition 3.2 that the new solution generated by the $N$-fold binary Darboux transformation is indeed real-valued.

Proposition 3.4 with the soliton parameters satisfying (a) and (b) generates a mixture of multisoliton and multi-breather solutions. To exclude breather solutions, we need only impose the following additional conditions for (a) [49, 56]:

(a’) $\langle c_j, c_j \rangle = 0$, $j = 1, 2, \ldots, M$.

Then, in the simplest nontrivial case of $M = 1$, $L = 0$ and $N = 2$, (3.20) gives the general one-soliton solution of the vector mKdV equation (2.24) [57]:

$$q = 2(\eta_1 + \eta_2) \frac{c_1 \eta_1 e^{-\eta_1 x + \eta_1^2 y} - c_2 \eta_2 e^{-\eta_2 x + \eta_2^2 y}}{\eta_1 - \eta_2 - \frac{4(\eta_1 \eta_2)}{\eta_1 - \eta_2} \langle c_1, c_2 \rangle e^{-(\eta_1 + \eta_2)x + (\eta_1^2 + \eta_2^2)y}}.$$  (3.23)

With $\eta_2 = \eta_1^*$ and $c_2 = c_1^*$, this is indeed a real solution and provides the vector analog of the complex mKdV soliton [7, 27, 32]. By redefining the constant vector as $2\eta_1 c_1 =: (\eta_1 - \eta_1^*)d_1$, the one-soliton solution (3.23) can
be rewritten in a more concise form:

\[ q = (\eta_1 + \eta_1^*) \frac{d_1 e^{-\eta_1 x + \eta_1^* y} + d_1^* e^{-\eta_1^* x + \eta_1 y}}{1 + \langle d_1, d_1^* \rangle e^{-(\eta_1 + \eta_1^*) x + (\eta_1^* + \eta_1) y}}, \quad \langle d_1, d_1 \rangle = 0. \]  \hfill (3.24)

Thus, in the limit \( \eta_1 \to \eta_1^* \), i.e., \( \text{Im} \eta_1 \to 0 \), this solution reduces to (3.21).

For general values of \( M \) and \( L \), (3.20) with the above conditions (a), (a’) and (b) provides the \((M + L)\)-soliton solution, a nonlinear superposition of \( M \) vector solitons of the oscillating type (3.24) and \( L \) vector solitons of the nonoscillating type (3.21). Because the nonoscillating-type soliton can be obtained from the oscillating-type soliton through the limiting procedure, we can somewhat loosely consider that the general \( M \)-soliton solution of the vector mKdV equation (2.24) is obtained by setting \( N = 2M \) and \( L = 0 \) and assuming the conditions (a) and (a’). Thus, Proposition 3.4 can be restated as follows.

**Proposition 3.5.** The general real-valued \( M \)-soliton solution of the vector mKdV equation (2.24) is given by

\[ q = \frac{1}{(1, 2, \ldots, 4M)} \left\{ \sum_{1 \leq j < k < 2M} \Gamma(j, k) \left( c_j e^{-\eta_j x + \eta_j^* y} - c_k e^{-\eta_k x + \eta_k^* y} \right) + \sum_{1 \leq j < k \leq 2M} \Gamma(j, 2M + k) \left( c_j e^{-\eta_j x + \eta_j^* y} + c_k e^{-\eta_k x + \eta_k^* y} \right) + \sum_{j=1}^{2M} \Gamma(j, 2M + j) c_j e^{-\eta_j x + \eta_j^* y} \right\}, \]  \hfill (3.25)

where \((1, 2, \ldots, 4M)\) is the Pfaffian (a square root of the determinant) of the skew-symmetric matrix with the entries

\[ (j, k) = -(2M + j, 2M + k) \]

\[ = -\frac{\langle c_j, c_k \rangle e^{-\eta_j x + \eta_j^* y}}{\eta_j - \eta_k}, \quad 1 \leq j < k \leq 2M, \]

\[ (j, 2M + k) = (k, 2M + j) \]

\[ = \frac{1 + \langle c_j, c_k \rangle e^{-(\eta_j + \eta_k) x + (\eta_j + \eta_k^*) y}}{\eta_j + \eta_k}, \quad 1 \leq j < k \leq 2M, \]

\[ (j, 2M + j) = \frac{1}{2\eta_j}, \quad 1 \leq j \leq 2M, \]

and the cofactors \( \Gamma(j, k) \) for \( 1 \leq j < k \leq 4M \) are defined as in (3.13). Here, \( \eta_{M+j} = \eta_j^* (\text{Re} \eta_j > 0) \), \( c_{M+j} = c_j^* \) and \( \langle c_j, c_j \rangle = 0 \) for \( j = 1, 2, \ldots, M \).
By extending the definition of the cofactors in (3.13) as $\Gamma(k, j) = -\Gamma(j, k)$ and $\Gamma(j, j) = 0$ [25] and noting the relation $\Gamma(j, 2M + k) = \Gamma(k, 2M + j)$, we can rewrite (3.25) in a more compact form as

\[
q = \frac{1}{(1, 2, \ldots, 4M)} \sum_{j=1}^{2M} \left( \sum_{k=1}^{4M} \Gamma(j, k) \right) \mathbf{c}_j e^{-\eta_j x + \eta_j^3 y}
\]

\[
= \frac{\sum_{j=1}^{2M} (1, 2, \ldots, j - 1, \beta, j + 1, \ldots, 4M) \mathbf{c}_j e^{-\eta_j x + \eta_j^3 y}}{(1, 2, \ldots, 4M)},
\]

with $(\beta, k) = 1$, $k = 1, 2, \ldots, 4M$. For the generalized vector mKdV equation (3.22) with a real symmetric and positive definite matrix $B$, the above formula with $\langle \mathbf{c}_j, \mathbf{c}_k \rangle \rightarrow \langle \mathbf{c}_j B, \mathbf{c}_k \rangle$ and $\langle \mathbf{c}_j, \mathbf{c}_j \rangle = 0 \rightarrow \langle \mathbf{c}_j B, \mathbf{c}_j \rangle = 0$ provides the general real-valued $M$-soliton solution.

If we generalize the time dependence as $\mathbf{c}_j e^{-\eta_j x + \eta_j^3 y} \rightarrow \mathbf{c}_j e^{-\eta_j x + \eta_j^3 y + \eta_j^{-1} z}$, Propositions 3.5 provides the general real-valued $M$-soliton solution of the vector sine-Gordon equation [33, 34] (up to a sign ambiguity of the square root) with the independent variables $x$ and $z$. Then, in the special case $\mathbf{c}_j = (\mathbf{a}_j, i\mathbf{a}_j)$ or

\[
\mathbf{c}_j = \left( c_j^{(1)}, ic_j^{(1)}, c_j^{(2)}, ic_j^{(2)}, \ldots \right),
\]

the condition $\langle \mathbf{c}_j, \mathbf{c}_j \rangle = 0$ is automatically satisfied and our $M$-soliton formula apparently reduces to the formula proposed by Feng [54]. He investigated the asymptotic behavior of the two-soliton solution in this special case and showed that the two-soliton collision in the vector sine-Gordon equation is highly nontrivial, reflecting the internal degrees of freedom of the solitons. This apparently disagrees with the conclusion of [49] that the soliton interactions in the vector sine-Gordon equation are exactly the same as in the scalar sine-Gordon equation. Propositions 3.5 could be used to resolve the discrepancy.

4 Concluding remarks

The vector NLS equation known as the Kulish–Sklyanin model [5] admits two different Lax representations; using the standard Lax representation based on the generators of the Clifford algebra, one can easily solve the Kulish–Sklyanin model by applying the inverse scattering method or the Darboux transformations. However, the obtained exact solutions such as the $N$-soliton solution naturally involve the generators of the Clifford algebra satisfying the anticommutation relations and thus are not so useful for further analysis.
In this paper, we translated the standard Lax representation for the Kulish–Sklyanin model into the nonstandard one, not involving the generators of the Clifford algebra, and then applied the binary Darboux transformation. The $N$-fold binary Darboux transformation can also be formulated in simple explicit form, so we could obtain a classical expression for the general $N$-soliton solution of the Kulish–Sklyanin model (3.8), which is more useful for further investigation. By changing the time dependence of the linear eigenfunctions and considering a natural reduction, we could also obtain a general formula for the multisoliton (or multi-breather) solutions of the vector mKdV equation (2.24); by imposing some additional conditions, we obtained the real $N$-soliton solution of the vector mKdV equation.

References


