Exact solutions of multicomponent nonlinear Schrödinger equations under general plane-wave boundary conditions

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Abstract

We construct exact soliton solutions of integrable multicomponent nonlinear Schrödinger (NLS) equations under general nonvanishing boundary conditions. Different components of the vector (or matrix) dependent variable can approach plane waves with different wavenumbers and frequencies at spatial infinity. We apply Bäcklund–Darboux transformations to the cubic NLS equations with a self-focusing nonlinearity, a self-defocusing nonlinearity or a mixed focusing-defocusing nonlinearity. Both bright-soliton solutions and dark-soliton solutions are obtained, depending on the signs of the nonlinear terms and the type of Bäcklund–Darboux transformation. The multicomponent solitons generally possess internal degrees of freedom and provide highly nontrivial generalizations of the scalar NLS solitons. The main step in the construction of the multicomponent solitons is to compute the matrix exponential of a constant non-diagonal matrix arising from the Lax pair. With a suitable re-parametrization of the non-diagonal matrix, the matrix exponential can be computed explicitly in closed form for the most interesting cases such as the two-component vector NLS equation. In particular, we do not resort to Cardano's formula in diagonalizing a 3×3 matrix, so our expressions for the multicomponent solitons are in some sense more explicit and useful than those obtained in [Q-H. Park and H. J. Shin, Phys. Rev. E **61** (2000) 3093].

Keywords: vector/matrix NLS, Lax pair, bright solitons, dark solitons, multicomponent solitons, Bäcklund–Darboux transformations

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1 Introduction

The cubic nonlinear Schrödinger (NLS) equation in $1+1$ dimensions was first solved by the inverse scattering method[1] under vanishing boundary conditions at spatial infinity [1–3]. The method was soon generalized to integrate the NLS equation under nonvanishing (or plane-wave[2]) boundary conditions at infinity and various interesting solutions were obtained [5–8] (also see [9–11]). In particular, dark-soliton solutions for the NLS equation with a self-defocusing nonlinearity [5,13–15] and bright-soliton solutions on a plane-wave background for the NLS equation with a self-focusing nonlinearity [6–8] are the most relevant and fundamental.

The integrability of the NLS equation is based on its Lax-pair[3] (or zero-curvature) representation [1–3, 5], which provides a starting point for applying the inverse scattering method. By generalizing the Lax pair, we can straightforwardly obtain a vector/matrix analog of the NLS equation [17,18] (also see [19]), which can be solved by the inverse scattering method under the vanishing boundary conditions. Thus, it is natural to expect that the inverse scattering method is also applicable to the vector/matrix NLS equation under general nonvanishing boundary conditions. However, in fact this is not the case. The main difficulty is that the background plane waves can have different wavenumbers and frequencies in different components of the vector/matrix dependent variable [20–23]. Such a general case is not amenable to the usual formulation of the inverse scattering method [22], so only a degenerate case in which all the components approach the same plane wave at spatial infinity, up to (possibly zero) proportionality factors, has been considered [24–27] (also see [28]). Although some soliton solutions in the degenerate case are certainly nontrivial and interesting [24, 29–31], they form a rather restricted subclass of the general soliton solutions satisfying the plane-wave boundary conditions.

The main objective of this paper is to construct new soliton solutions of the vector/matrix NLS equation, which have not been presented explicitly in the literature. To deal with the non-degenerate case of the nonvanishing boundary conditions involving more than one background plane wave, we use Bäcklund–Darboux transformations [32, 33] instead of the inverse scattering method. The main advantage of the Bäcklund–Darboux transformations is that a new solution of the nonlinear equation considered can be constructed

[1] The inverse scattering method was originally devised to solve the KdV equation [4].

[2] The Galilean invariance of the NLS equation [12] allows us to set the wavenumber of the background plane wave as zero without loss of generality. However, the complex phase of the dependent variable can take different values for $x \to -\infty$ and $x \to +\infty$ [5, 7].

[3] The term "Lax pair" is named after the work of Peter Lax on the KdV hierarchy [16].

from a known solution using only linear operations. More specifically, we need to find a linear eigenfunction of the Lax-pair representation for a given seed solution of the nonlinear equation. In essence, this task is equivalent to solving a linear system of ordinary differential equations in the spatial variable. For the vector/matrix NLS equation under the general plane-wave boundary conditions, the construction of soliton solutions reduces to the computation of the exponential of a constant non-diagonal matrix arising from the Lax pair [34, 35]; this also applies for the higher flows of the vector/matrix NLS hierarchy (cf. [36]). A straightforward way to compute the matrix exponential is to diagonalize the non-diagonal matrix by solving its characteristic equation.[4] However, even in the simplest nontrivial case of the two-component vector NLS equation, it requires to solve a cubic equation with generally complex coefficients using Cardano's formula[5] [34, 38] (also see [22]). Unfortunately, the obtained expression for the matrix exponential is too cumbersome and not suitable for investigating the asymptotic behaviors of the solutions. Moreover, such an expression intrinsically has an ambiguity, because Cardano's formula generally involves the square and cubic roots of a complex number, which are not uniquely determined. To fix this shortcoming, we re-parametrize the non-diagonal matrix so that the matrix exponential can be computed in a more easy-to-read form without any ambiguity. Consequently, for the two-component vector NLS equation, we can express the general one-soliton solution explicitly by using the solutions of a quadratic equation with real coefficients.

The nature and the solution space of the integrable NLS equation depend critically on the sign of the cubic term. In the scalar case, there exist only two choices corresponding to the self-focusing NLS and the self-defocusing NLS, respectively; these names can be justified by considering the physical meaning of the potential energy term in the canonical Hamiltonian formalism [39]. In the multicomponent case, *e.g.*, when the dependent variable is vector- or matrix-valued [17, 18], we have the third possibility: a mixed focusing-defocusing nonlinearity that contains both terms with a plus sign and terms with a minus sign [20, 21, 29, 40–42].

In the scalar case, the self-defocusing NLS equation admits dark-soliton solutions. A dark soliton is a dip of the density in the background plane wave, which can be obtained by applying an elementary Bäcklund–Darboux transformation [43, 44] to the plane-wave solution. Each dark soliton is associated

[4] In this paper, we do not separately discuss the non-generic case wherein this non-diagonal matrix cannot be diagonalized, i.e., it has a non-diagonal Jordan normal form (cf. [37]).

[5] The formula for finding the roots of a cubic equation has a rather complicated history (see, *e.g.*, `http://en.wikipedia.org/wiki/Cubic_function`).

with a square-integrable eigenfunction (i.e., a bound state) of the Lax pair with a real eigenvalue, because the spatial Lax operator for the self-defocusing NLS equation is self-adjoint [2,3,5]. In contrast, the self-focusing NLS equation admits bright-soliton solutions. A bright soliton is generally a hump of the density, which can be obtained by applying a binary Bäcklund–Darboux transformation [32,33] (also see [45–47]) that is equivalent to a suitable composition of two elementary Bäcklund–Darboux transformations [43,44]. Each bright soliton is associated with two bound states of the Lax pair, wherein the spatial Lax operator is not self-adjoint and the bound-state eigenvalues form a complex conjugate pair [1,3].

In a similar way, we can obtain multicomponent dark-soliton solutions in the self-defocusing case and multicomponent bright-soliton solutions in the self-focusing case using an elementary Bäcklund–Darboux transformation and a binary Bäcklund–Darboux transformation, respectively. Indeed, the general formulation of the Bäcklund–Darboux transformations is valid irrespective of the matrix size of the Lax-pair representation [34,35,48,49], which is related to the number of components of the vector/matrix dependent variable. A vector dark soliton [22] in the self-defocusing case does not admit any essential parameter representing the internal degrees of freedom, although it provides a nontrivial generalization of the scalar dark soliton. Contrastingly, a matrix dark soliton in the self-defocusing case and a vector/matrix bright soliton in the self-focusing case contain free parameters representing the internal degrees of freedom. The case of a mixed focusing-defocusing nonlinearity is more complicated, because it admits both a dark soliton [20–22] and a bright soliton [22]; they can be obtained by applying an elementary Bäcklund–Darboux transformation and a binary Bäcklund–Darboux transformation, respectively. Moreover, in this case, we can also consider a limiting case of the binary Bäcklund–Darboux transformation in such a way that the associated two bound-state eigenvalues coalesce into a real eigenvalue (cf. §2.4 in [33] and [35]); as a result of its application, a vector dark soliton with internal degrees of freedom can be constructed. In particular, the three-component vector NLS equation with one focusing and two defocusing components admits a non-stationary dark soliton that exhibits a soliton mutation phenomenon; that is, the shape and velocity of the dark soliton can change spontaneously in the time evolution.

In the multicomponent case, it is generally impossible to distinguish between a dark soliton and a bright soliton in a rigorous manner just by looking at their wave profiles. Indeed, each soliton may exhibit a rather complicated behavior in individual components [30,31,34,49], which cannot be identified as a simple dip or hump. In addition, there exists a group of linear transformations, called the symmetry group, which mixes the components of the

vector/matrix dependent variable but leaves the equation of motion invariant [21,29,50]; for example, the self-focusing/defocusing vector NLS equation is invariant under the group of (special) unitary transformations. Such linear transformations can drastically change both the boundary conditions and the wave profile of a soliton observed in individual components. Thus, instead of distinguishing between a dark soliton and a bright soliton intuitively, in this paper we use these terms in correspondence with the type of Bäcklund–Darboux transformation generating each soliton solution. That is, in analogy with the case of the scalar NLS equation, a dark soliton is associated with a square-integrable eigenfunction of the Lax pair with a real eigenvalue, while a bright soliton entails a pair of bound states with eigenvalues that are complex conjugates of each other. Note that this definition is more stringent than that commonly used in the literature. In particular, the spatial Lax operator for the vector/matrix NLS equation with a self-defocusing nonlinearity is self-adjoint [26, 27], so it does not admit any bright-soliton solution. This is in contrast to the work of Park and Shin [34, 49], wherein they applied a binary Bäcklund–Darboux transformation to the two-component vector NLS equation with a self-defocusing nonlinearity and constructed "dark-bright" and "dark-dark" soliton solutions (cf. "double solitons" in [22]). Their solutions are not associated with square-integrable eigenfunctions of the Lax pair, so we call them "soliton-like" solutions, which are not genuine soliton solutions in our definition.

The remainder of this paper is organized as follows. In section 2, we introduce step by step three types of Bäcklund–Darboux transformations for the vector/matrix NLS system. First, we introduce two elementary Bäcklund–Darboux transformations on the basis of the Lax-pair representation and its adjoint problem. Second, we combine the two elementary Bäcklund–Darboux transformations to define a binary Bäcklund–Darboux transformation. Third, we consider a limiting case of the binary Bäcklund–Darboux transformation in such a way that the associated two eigenvalues merge into a single eigenvalue. In section 3, by applying an elementary Bäcklund–Darboux transformation to a general plane-wave solution, we obtain dark-soliton solutions of the multicomponent NLS equations with a self-defocusing or mixed focusing-defocusing nonlinearity. In section 4, we apply the limiting case of the binary Bäcklund–Darboux transformation to the vector NLS equation with a mixed focusing-defocusing nonlinearity. Thus, we obtain a dark-soliton solution with internal degrees of freedom, which is more general than the vector dark-soliton solution obtained in section 3. In section 5, using the binary Bäcklund–Darboux transformation, we construct bright-soliton solutions on a general plane-wave background for the multicomponent NLS equations with a self-focusing or mixed focusing-defocusing nonlinearity; a

"soliton-like" solution in the self-defocusing case given in [34] (also see [49]) can also be obtained in a more explicit form. The last section, section 6, is devoted to concluding remarks.

2 General formulation of Bäcklund–Darboux transformations

In this section, we formulate three types of Bäcklund–Darboux transformations for the vector/matrix NLS system using the Lax-pair representation and its adjoint problem.

2.1 Multicomponent NLS systems

The matrix generalization [18] of the nonreduced NLS system [2, 3] is given by

$$\begin{cases} \mathrm{i}Q_t + Q_{xx} - 2QRQ = O, & (2.1\mathrm{a}) \\ \mathrm{i}R_t - R_{xx} + 2RQR = O. & (2.1\mathrm{b}) \end{cases}$$

Here, Q and R are $l \times m$ and $m \times l$ matrices, respectively, and the subscripts t and x denote the partial differentiation; we use the symbol O instead of 0 to stress that the dependent variables can take their values in matrices. A list of references on the matrix NLS system (2.1) and related equations can be found in [51].

Note that (2.1) allows a Hermitian conjugation reduction [21] (also see [29, 40–42]):

$$R = \Sigma Q^{\dagger} \Omega. \tag{2.2}$$

Here, Σ and Ω are $m \times m$ and $l \times l$ constant Hermitian matrices, respectively, and the dagger denotes the Hermitian conjugation. The reduction (2.2) simplifies the nonreduced matrix NLS system (2.1) to the matrix NLS equation:

$$\mathrm{i}Q_t + Q_{xx} - 2Q\Sigma Q^{\dagger} \Omega Q = O. \tag{2.3}$$

Considering a linear transformation $Q \to U_1 Q U_2$ with nonsingular constant matrices U_1 and U_2, we can recast Σ and Ω in their canonical forms, i.e., diagonal matrices whose diagonal entries are $+1$, -1 or 0. Because we are more interested in truly coupled systems than in triangular systems, we require that the diagonal entries of (the canonical forms of) Σ and Ω are $+1$ or -1. In particular, in the subsequent sections, we mainly consider the following two cases.

- The case $l = 1$: vector NLS equation [21, 29, 40–42],

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} - 2\langle \boldsymbol{q}\Sigma, \boldsymbol{q}^* \rangle \boldsymbol{q} = \boldsymbol{0}, \tag{2.4}$$

  which generalizes the Manakov model [17]. Here, $\boldsymbol{q} := (q_1, q_2, \ldots, q_m)$ is an m-component row vector and $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$ is a diagonal matrix with entries $\sigma_j = +1$ or -1. Thus, after a suitable re-numbering of the components, the scalar product can be written explicitly as

$$\langle \boldsymbol{q}\Sigma, \boldsymbol{q}^* \rangle := \sum_{j=1}^{n} |q_j|^2 - \sum_{k=n+1}^{m} |q_k|^2,$$

  where n $(0 \le n \le m)$ is the number of defocusing components and $m - n$ is the number of focusing components.

- The case $l = m$: square matrix NLS equation [18],

$$\mathrm{i}Q_t + Q_{xx} - 2\sigma Q Q^\dagger Q = O, \qquad \sigma = +1 \text{ or } -1. \tag{2.5}$$

  Here, the choice of $\sigma = +1$ and the choice of $\sigma = -1$ correspond to the self-defocusing case and the self-focusing case, respectively.

2.2 Lax-pair representation and Miura maps

The Lax-pair representation [16] for the nonreduced matrix NLS system (2.1) is given by [52, 53]

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I_l & Q \\ R & \mathrm{i}\zeta I_m \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{2.6a}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = \begin{bmatrix} -2\mathrm{i}\zeta^2 I_l - \mathrm{i}QR & 2\zeta Q + \mathrm{i}Q_x \\ 2\zeta R - \mathrm{i}R_x & 2\mathrm{i}\zeta^2 I_m + \mathrm{i}RQ \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{2.6b}$$

Here, ζ is the spectral parameter, which is an arbitrary constant, and I_l and I_m are the $l \times l$ and $m \times m$ identity matrices, respectively. The compatibility condition for the overdetermined linear system (2.6) indeed implies (2.1). Note that (2.6a) can be rewritten explicitly as an eigenvalue problem [1, 5]:

$$\begin{bmatrix} \mathrm{i}\partial_x & -\mathrm{i}Q \\ \mathrm{i}R & -\mathrm{i}\partial_x \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \zeta \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}.$$

In the self-defocusing case $R = Q^\dagger$, this eigenvalue problem is self-adjoint, so the eigenvalue ζ is restricted to be real-valued for any square-integrable (i.e., bound-state) eigenfunction; note that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \Psi_1^\dagger & \Psi_2^\dagger \end{bmatrix} \begin{bmatrix} \mathrm{i}\partial_x & -\mathrm{i}Q \\ \mathrm{i}Q^\dagger & -\mathrm{i}\partial_x \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \mathrm{d}x = \zeta \int_{-\infty}^{\infty} \begin{bmatrix} \Psi_1^\dagger & \Psi_2^\dagger \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \mathrm{d}x.$$

We also consider the adjoint Lax-pair representation:[6]

$$\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]_x = -\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]\left[\begin{array}{cc}-\mathrm{i}\zeta I_l & Q \\ R & \mathrm{i}\zeta I_m\end{array}\right], \tag{2.7a}$$

$$\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]_t = -\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]\left[\begin{array}{cc}-2\mathrm{i}\zeta^2 I_l - \mathrm{i}QR & 2\zeta Q + \mathrm{i}Q_x \\ 2\zeta R - \mathrm{i}R_x & 2\mathrm{i}\zeta^2 I_m + \mathrm{i}RQ\end{array}\right]. \tag{2.7b}$$

Indeed, a Lagrange-like identity,

$$\begin{aligned}&\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]\left\{\left[\begin{array}{c}\Psi_1 \\ \Psi_2\end{array}\right]_x - \left[\begin{array}{cc}-\mathrm{i}\zeta I_l & Q \\ R & \mathrm{i}\zeta I_m\end{array}\right]\left[\begin{array}{c}\Psi_1 \\ \Psi_2\end{array}\right]\right\} \\ &+\left\{\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]_x + \left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]\left[\begin{array}{cc}-\mathrm{i}\zeta I_l & Q \\ R & \mathrm{i}\zeta I_m\end{array}\right]\right\}\left[\begin{array}{c}\Psi_1 \\ \Psi_2\end{array}\right] \\ &=\left\{\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]\left[\begin{array}{c}\Psi_1 \\ \Psi_2\end{array}\right]\right\}_x,\end{aligned}$$

and a similar relation for the t-differentiation imply that (2.7) is the adjoint problem of (2.6). In particular, the product of solutions of (2.7) and (2.6) is (x,t)-independent:

$$\left[\begin{array}{cc}\Phi_1 & \Phi_2\end{array}\right]\left[\begin{array}{c}\Psi_1 \\ \Psi_2\end{array}\right] = \text{const.} \tag{2.8}$$

We consider an $(l+m)\times l$ matrix-valued linear eigenfunction of the Lax-pair representation (2.6) such that Ψ_1 is an $l\times l$ nonsingular matrix and define an $m\times l$ matrix $\widetilde{R}$ as $\widetilde{R} := \Psi_2\Psi_1^{-1}$. Then, (2.6) implies the relations (cf. [46, 47, 55]):

$$R = \widetilde{R}_x - 2\mathrm{i}\zeta\widetilde{R} + \widetilde{R}Q\widetilde{R}, \tag{2.9a}$$

$$\widetilde{R}_t = 2\zeta R - \mathrm{i}R_x + 4\mathrm{i}\zeta^2\widetilde{R} + \mathrm{i}RQ\widetilde{R} + \mathrm{i}\widetilde{R}QR - 2\zeta\widetilde{R}Q\widetilde{R} - \mathrm{i}\widetilde{R}Q_x\widetilde{R}. \tag{2.9b}$$

Relation (2.9a) enables us to express R in (2.1a) and (2.9b) in terms of Q and $\widetilde{R}$ [56]. Thus, we obtain a closed two-component system for Q and $\widetilde{R}$:

$$\left\{\begin{array}{ll}\mathrm{i}Q_t + Q_{xx} + 4\mathrm{i}\zeta Q\widetilde{R}Q - 2Q\widetilde{R}_xQ - 2Q\widetilde{R}Q\widetilde{R}Q = O, & (2.10a) \\ \mathrm{i}\widetilde{R}_t - \widetilde{R}_{xx} - 4\mathrm{i}\zeta\widetilde{R}Q\widetilde{R} - 2\widetilde{R}Q_x\widetilde{R} + 2\widetilde{R}Q\widetilde{R}Q\widetilde{R} = O. & (2.10b)\end{array}\right.$$

This is a matrix generalization [57–62] of the derivative NLS system studied by Ablowitz, Ramani and Segur [63] (and later by Gerdjikov and Ivanov [64] without citing the paper [63]); the cubic terms involving the parameter ζ can

[6] If we have a square-matrix solution Ψ of the linear system $\Psi_x = U\Psi,\ \Psi_t = V\Psi$, then its inverse $\Phi := \Psi^{-1}$ solves the adjoint system $\Phi_x = -\Phi U,\ \Phi_t = -\Phi V$ [41, 54].

be removed by a simple point transformation (cf. [65–67]). Note that (2.9a) defines a Miura map from (2.10) to (2.1): $(Q, \widetilde{R}) \mapsto (Q, R)$.

Similarly, using a linear eigenfunction of the adjoint Lax-pair representation (2.7), we define an $l \times m$ matrix $\widehat{Q}$ as $\widehat{Q} := \Phi_1^{-1}\Phi_2$. Then, we obtain the relations:

$$Q = -\widehat{Q}_x - 2\mathrm{i}\zeta\widehat{Q} + \widehat{Q}R\widehat{Q}, \tag{2.11a}$$

$$\widehat{Q}_t = -2\zeta Q - \mathrm{i}Q_x - 4\mathrm{i}\zeta^2\widehat{Q} - \mathrm{i}QR\widehat{Q} - \mathrm{i}\widehat{Q}RQ + 2\zeta\widehat{Q}R\widehat{Q} - \mathrm{i}\widehat{Q}R_x\widehat{Q}. \tag{2.11b}$$

Relation (2.11a) allows us to express Q in (2.11b) and (2.1b) in terms of $\widehat{Q}$ and R. Thus, a closed system for $\widehat{Q}$ and R is obtained:

$$\begin{cases} \mathrm{i}\widehat{Q}_t + \widehat{Q}_{xx} + 4\mathrm{i}\zeta\widehat{Q}R\widehat{Q} - 2\widehat{Q}R_x\widehat{Q} - 2\widehat{Q}R\widehat{Q}R\widehat{Q} = O, & (2.12a) \\ \mathrm{i}R_t - R_{xx} - 4\mathrm{i}\zeta R\widehat{Q}R - 2R\widehat{Q}_xR + 2R\widehat{Q}R\widehat{Q}R = O. & (2.12b) \end{cases}$$

Note that (2.10) and (2.12) are identical, so we have another Miura map (2.11a) from (2.12) to (2.1): $(\widehat{Q}, R) \mapsto (Q, R)$.

2.3 Elementary Bäcklund–Darboux transformations

We have obtained two distinct Miura maps from the matrix derivative NLS system to the matrix NLS system [58, 67]; the inverse of each Miura map can be expressed in terms of a linear eigenfunction of the (adjoint) Lax-pair representation for the matrix NLS system. Thus, combining the inverse of one Miura map with the other Miura map, we can construct an elementary Bäcklund–Darboux transformation for the matrix NLS system. Moreover, we can also identify the gauge transformation of the linear eigenfunction of the Lax-pair representation, which generates each elementary Bäcklund–Darboux transformation [68–70]. In the following, we often fix the arbitrary parameter ζ in the Bäcklund–Darboux transformation at some specific value, say $\zeta = \mu$, to distinguish it from the original spectral parameter ζ in the Lax-pair representation.

Proposition 2.1. The spatial part of an elementary auto-Bäcklund transformation for the matrix NLS system (2.1), which connects two solutions (Q, R) and $(\widetilde{Q}, \widetilde{R})$, is given by (see [43, 44] for the scalar case and [58, 70] for the matrix case)

$$\widetilde{Q} = -Q_x - 2\mathrm{i}\mu Q + Q\widetilde{R}Q,$$
$$R = \widetilde{R}_x - 2\mathrm{i}\mu\widetilde{R} + \widetilde{R}Q\widetilde{R}.$$

The transformation $(Q,R)\mapsto(\widetilde{Q},\widetilde{R})$ can be expressed explicitly in terms of a linear eigenfunction of the Lax-pair representation (2.6) at $\zeta=\mu$ as [43,44]

$$\widetilde{R}=\Psi_2\Psi_1^{-1}\big|_{\zeta=\mu}, \qquad \widetilde{Q}=-Q_x-2\mathrm{i}\mu Q+Q\widetilde{R}Q. \tag{2.13}$$

The corresponding gauge transformation [68–70],

$$\begin{bmatrix}\widetilde{\Psi}_1\\ \widetilde{\Psi}_2\end{bmatrix}:=G(Q,\widetilde{R};\mu)\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}, \qquad G(Q,\widetilde{R};\mu):=g(\zeta,\mu)\begin{bmatrix}2\mathrm{i}(\zeta-\mu)I_l+Q\widetilde{R} & -Q\\ -\widetilde{R} & I_m\end{bmatrix}, \tag{2.14}$$

where g is an arbitrary (but not identically zero) scalar function of ζ and μ, changes the original Lax-pair representation (2.6) with $R=\widetilde{R}_x-2\mathrm{i}\mu\widetilde{R}+\widetilde{R}Q\widetilde{R}$ into a new Lax-pair representation of the same form. That is,

$$\begin{bmatrix}\widetilde{\Psi}_1\\ \widetilde{\Psi}_2\end{bmatrix}_x=\begin{bmatrix}-\mathrm{i}\zeta I_l & \widetilde{Q}\\ \widetilde{R} & \mathrm{i}\zeta I_m\end{bmatrix}\begin{bmatrix}\widetilde{\Psi}_1\\ \widetilde{\Psi}_2\end{bmatrix}, \qquad \widetilde{Q}=-Q_x-2\mathrm{i}\mu Q+Q\widetilde{R}Q,$$

and similar for the t-differentiation. In the same manner, the gauge transformation preserving the form of the adjoint Lax-pair representation (2.7) is given by

$$\begin{bmatrix}\widetilde{\Phi}_1 & \widetilde{\Phi}_2\end{bmatrix}:=\begin{bmatrix}\Phi_1 & \Phi_2\end{bmatrix}H(Q,\widetilde{R};\mu), \qquad H(Q,\widetilde{R};\mu):=h(\zeta,\mu)\begin{bmatrix}I_l & Q\\ \widetilde{R} & 2\mathrm{i}(\zeta-\mu)I_m+\widetilde{R}Q\end{bmatrix},$$

where h is an arbitrary (but not identically zero) scalar function of ζ and μ. Note that $G(Q,\widetilde{R};\mu)H(Q,\widetilde{R};\mu)=2\mathrm{i}(\zeta-\mu)g(\zeta,\mu)h(\zeta,\mu)I_{l+m}$.

The elementary Bäcklund–Darboux transformation $(Q,R)\mapsto(\widetilde{Q},\widetilde{R})$ defined by (2.13) is not a map determining $(\widetilde{Q},\widetilde{R})$ uniquely; indeed, it involves additional free parameters, because each column of the linear eigenfunction used in (2.13) is an arbitrary linear combination of a fundamental set of solutions. The transformation admits the following conservation law:

$$\begin{aligned}\widetilde{Q}\widetilde{R}-QR&=-\left(Q\widetilde{R}\right)_x\\ &=-\left(Q\ \Psi_2\Psi_1^{-1}\big|_{\zeta=\mu}\right)_x\\ &=-\left(\Psi_{1,x}\Psi_1^{-1}\big|_{\zeta=\mu}\right)_x.\end{aligned}$$

Thus, if we can choose the linear eigenfunction such that $\lim_{x\to-\infty}\Psi_{1,x}\Psi_1^{-1}$ and $\lim_{x\to+\infty}\Psi_{1,x}\Psi_1^{-1}$ exist and $\Psi_{1,x}\Psi_1^{-1}$ does not oscillate extraordinarily

rapidly as $x \to \pm\infty$, then $\widetilde{Q}\widetilde{R}$ and QR exhibit the same asymptotic behavior at spatial infinity. In addition, there exists another conservation law:

$$\begin{aligned}\widetilde{R}\widetilde{Q} - RQ &= -\left(\widetilde{R}Q\right)_x \\ &= -\left(\Psi_2\Psi_1^{-1}\big|_{\zeta=\mu} Q\right)_x .\end{aligned}$$

These conservation laws give a useful hint on how to realize a Hermitian conjugation reduction between $\widetilde{Q}$ and $\widetilde{R}$.

Combining the inverse of one Miura map with the other Miura map in the other way, we obtain another elementary Bäcklund–Darboux transformation; by construction, it provides the inverse of the elementary Bäcklund–Darboux transformation defined in Proposition 2.1 if appropriate boundary conditions are imposed (cf. [43]).

Proposition 2.2. The spatial part of an elementary auto-Bäcklund transformation for the matrix NLS system (2.1), which connects two solutions (Q, R) and $(\widehat{Q}, \widehat{R})$, is given by (see [43, 44] for the scalar case and [58, 70] for the matrix case)

$$\begin{aligned}Q &= -\widehat{Q}_x - 2\mathrm{i}\nu\widehat{Q} + \widehat{Q}R\widehat{Q}, \\ \widehat{R} &= R_x - 2\mathrm{i}\nu R + R\widehat{Q}R.\end{aligned}$$

The transformation $(Q, R) \mapsto (\widehat{Q}, \widehat{R})$ can be expressed explicitly in terms of a linear eigenfunction of the adjoint Lax-pair representation (2.7) at $\zeta = \nu$ as

$$\widehat{Q} = \Phi_1^{-1}\Phi_2\big|_{\zeta=\nu}, \qquad \widehat{R} = R_x - 2\mathrm{i}\nu R + R\widehat{Q}R.$$

The corresponding gauge transformation [68–70],

$$\begin{bmatrix} \widehat{\Psi}_1 \\ \widehat{\Psi}_2 \end{bmatrix} := H(\widehat{Q}, R; \nu) \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \qquad H(\widehat{Q}, R; \nu) = h(\zeta, \nu) \begin{bmatrix} I_l & \widehat{Q} \\ R & 2\mathrm{i}(\zeta - \nu)I_m + R\widehat{Q} \end{bmatrix},$$

changes the original Lax-pair representation (2.6) with $Q = -\widehat{Q}_x - 2\mathrm{i}\nu\widehat{Q} + \widehat{Q}R\widehat{Q}$ into a new Lax-pair representation of the same form. That is,

$$\begin{bmatrix} \widehat{\Psi}_1 \\ \widehat{\Psi}_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I_l & \widehat{Q} \\ \widehat{R} & \mathrm{i}\zeta I_m \end{bmatrix} \begin{bmatrix} \widehat{\Psi}_1 \\ \widehat{\Psi}_2 \end{bmatrix}, \qquad \widehat{R} = R_x - 2\mathrm{i}\nu R + R\widehat{Q}R,$$

and similar for the t-differentiation. The gauge transformation preserving the form of the adjoint Lax-pair representation (2.7) is given by

$$\begin{bmatrix} \widehat{\Phi}_1 & \widehat{\Phi}_2 \end{bmatrix} := \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} G(\widehat{Q}, R; \nu), \qquad G(\widehat{Q}, R; \nu) = g(\zeta, \nu) \begin{bmatrix} 2\mathrm{i}(\zeta - \nu)I_l + \widehat{Q}R & -\widehat{Q} \\ -R & I_m \end{bmatrix}.$$

The elementary Bäcklund–Darboux transformation $(Q,R)\mapsto(\widehat{Q},\widehat{R})$ admits the following conservation laws:

$$\begin{aligned}\widehat{Q}\widehat{R}-QR&=\Big(\widehat{Q}R\Big)_x\\&=-\Big(\Phi_1^{-1}\Phi_{1,x}\big|_{\zeta=\nu}\Big)_x,\\\widehat{R}\widehat{Q}-RQ&=\Big(R\widehat{Q}\Big)_x\\&=\Big(R\,\Phi_1^{-1}\Phi_2\big|_{\zeta=\nu}\Big)_x.\end{aligned}$$

2.4 Binary Bäcklund–Darboux transformation

We can further combine the two elementary Bäcklund–Darboux transformations defined in Propositions 2.1 and 2.2 involving the Bäcklund parameters μ and ν, respectively. In the following, we consider the composition in the order

$$(Q,R)\mapsto(\widetilde{Q},\widetilde{R})\mapsto(\widehat{\widetilde{Q}},\widehat{\widetilde{R}})\tag{2.15}$$

because the composition in the opposite order

$$(Q,R)\mapsto(\widehat{Q},\widehat{R})\mapsto(\widetilde{\widehat{Q}},\widetilde{\widehat{R}})$$

provides the same final result; that is, the two elementary Bäcklund–Darboux transformations commute with each other [43, 50, 71].

Thus, by applying Propositions 2.1 and 2.2 in the order (2.15), we obtain

$$\begin{aligned}\widehat{\widetilde{Q}}&=\Big[\Phi_1(\nu)+\Phi_2(\nu)\widetilde{R}\Big]^{-1}\Big[\Phi_1(\nu)Q+2\mathrm{i}(\nu-\mu)\Phi_2(\nu)+\Phi_2(\nu)\widetilde{R}Q\Big]\\&=Q+2\mathrm{i}(\nu-\mu)\left[\Phi_1(\nu)+\Phi_2(\nu)\Psi_2(\mu)\Psi_1(\mu)^{-1}\right]^{-1}\Phi_2(\nu)\\&=Q+2\mathrm{i}(\nu-\mu)\Psi_1(\mu)\left[\Phi_1(\nu)\Psi_1(\mu)+\Phi_2(\nu)\Psi_2(\mu)\right]^{-1}\Phi_2(\nu),\end{aligned}\tag{2.16a}$$

and

$$\begin{aligned}\widehat{\widetilde{R}}&=\widetilde{R}_x-2\mathrm{i}\nu\widetilde{R}+\widetilde{R}\widehat{\widetilde{Q}}\widetilde{R}\\&=R-2\mathrm{i}(\nu-\mu)\widetilde{R}+\widetilde{R}\Big(\widehat{\widetilde{Q}}-Q\Big)\widetilde{R}\\&=R-2\mathrm{i}(\nu-\mu)\Psi_2(\mu)\Psi_1(\mu)^{-1}\\&\quad+2\mathrm{i}(\nu-\mu)\Psi_2(\mu)\left[\Phi_1(\nu)\Psi_1(\mu)+\Phi_2(\nu)\Psi_2(\mu)\right]^{-1}\Phi_2(\nu)\Psi_2(\mu)\Psi_1(\mu)^{-1}\\&=R-2\mathrm{i}(\nu-\mu)\Psi_2(\mu)\left[\Phi_1(\nu)\Psi_1(\mu)+\Phi_2(\nu)\Psi_2(\mu)\right]^{-1}\Phi_1(\nu).\end{aligned}\tag{2.16b}$$

The corresponding gauge transformation of the linear eigenfunction can be written as

$$
\begin{aligned}
\begin{bmatrix} \widehat{\widetilde{\Psi}}_1 \\ \widehat{\widetilde{\Psi}}_2 \end{bmatrix} &:= H(\widehat{\widetilde{Q}}, \widetilde{R}; \nu) G(Q, \widetilde{R}; \mu) \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \\
&= h(\zeta, \nu) g(\zeta, \mu) \begin{bmatrix} I_l & \widehat{\widetilde{Q}} \\ \widetilde{R} & 2\mathrm{i}(\zeta - \nu) I_m + \widetilde{R}\widehat{\widetilde{Q}} \end{bmatrix} \begin{bmatrix} 2\mathrm{i}(\zeta - \mu) I_l + Q\widetilde{R} & -Q \\ -\widetilde{R} & I_m \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \\
&= h(\zeta, \nu) g(\zeta, \mu) \begin{bmatrix} 2\mathrm{i}(\zeta - \mu) I_l + \Big(Q - \widehat{\widetilde{Q}}\Big)\widetilde{R} & \widehat{\widetilde{Q}} - Q \\ 2\mathrm{i}(\nu - \mu)\widetilde{R} + \widetilde{R}\Big(Q - \widehat{\widetilde{Q}}\Big)\widetilde{R} & 2\mathrm{i}(\zeta - \nu) I_m + \widetilde{R}\Big(\widehat{\widetilde{Q}} - Q\Big) \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \\
&\propto \left\{ I_{l+m} + \frac{\nu - \mu}{\zeta - \nu} \begin{bmatrix} \Psi_1(\mu) \\ \Psi_2(\mu) \end{bmatrix} [\Phi_1(\nu)\Psi_1(\mu) + \Phi_2(\nu)\Psi_2(\mu)]^{-1} \begin{bmatrix} \Phi_1(\nu) & \Phi_2(\nu) \end{bmatrix} \right\} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}.
\end{aligned}
$$

Here, we omit the unessential proportionality factor in the last line. In a similar way, the gauge transformation of the adjoint linear eigenfunction can be written as

$$
\begin{aligned}
&\begin{bmatrix} \widehat{\widetilde{\Phi}}_1 & \widehat{\widetilde{\Phi}}_2 \end{bmatrix} \\
&\propto \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \left\{ I_{l+m} + \frac{\mu - \nu}{\zeta - \mu} \begin{bmatrix} \Psi_1(\mu) \\ \Psi_2(\mu) \end{bmatrix} [\Phi_1(\nu)\Psi_1(\mu) + \Phi_2(\nu)\Psi_2(\mu)]^{-1} \begin{bmatrix} \Phi_1(\nu) & \Phi_2(\nu) \end{bmatrix} \right\}.
\end{aligned}
$$

Note that the commutativity of the two elementary Bäcklund transformations can be expressed in the matrix product form (cf. [72–74]):

$$
\begin{aligned}
&\begin{bmatrix} I_l & \widehat{\widetilde{Q}} \\ \widetilde{R} & 2\mathrm{i}(\zeta - \nu) I_m + \widetilde{R}\widehat{\widetilde{Q}} \end{bmatrix} \begin{bmatrix} 2\mathrm{i}(\zeta - \mu) I_l + Q\widetilde{R} & -Q \\ -\widetilde{R} & I_m \end{bmatrix} \\
&= \begin{bmatrix} 2\mathrm{i}(\zeta - \mu) I_l + \widehat{Q}\widetilde{\widehat{R}} & -\widehat{Q} \\ -\widetilde{\widehat{R}} & I_m \end{bmatrix} \begin{bmatrix} I_l & \widehat{Q} \\ R & 2\mathrm{i}(\zeta - \nu) I_m + R\widehat{Q} \end{bmatrix},
\end{aligned}
$$

which provides the nonlinear superposition principle for the solutions of the matrix NLS hierarchy. It can be described by the pair of relations [43, 44, 70]:

$$
\begin{aligned}
\widehat{\widetilde{Q}} &= Q + 2\mathrm{i}(\nu - \mu)\widehat{Q}\left(I_m + \widetilde{R}\widehat{Q}\right)^{-1}, \\
\widetilde{\widehat{R}} &= R + 2\mathrm{i}(\mu - \nu)\widetilde{R}\left(I_l + \widehat{Q}\widetilde{R}\right)^{-1}.
\end{aligned}
$$

These relations can be reinterpreted as a fully discrete integrable system defined on the two-dimensional lattice [68, 73–75].

The composition of the two elementary Bäcklund–Darboux transformations defines a binary Bäcklund–Darboux transformation [32,33] that can be formulated as follows.

Proposition 2.3. For a linear eigenfunction satisfying (2.6) at $\zeta = \mu$ and an adjoint linear eigenfunction satisfying (2.7) at $\zeta = \nu$, we define a projection matrix $P(\mu, \nu)$ in the 2×2 block-matrix form:

$$P(\mu,\nu) := \begin{bmatrix} \Psi_1(\mu) \\ \Psi_2(\mu) \end{bmatrix} [\Phi_1(\nu)\Psi_1(\mu) + \Phi_2(\nu)\Psi_2(\mu)]^{-1} \begin{bmatrix} \Phi_1(\nu) & \Phi_2(\nu) \end{bmatrix}.$$

Indeed, $P^2 = P$. Then, the binary Bäcklund–Darboux transformation $(Q, R) \mapsto (\widehat{\widetilde{Q}}, \widehat{\widetilde{R}})$ for the matrix NLS system (2.1) can be expressed as (see [46,47] for the scalar case and [34,38,48,49,76] for the vector case)

$$\begin{aligned} \widehat{\widetilde{Q}} &= Q + 2\mathrm{i}(\nu - \mu)P_{12}(\mu,\nu), \\ \widehat{\widetilde{R}} &= R - 2\mathrm{i}(\nu - \mu)P_{21}(\mu,\nu). \end{aligned}$$

Here, P_{12} and P_{21} denote the upper-right and lower-left block of the matrix P, respectively. The corresponding gauge transformation of the linear eigenfunction is given as

$$\begin{bmatrix} \widehat{\widetilde{\Psi}}_1 \\ \widehat{\widetilde{\Psi}}_2 \end{bmatrix} \propto \left\{ I_{l+m} - \frac{\mu - \nu}{\zeta - \nu} P(\mu,\nu) \right\} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix},$$

up to an overall constant, which coincides with the form suggested by the Zakharov–Shabat dressing method [77]. The gauge transformation of the adjoint linear eigenfunction is given using the inverse of the above dressing operator as

$$\begin{bmatrix} \widehat{\widetilde{\Phi}}_1 & \widehat{\widetilde{\Phi}}_2 \end{bmatrix} \propto \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \left\{ I_{l+m} - \frac{\nu - \mu}{\zeta - \mu} P(\mu,\nu) \right\}.$$

In the limit $\nu \to \mu$, the binary Bäcklund–Darboux transformation reduces to the identity transformation as long as $\Phi_1(\mu)\Psi_1(\mu) + \Phi_2(\mu)\Psi_2(\mu)$ is an invertible matrix (cf. (2.8)).

Thus, under a suitable choice of the (adjoint) linear eigenfunctions used in the transformations, the two elementary Bäcklund–Darboux transformations defined in Propositions 2.1 and 2.2 are the inverse of each other at $\mu = \nu$. However, we can consider a more interesting choice of the (adjoint) linear eigenfunctions such that the binary Bäcklund–Darboux transformation reduces to a nontrivial transformation in the limit $\nu \to \mu$.

2.5 Coalescence limit of the binary Bäcklund–Darboux transformation

From the Lax-pair representation (2.6) and its adjoint (2.7), we can show that the product of an adjoint linear eigenfunction and a linear eigenfunction satisfies (cf. [1,3])

$$\left\{\frac{1}{\mathrm{i}(\nu-\mu)}\left[\Phi_1(\nu)\Psi_1(\mu)+\Phi_2(\nu)\Psi_2(\mu)\right]\right\}_x=\Phi_1(\nu)\Psi_1(\mu)-\Phi_2(\nu)\Psi_2(\mu)$$

and

$$\begin{aligned}&\left\{\frac{1}{\mathrm{i}(\nu-\mu)}\left[\Phi_1(\nu)\Psi_1(\mu)+\Phi_2(\nu)\Psi_2(\mu)\right]\right\}_t\\&=2(\mu+\nu)\left[\Phi_1(\nu)\Psi_1(\mu)-\Phi_2(\nu)\Psi_2(\mu)\right]+2\mathrm{i}\left[\Phi_1(\nu)Q\Psi_2(\mu)+\Phi_2(\nu)R\Psi_1(\mu)\right].\end{aligned}$$

By construction, these two relations are compatible, i.e.

$$\begin{aligned}&\left\{\Phi_1(\nu)\Psi_1(\mu)-\Phi_2(\nu)\Psi_2(\mu)\right\}_t\\&=\left\{2(\mu+\nu)\left[\Phi_1(\nu)\Psi_1(\mu)-\Phi_2(\nu)\Psi_2(\mu)\right]+2\mathrm{i}\left[\Phi_1(\nu)Q\Psi_2(\mu)+\Phi_2(\nu)R\Psi_1(\mu)\right]\right\}_x.\end{aligned}$$

Thus, if we introduce an $l\times l$ matrix F as

$$F(\mu):=\lim_{\nu\to\mu}\frac{1}{\mathrm{i}(\nu-\mu)}\left[\Phi_1(\nu)\Psi_1(\mu)+\Phi_2(\nu)\Psi_2(\mu)\right], \tag{2.17}$$

it satisfies the pair of relations:

$$F_x=\Phi_1(\mu)\Psi_1(\mu)-\Phi_2(\mu)\Psi_2(\mu), \tag{2.18a}$$

$$F_t=4\mu F_x+2\mathrm{i}\left[\Phi_1(\mu)Q\Psi_2(\mu)+\Phi_2(\mu)R\Psi_1(\mu)\right]. \tag{2.18b}$$

Moreover, F^{-1} should satisfy

$$F^{-1}\left[\Phi_1(\mu)\Psi_1(\mu)+\Phi_2(\mu)\Psi_2(\mu)\right]=\left[\Phi_1(\mu)\Psi_1(\mu)+\Phi_2(\mu)\Psi_2(\mu)\right]F^{-1}=O.$$

In this paper, we consider the simplest case where $\Phi_1(\mu)\Psi_1(\mu)+\Phi_2(\mu)\Psi_2(\mu)$ vanishes (cf. (2.8)). Thus, considering the $\nu\to\mu$ limit of Proposition 2.3, we arrive at a nontrivial limiting case of the binary Bäcklund–Darboux transformation.[7] Relavant results were obtained independently by Degasperis and Lombardo [35] using a different approach.

[7] Such a limiting procedure is already well-known for the linear Schrödinger equation associated with the KdV equation (see §2.4 of [33]); the resulting transformation is sometimes referred to as the Abraham–Moses transformation [78] (see [79] for details).

Proposition 2.4. We choose a linear eigenfunction and an adjoint linear eigenfunction satisfying (2.6) and (2.7), respectively, at $\zeta = \mu$ in such a way that their (x,t)-independent product is equal to zero:

$$\Phi_1(\mu)\Psi_1(\mu) + \Phi_2(\mu)\Psi_2(\mu) = O.$$

We can introduce $F(\mu)$ that satisfies both (2.18a) and (2.18b), because they are compatible. Thus, $F(\mu)$ is defined up to the addition of an integration constant, which is an arbitrary $l \times l$ matrix. Then, the limiting case of the binary Bäcklund–Darboux transformation, $(Q,R) \mapsto (\breve{Q},\breve{R})$, for the matrix NLS system (2.1) can be expressed as

$$\breve{Q} = Q + 2\mathcal{P}_{12}(\mu), \tag{2.19a}$$

$$\breve{R} = R - 2\mathcal{P}_{21}(\mu). \tag{2.19b}$$

Here, $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ are the upper-right and lower-left block, respectively, of the nilpotent matrix $\mathcal{P}(\mu)$ defined as

$$\mathcal{P}(\mu) := \begin{bmatrix} \Psi_1(\mu) \\ \Psi_2(\mu) \end{bmatrix} F(\mu)^{-1} \begin{bmatrix} \Phi_1(\mu) & \Phi_2(\mu) \end{bmatrix}.$$

Note that $\mathcal{P}^2 = O$. The corresponding gauge transformation that preserves the form of the Lax-pair representation (2.6) is given as

$$\begin{bmatrix} \breve{\Psi}_1 \\ \breve{\Psi}_2 \end{bmatrix} \propto \left\{ I_{l+m} - \frac{\mathrm{i}}{\zeta - \mu}\mathcal{P}(\mu) \right\} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix},$$

up to an overall constant. The gauge transformation that preserves the form of the adjoint Lax-pair representation (2.7) is given using the inverse of the above operator as

$$\begin{bmatrix} \breve{\Phi}_1 & \breve{\Phi}_2 \end{bmatrix} \propto \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \left\{ I_{l+m} + \frac{\mathrm{i}}{\zeta - \mu}\mathcal{P}(\mu) \right\}.$$

The validity of Proposition 2.4 can be checked by a direct calculation. Note that $F(\mu)$ satisfying (2.18) can be informally written as

$$\begin{aligned} F(\mu) &= 2C + \int [\Phi_1(\mu)\Psi_1(\mu) - \Phi_2(\mu)\Psi_2(\mu)]\,\mathrm{d}x \\ &= 2\left[C + \int \Phi_1(\mu)\Psi_1(\mu)\,\mathrm{d}x\right], \end{aligned}$$

where C is an arbitrary $l \times l$ constant matrix. More precisely, the time independence of C should be checked by using (2.18b); however, this process can be skipped by choosing the linear eigenfunction and the adjoint linear eigenfunction so that the right-hand sides of (2.18a) and (2.18b) decay rapidly as $x \to -\infty$ (or $+\infty$). This expression involves one integration, but it is generally much easier to compute than taking the limit following the original definition (2.17) (cf. [34, 80]).

2.6 Seed solution

A general plane-wave solution of the nonreduced matrix NLS system (2.1) is given by

$$Q = P_1 \mathrm{e}^{-2\mathrm{i}tAB} A \mathrm{e}^{\mathrm{i}x\Gamma - \mathrm{i}t\Gamma^2} P_2, \qquad R = P_2^{-1} \mathrm{e}^{-\mathrm{i}x\Gamma + \mathrm{i}t\Gamma^2} B \mathrm{e}^{2\mathrm{i}tAB} P_1^{-1}, \tag{2.20}$$

where Γ, A, B, P_1 and P_2 are constant matrices; P_1 is an $l \times l$ matrix, Γ and P_2 are $m \times m$ matrices, A is an $l \times m$ matrix and B is an $m \times l$ matrix. Actually, P_1 and P_2 can be removed by redefining Γ, A and B; note, however, that they can be used to cast Γ and AB in Jordan normal form. A general plane-wave solution of the matrix NLS equation (2.3) can be obtained by imposing the reduction conditions

$$P_1^\dagger \Omega P_1 = \Omega, \qquad P_2 \Sigma P_2^\dagger = \Sigma, \qquad \Sigma \Gamma^\dagger \Sigma^{-1} = \Gamma, \qquad B = \Sigma A^\dagger \Omega$$

on (2.20) so that the Hermitian conjugation reduction (2.2) is realized. The third condition is satisfied for a diagonal matrix Σ by setting

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m),$$

where γ_j are real constants. In sections 4 and 5, we apply a binary Bäcklund–Darboux transformation and its limiting version to the reduced plane-wave solution, which can maintain the Hermitian conjugation reduction.

To obtain the linear eigenfunction of the Lax-pair representation (2.6) with Q and R given by the general plane-wave solution (2.20), we consider a gauge transformation,

$$\Psi_1 = P_1 \mathrm{e}^{-2\mathrm{i}tAB} \boldsymbol{\Psi}_1, \qquad \Psi_2 = \mathrm{i} P_2^{-1} \mathrm{e}^{-\mathrm{i}x\Gamma + \mathrm{i}t\Gamma^2} \boldsymbol{\Psi}_2. \tag{2.21}$$

Then, the Lax-pair representation at $\zeta = \mu$ reduces to the form:

$$\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix}_x = \mathrm{i} \begin{bmatrix} -\mu I_l & A \\ -B & \mu I_m + \Gamma \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix}, \tag{2.22a}$$

$$\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix}_t = \mathrm{i} \begin{bmatrix} -2\mu^2 I_l + AB & 2\mu A - A\Gamma \\ -2\mu B + \Gamma B & 2\mu^2 I_m + BA - \Gamma^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix}. \tag{2.22b}$$

The general solution of this linear system can be expressed using the matrix exponential of an $(l+m)\times(l+m)$ matrix as

$$\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} = \mathrm{e}^{\mathrm{i}x(\boldsymbol{Z}+\mu I)+\mathrm{i}t(-\boldsymbol{Z}^2+2\mu^2 I)} \begin{bmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \end{bmatrix}, \qquad \boldsymbol{Z}(\mu) := \begin{bmatrix} -2\mu I_l & A \\ -B & \Gamma \end{bmatrix}. \tag{2.23}$$

Here, I denotes the identity matrix I_{l+m} and $\mathscr{C}_1$ and $\mathscr{C}_2$ are $l\times l$ and $m\times l$ constant matrices, respectively. Assume that the constant matrix $\boldsymbol{Z}(\mu)$ can be diagonalized as

$$\boldsymbol{Z}X = X\Lambda, \qquad \Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{l+m}),$$

using a nonsingular matrix X whose jth column is an eigenvector of $\boldsymbol{Z}(\mu)$ with an eigenvalue λ_j. Then, by setting

$$\begin{bmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \end{bmatrix} = X \begin{bmatrix} \mathscr{U}_1 \\ \mathscr{U}_2 \end{bmatrix},$$

we can compute the matrix exponential in (2.23) explicitly. Thus, in view of (2.21), we obtain

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} P_1\mathrm{e}^{-2\mathrm{i}tAB} & O \\ O & \mathrm{i}P_2^{-1}\mathrm{e}^{-\mathrm{i}x\Gamma+\mathrm{i}t\Gamma^2} \end{bmatrix} X\mathrm{e}^{\mathrm{i}x(\Lambda+\mu I)+\mathrm{i}t(-\Lambda^2+2\mu^2 I)} \begin{bmatrix} \mathscr{U}_1 \\ \mathscr{U}_2 \end{bmatrix}. \tag{2.24}$$

In a similar way, we can also obtain the linear eigenfunction of the adjoint Lax-pair representation (2.7) with the potentials given by (2.20). The constant matrices P_1 and P_2 play no essential role in the Bäcklund–Darboux transformations, i.e., the order of applying a Bäcklund–Darboux transformation and a constant linear transformation using P_1 and P_2 is irrelevant. Thus, in the following, we usually drop P_1 and P_2, keeping in mind the invariance of the matrix NLS system under the action of the symmetry group.

To express the matrix exponential explicitly, we need to diagonalize the constant matrix $\boldsymbol{Z}(\mu)$; throughout this paper, we do not separately discuss the non-generic case where $\boldsymbol{Z}(\mu)$ is non-diagonalizable (see, *e.g.*, [37]). For a general choice of μ, Γ, A and B, this requires to solve the characteristic equation for $\boldsymbol{Z}(\mu)$ of degree $l+m$. Thus, even for the simple case of the two-component vector NLS equation, we generally need to resort to Cardano's formula [34, 38], which is too complicated for practical purposes. However, as we will see below, with a suitable re-parametrization of μ, Γ, A and B, it is possible to compute the matrix exponential explicitly without using Cardano's formula.

3 Dark-soliton solutions

In this section, we apply the elementary Bäcklund–Darboux transformation defined in Proposition 2.1 to the matrix NLS system (2.1), which includes the vector NLS system as a special case. Taking a plane-wave solution as the seed solution, we obtain dark-soliton solutions of the multicomponent NLS equations with a self-defocusing or mixed focusing-defocusing nonlinearity. The Hermitian conjugation reduction between the transformed potentials $\widetilde{Q}$ and $\widetilde{R}$ can be realized through a suitable choice of the parameters in the solution. In particular, the Bäcklund parameter μ, which has its origin in the spectral parameter ζ in the Lax-pair representation, is restricted to be real.

3.1 Vector dark soliton without internal degrees of freedom

We first consider the nonreduced vector NLS system, which can be obtained from (2.1) by setting $l = 1$, $Q = \boldsymbol{q}$ and $R = \boldsymbol{r}^T$, i.e.

$$\begin{cases} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} - 2\langle \boldsymbol{q}, \boldsymbol{r}\rangle \boldsymbol{q} = \boldsymbol{0}, & (3.1\mathrm{a}) \\ \mathrm{i}\boldsymbol{r}_t - \boldsymbol{r}_{xx} + 2\langle \boldsymbol{r}, \boldsymbol{q}\rangle \boldsymbol{r} = \boldsymbol{0}. & (3.1\mathrm{b}) \end{cases}$$

Here, $\boldsymbol{q} := (q_1, \ldots, q_m)$ and $\boldsymbol{r} := (r_1, \ldots, r_m)$ are m-component row vectors. Note that the simplest case of $m = 1$ corresponds to the scalar NLS system:

$$\begin{cases} \mathrm{i}q_t + q_{xx} - 2q^2 r = 0, & (3.2\mathrm{a}) \\ \mathrm{i}r_t - r_{xx} + 2r^2 q = 0. & (3.2\mathrm{b}) \end{cases}$$

A general plane-wave solution [21] of the vector NLS system (3.1) is (cf. (2.20))

$$q_j(x,t) = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left(\gamma_j^2 + 2\sum_{k=1}^m a_k b_k\right)t}, \qquad r_j(x,t) = b_j \mathrm{e}^{-\mathrm{i}\gamma_j x + \mathrm{i}\left(\gamma_j^2 + 2\sum_{k=1}^m a_k b_k\right)t}, \tag{3.3}$$

where $j = 1, 2, \ldots, m$. We consider the generic case where all γ_j are pairwise distinct. The matrix $\boldsymbol{Z}(\mu)$ (cf. (2.23)) in this case is given by

$$\boldsymbol{Z}(\mu) = \begin{bmatrix} -2\mu & a_1 & \cdots & a_m \\ -b_1 & \gamma_1 & & O \\ \vdots & & \ddots & \\ -b_m & O & & \gamma_m \end{bmatrix}. \tag{3.4}$$

The characteristic equation (up to an overall sign) reads

$$\det(\lambda I - \boldsymbol{Z}) = (\lambda + 2\mu)\prod_{j=1}^{m}(\lambda - \gamma_j) + \sum_{k=1}^{m} a_k b_k \prod_{\substack{j=1\\ j\neq k}}^{m}(\lambda - \gamma_j) = 0, \tag{3.5}$$

which can be rewritten as (cf. [22])

$$\lambda + \sum_{k=1}^{m} \frac{a_k b_k}{\lambda - \gamma_k} = -2\mu. \tag{3.6}$$

For each eigenvalue λ that solves (3.6), the corresponding eigenvector is given up to an overall constant as

$$\boldsymbol{Z}(\mu)\begin{bmatrix} 1 \\ -\frac{b_1}{\lambda-\gamma_1} \\ \vdots \\ -\frac{b_m}{\lambda-\gamma_m} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -\frac{b_1}{\lambda-\gamma_1} \\ \vdots \\ -\frac{b_m}{\lambda-\gamma_m} \end{bmatrix}.$$

To obtain dark-soliton solutions satisfying a complex conjugation reduction between $\boldsymbol{q}$ and $\boldsymbol{r}$, we assume that the parameters γ_k, $a_k b_k$ and μ are all real. Then, considering the behavior of the left-hand side of (3.6) as a function of λ, we can identify conditions for the algebraic equation (3.6) in λ to have a pair of complex conjugate roots. In particular, at least one of the $a_k b_k$ must be positive. Then, among the $m+1$ solutions of (3.6), we express the pair of complex conjugate roots as $\lambda = \xi \pm \mathrm{i}\eta$ with $\eta > 0$ to obtain

$$\xi \pm \mathrm{i}\eta + \sum_{k=1}^{m} \frac{a_k b_k}{\xi \pm \mathrm{i}\eta - \gamma_k} = -2\mu. \tag{3.7}$$

This can be rewritten as

$$\xi + \sum_{k=1}^{m} \frac{(\xi - \gamma_k) a_k b_k}{(\xi - \gamma_k)^2 + \eta^2} = -2\mu, \qquad \sum_{k=1}^{m} \frac{a_k b_k}{(\xi - \gamma_k)^2 + \eta^2} = 1. \tag{3.8}$$

Using the second equation, it is in principle possible to determine $\eta\,(>0)$ from ξ, γ_k and $a_k b_k$ by solving an algebraic equation of degree m for η^2. Subsequently, the first equation can be used to determine μ from ξ, γ_k and $a_k b_k$. However, in practice, this procedure is not effective for $m \geq 3$, because we need to use Cardano's formula even for $m = 3$.

3.1.1 Scalar NLS equation

It would be instructive to study the scalar NLS system by setting $m = 1$, before considering the multicomponent case. From (3.8), we obtain

$$2\xi - \gamma_1 = -2\mu, \quad a_1 b_1 = (\xi - \gamma_1)^2 + \eta^2. \tag{3.9}$$

Thus, we have $\eta = \sqrt{a_1 b_1 - (\xi - \gamma_1)^2}$ with the condition $a_1 b_1 > (\xi - \gamma_1)^2$ and

$$\boldsymbol{Z} = \begin{bmatrix} 2\xi - \gamma_1 & a_1 \\ -b_1 & \gamma_1 \end{bmatrix}.$$

We choose the eigenvectors corresponding to the eigenvalues $\xi \pm \mathrm{i}\eta$ as

$$\boldsymbol{Z} \begin{bmatrix} a_1 \\ -\xi \pm \mathrm{i}\eta + \gamma_1 \end{bmatrix} = (\xi \pm \mathrm{i}\eta) \begin{bmatrix} a_1 \\ -\xi \pm \mathrm{i}\eta + \gamma_1 \end{bmatrix}.$$

The general solution (2.23) of the linear problem (2.22) in the scalar NLS case can be written explicitly as

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} &= c_1 \mathrm{e}^{\mathrm{i}x\left(\frac{\gamma}{2} + \mathrm{i}\eta\right) + \mathrm{i}t\left(\xi^2 - 2\mathrm{i}\xi\eta + \eta^2 - 2\xi\gamma_1 + \frac{\gamma_1^2}{2}\right)} \begin{bmatrix} a_1 \\ -\xi + \mathrm{i}\eta + \gamma_1 \end{bmatrix} \\ &\quad + c_2 \mathrm{e}^{\mathrm{i}x\left(\frac{\gamma}{2} - \mathrm{i}\eta\right) + \mathrm{i}t\left(\xi^2 + 2\mathrm{i}\xi\eta + \eta^2 - 2\xi\gamma_1 + \frac{\gamma_1^2}{2}\right)} \begin{bmatrix} a_1 \\ -\xi - \mathrm{i}\eta + \gamma_1 \end{bmatrix}, \end{aligned}$$

where c_1 and c_2 are arbitrary constants; they should be nonzero so that a nontrivial solution can be generated by applying the elementary Bäcklund–Darboux transformation. In view of (2.21) and (3.9), we obtain

$$\begin{aligned} \frac{\Psi_2}{\Psi_1} &= \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma_1 x + \mathrm{i}(\gamma_1^2 + 2a_1 b_1)t} \frac{\boldsymbol{\Psi}_2}{\boldsymbol{\Psi}_1} \\ &= \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma_1 x + \mathrm{i}\left[\gamma_1^2 + 2(\xi - \gamma_1)^2 + 2\eta^2\right]t} \frac{c_1(-\xi + \mathrm{i}\eta + \gamma_1)\mathrm{e}^{-\eta x + 2\xi\eta t} + c_2(-\xi - \mathrm{i}\eta + \gamma_1)\mathrm{e}^{\eta x - 2\xi\eta t}}{a_1\left(c_1 \mathrm{e}^{-\eta x + 2\xi\eta t} + c_2 \mathrm{e}^{\eta x - 2\xi\eta t}\right)}. \end{aligned}$$

Then, formulas (2.13) provides the solution of the scalar NLS system (3.2) as

$$\begin{aligned} r &= \frac{\mathrm{i}}{a_1} \mathrm{e}^{-\mathrm{i}\gamma_1 x + \mathrm{i}\left[\gamma_1^2 + 2(\xi - \gamma_1)^2 + 2\eta^2\right]t} \frac{c_1(-\xi + \mathrm{i}\eta + \gamma_1)\mathrm{e}^{-\eta x + 2\xi\eta t} + c_2(-\xi - \mathrm{i}\eta + \gamma_1)\mathrm{e}^{\eta x - 2\xi\eta t}}{c_1 \mathrm{e}^{-\eta x + 2\xi\eta t} + c_2 \mathrm{e}^{\eta x - 2\xi\eta t}}, \\ q &= \mathrm{i}a_1 \mathrm{e}^{\mathrm{i}\gamma_1 x - \mathrm{i}\left[\gamma_1^2 + 2(\xi - \gamma_1)^2 + 2\eta^2\right]t} \frac{c_1(\xi + \mathrm{i}\eta - \gamma_1)\mathrm{e}^{-\eta x + 2\xi\eta t} + c_2(\xi - \mathrm{i}\eta - \gamma_1)\mathrm{e}^{\eta x - 2\xi\eta t}}{c_1 \mathrm{e}^{-\eta x + 2\xi\eta t} + c_2 \mathrm{e}^{\eta x - 2\xi\eta t}}, \end{aligned}$$

where we omit the tilde to denote the application of the Bäcklund–Darboux transformation. Thus, if we re-parametrize some parameters as

$$\mathrm{i}a_1 =: \mathrm{e}^{\mathrm{i}\theta}, \quad \frac{c_2}{c_1} =: \mathrm{e}^{2\delta}$$

with real θ and δ, we can realize the complex conjugation reduction $r = q^*$ and obtain the one-dark soliton solution of the self-defocusing NLS equation $\mathrm{i}q_t + q_{xx} - 2|q|^2 q = 0$ in the form [5, 13–15]:

$$
\begin{aligned}
q(x,t) &= \mathrm{e}^{\mathrm{i}\gamma_1 x - \mathrm{i}\left[\gamma_1^2 + 2(\xi-\gamma_1)^2 + 2\eta^2\right]t + \mathrm{i}\theta} \frac{(\xi + \mathrm{i}\eta - \gamma_1)\mathrm{e}^{-\eta(x-2\xi t)-\delta} + (\xi - \mathrm{i}\eta - \gamma_1)\mathrm{e}^{\eta(x-2\xi t)+\delta}}{\mathrm{e}^{-\eta(x-2\xi t)-\delta} + \mathrm{e}^{\eta(x-2\xi t)+\delta}} \\
&= \mathrm{e}^{\mathrm{i}\gamma_1 x - \mathrm{i}\left[\gamma_1^2 + 2(\xi-\gamma_1)^2 + 2\eta^2\right]t + \mathrm{i}\theta} \left\{(\xi - \gamma_1) - \mathrm{i}\eta \tanh\left[\eta(x - 2\xi t) + \delta\right]\right\}.
\end{aligned}
$$

This solution represents a simple dip in the background plane wave and the asymptotic value at infinity is given as $\lim_{x\to\pm\infty} |q(x,t)| = \sqrt{(\xi-\gamma_1)^2 + \eta^2}$. The elementary Bäcklund–Darboux transformation defined in Proposition 2.1 does not retain the complex conjugation reduction in general; however, if it is applied to a plane-wave solution in combination with a simple rescaling of the dependent variables, we can realize the complex conjugation reduction in both the old and new pair of variables.

Using Proposition 2.1, we can confirm that the one-dark soliton solution is associated with a bound state of the Lax-pair representation (2.6) at $\zeta = \mu$, i.e., when the spectral parameter coincides with the parameter of the applied Bäcklund–Darboux transformation. Indeed, the gauge transformation (2.14) provides a generic linear eigenfunction at $\zeta = \mu$ for the transformed potentials as

$$
\begin{bmatrix} \widetilde{\Psi}_1 \\ \widetilde{\Psi}_2 \end{bmatrix} \propto \frac{\begin{vmatrix} \Psi_1^{(1)} & \Psi_1 \\ \Psi_2^{(1)} & \Psi_2 \end{vmatrix}}{\Psi_1^{(1)}} \begin{bmatrix} -q \\ 1 \end{bmatrix},
$$

up to an overall constant. Here, the superscript $^{(1)}$ means the linear eigenfunction used to define the elementary Bäcklund–Darboux transformation (2.13). In the above expression, the determinant in the numerator is the Wronskian of two linearly independent eigenfunctions, which is a nonzero constant; $q(x,t)$ in the seed solution is a plane wave and $1/\Psi_1^{(1)}$ is nonsingular everywhere and decays exponentially fast as $x \to \pm\infty$. Thus, the application of the elementary Bäcklund–Darboux transformation indeed generates a bound state of the associated linear problem. More generally, the elementary Bäcklund–Darboux transformation increases the number of bound states by one, by adding a new bound state with a real eigenvalue μ, when it is applied to a seed solution that does not have a bound state at $\zeta = \mu$. Thus, one can intuitively think of the elementary Bäcklund–Darboux transformation as a classical analog of the creation operator.

3.1.2 Vector NLS equation

Let us derive a one-dark soliton solution of the vector NLS equation (2.4) in the general m-component case. Actually, using the elementary Bäcklund–Darboux transformation, we can construct more general solutions of the nonreduced vector NLS system (3.1), which are not compatible with the complex conjugation reduction; however, such solutions apparently have no physical significance, so we do not discuss them in this paper. Thus, among the general solution (2.23) of the linear problem (2.22) in the vector NLS case, we consider a particular solution,

$$\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} = c_1 \mathrm{e}^{\mathrm{i}x(\xi+\mathrm{i}\eta+\mu)+\mathrm{i}t\left[-(\xi+\mathrm{i}\eta)^2+2\mu^2\right]} \begin{bmatrix} 1 \\ -\frac{b_1}{\xi+\mathrm{i}\eta-\gamma_1} \\ \vdots \\ -\frac{b_m}{\xi+\mathrm{i}\eta-\gamma_m} \end{bmatrix} + c_2 \mathrm{e}^{\mathrm{i}x(\xi-\mathrm{i}\eta+\mu)+\mathrm{i}t\left[-(\xi-\mathrm{i}\eta)^2+2\mu^2\right]} \begin{bmatrix} 1 \\ -\frac{b_1}{\xi-\mathrm{i}\eta-\gamma_1} \\ \vdots \\ -\frac{b_m}{\xi-\mathrm{i}\eta-\gamma_m} \end{bmatrix}.$$

Here, c_1 and c_2 are arbitrary constants, which should be nonzero so that the above solution can provide a nontrivial solution of the vector NLS system. Recall that the seed solution of the vector NLS system (3.1) is the general plane-wave solution (3.3) and the parameters satisfy the relations (3.8).

Then, by applying the elementary Bäcklund–Darboux transformation (2.13), we obtain a new solution of the vector NLS system (3.1) as

$$\begin{aligned} \widetilde{\boldsymbol{r}}^T &= \mathrm{i}\mathrm{e}^{-\mathrm{i}x\Gamma+\mathrm{i}t\Gamma^2} \boldsymbol{\Psi}_2 \boldsymbol{\Psi}_1^{-1} \mathrm{e}^{2\mathrm{i}t\sum_{k=1}^m a_k b_k} \\ &= -\mathrm{i}\frac{\mathrm{e}^{2\mathrm{i}t\sum_{k=1}^m a_k b_k}}{c_1\mathrm{e}^{-\eta x+2\xi\eta t}+c_2\mathrm{e}^{\eta x-2\xi\eta t}} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\gamma_1 x+\mathrm{i}\gamma_1^2 t}\left(c_1\frac{b_1}{\xi+\mathrm{i}\eta-\gamma_1}\mathrm{e}^{-\eta x+2\xi\eta t}+c_2\frac{b_1}{\xi-\mathrm{i}\eta-\gamma_1}\mathrm{e}^{\eta x-2\xi\eta t}\right) \\ \vdots \\ \mathrm{e}^{-\mathrm{i}\gamma_m x+\mathrm{i}\gamma_m^2 t}\left(c_1\frac{b_m}{\xi+\mathrm{i}\eta-\gamma_m}\mathrm{e}^{-\eta x+2\xi\eta t}+c_2\frac{b_m}{\xi-\mathrm{i}\eta-\gamma_m}\mathrm{e}^{\eta x-2\xi\eta t}\right) \end{bmatrix}, \end{aligned}$$

where $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m)$, and

$$\begin{aligned} \widetilde{\boldsymbol{q}} &= -\boldsymbol{q}_x - 2\mathrm{i}\mu\boldsymbol{q} + \langle \boldsymbol{q}, \widetilde{\boldsymbol{r}}\rangle \boldsymbol{q} \\ &= (\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_m), \end{aligned}$$

with

$$\widetilde{q}_j = \mathrm{i}a_j \mathrm{e}^{\mathrm{i}\gamma_j x-\mathrm{i}\left(\gamma_j^2+2\sum_{k=1}^m a_k b_k\right)t} \frac{c_1(\xi+\mathrm{i}\eta-\gamma_j)\mathrm{e}^{-\eta x+2\xi\eta t}+c_2(\xi-\mathrm{i}\eta-\gamma_j)\mathrm{e}^{\eta x-2\xi\eta t}}{c_1\mathrm{e}^{-\eta x+2\xi\eta t}+c_2\mathrm{e}^{\eta x-2\xi\eta t}}, \quad j = 1, 2, \ldots, m.$$

Here, the Bäcklund parameter μ has been eliminated with the aid of (3.7). The complex conjugation reduction $\widetilde{r}_j = \sigma_j \widetilde{q}_j^*$ can be realized by setting

$$b_j = \sigma_j \left[(\xi - \gamma_j)^2 + \eta^2\right] a_j^*,$$
$$\frac{c_2}{c_1} = \mathrm{e}^{2\delta}, \quad \delta \in \mathbb{R},$$

where $\sigma_j = +1$ or -1. Thus, omitting the tilde, we obtain a one-dark soliton solution of the vector NLS equation (2.4) as

$$q_j(x,t) = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left\{\gamma_j^2 + 2\sum_{k=1}^m \sigma_k |a_k|^2 \left[(\xi-\gamma_j)^2+\eta^2\right]\right\} t} \left\{\mathrm{i}(\xi - \gamma_j) + \eta \tanh\left[\eta(x - 2\xi t) + \delta\right]\right\}, \quad j = 1, 2, \ldots, m. \tag{3.10}$$

Note that the second relation in (3.8) reduces to the normalization condition:

$$\sum_{j=1}^m \sigma_j |a_j|^2 = 1.$$

This implies that at least one of the σ_j must be positive, i.e., there must exist one or more defocusing components in the vector variable $\boldsymbol{q}(x,t)$. This is consistent with the previous observation that at least one of the $a_k b_k$ must be positive for (3.6) to have a pair of complex conjugate roots.

The vector dark soliton (3.10) has an essentially time-independent shape and contains no free parameters corresponding to the internal degrees of freedom; in the two-component case ($m = 2$), this solution has been extensively studied in the literature [20–22, 81, 82]. In section 4, we will derive a more general vector dark-soliton solution that can exist for $m \geq 3$ and admits the internal degrees of freedom.

3.2 Matrix dark soliton

In this subsection, we construct new nontrivial dark-soliton solutions of the defocusing matrix NLS equation (2.5) with $\sigma = +1$. Before focusing on the square matrix case $l = m$, we consider the more general case of $l \leq m$ and identify the conditions under which the new potentials generated by the elementary Bäcklund–Darboux transformation admit the Hermitian conjugation reduction.

By applying the elementary Bäcklund–Darboux transformation defined in Proposition 2.1 to the plane-wave solution (2.20) with P_1 and P_2 dropped,

we obtain the transformed potentials in the form:

$$\begin{aligned}\widetilde{R} &= \Psi_2\Psi_1^{-1}\\ &= \mathrm{i}\mathrm{e}^{-\mathrm{i}x\Gamma+\mathrm{i}t\Gamma^2}\,\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}\mathrm{e}^{2\mathrm{i}tAB},\end{aligned} \tag{3.11a}$$

$$\begin{aligned}\widetilde{Q} &= -Q_x - 2\mathrm{i}\mu Q + Q\widetilde{R}Q\\ &= \mathrm{i}\mathrm{e}^{-2\mathrm{i}tAB}A\left(\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}A - \Gamma - 2\mu I_m\right)\mathrm{e}^{\mathrm{i}x\Gamma-\mathrm{i}t\Gamma^2}.\end{aligned} \tag{3.11b}$$

Here, the $l\times l$ matrix $\boldsymbol{\Psi}_1$ and the $m\times l$ matrix $\boldsymbol{\Psi}_2$ satisfy the linear problem (2.22). From (2.22a), we obtain the matrix Riccati equation for $\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}$:

$$-\mathrm{i}\left(\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}\right)_x = -B + 2\mu\,\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1} + \Gamma\,\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1} - \boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}A\,\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}. \tag{3.12}$$

To realize the Hermitian conjugation reduction $\widetilde{R} = \widetilde{Q}^\dagger$ in (3.11), we require that

$$\Gamma^\dagger = \Gamma, \qquad (AB)^\dagger = AB,$$

and

$$\left(\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}\right)^\dagger = -A\left(\boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1}A - \Gamma - 2\mu I\right). \tag{3.13}$$

A direct calculation shows that the matrix Riccati equation (3.12) is consistent with the reduction (3.13) if

$$AA^\dagger A = A, \qquad B = \Gamma\left(I_m - A^\dagger A\right)\Gamma A^\dagger + 2\mu\Gamma A^\dagger + A^\dagger AKA^\dagger, \qquad \boldsymbol{\Psi}_2\,\boldsymbol{\Psi}_1^{-1} = \Gamma A^\dagger + A^\dagger\boldsymbol{\Phi}, \tag{3.14}$$

where K is an $m\times m$ constant Hermitian matrix, $K^\dagger = K$, $\boldsymbol{\Phi}$ is an $l\times l$ (x,t)-dependent matrix, and the Bäcklund parameter μ is real. Note that for B given above, the relation $(AB)^\dagger = AB$ is automatically satisfied. With the aid of the singular value decomposition, the first relation in (3.14) implies that the singular values of A are either 0 or 1. To satisfy the last relation in (3.14), we set

$$\boldsymbol{\Psi}_2 = \Gamma A^\dagger\,\boldsymbol{\Phi}_1 + A^\dagger\,\boldsymbol{\Phi}_2, \qquad \boldsymbol{\Psi}_1 = \boldsymbol{\Phi}_1,$$

where $\boldsymbol{\Phi} = \boldsymbol{\Phi}_2\,\boldsymbol{\Phi}_1^{-1}$. Then, the linear problem (2.22) for $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ is satisfied if the $l\times l$ matrices $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ satisfy

$$\begin{bmatrix}\boldsymbol{\Phi}_1\\ \boldsymbol{\Phi}_2\end{bmatrix}_x = \mathrm{i}\begin{bmatrix}-\mu I_l + A\Gamma A^\dagger & AA^\dagger\\ -AKA^\dagger & \mu I_l\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}_1\\ \boldsymbol{\Phi}_2\end{bmatrix}, \tag{3.15a}$$

$$\begin{bmatrix}\boldsymbol{\Phi}_1\\ \boldsymbol{\Phi}_2\end{bmatrix}_t = \mathrm{i}\left(\partial_x^2 - 2\mathrm{i}\mu\partial_x + \mu^2\right)\begin{bmatrix}\boldsymbol{\Phi}_1\\ \boldsymbol{\Phi}_2\end{bmatrix}. \tag{3.15b}$$

Thus, the problem of computing the exponential of an $(l+m)\times(l+m)$ matrix is reduced to that of a $2l\times 2l$ matrix, which is a meaningful simplification for $l<m$. In particular, in the case of $l=1$ corresponding to the vector NLS equation considered in the previous subsection, we need only diagonalize a 2×2 matrix to construct a one-dark soliton solution.

Now, we consider the case of $l=m$ corresponding to the square matrix NLS equation and assume that A is a unitary matrix, i.e., $AA^\dagger = A^\dagger A = I_l$. Thus, the first relation in (3.14) is automatically satisfied and $B=(2\mu\Gamma+K)\,A^\dagger$, where $2\mu\Gamma+K$ is a constant Hermitian matrix. Actually, we can reduce the seed plane-wave solution (2.20) in this case to the simpler case of $A=I_l$ by redefining P_1. Thus, in the following, we simply set $A=I_l$ and $B=B^\dagger$. Then, it can be checked directly that (3.12) is indeed consistent with the reduction (3.13). Because $l=m$, it is not so meaningful to consider the linear problem (3.15) for $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$. Thus, we consider the original linear problem (2.22) for $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ in the form:

$$\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1 \\ \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2 \end{bmatrix}_x = \mathrm{i}\begin{bmatrix} -2\mu I_l & I_l \\ -B & \Gamma \end{bmatrix}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1 \\ \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2 \end{bmatrix}, \tag{3.16a}$$

$$\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1 \\ \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2 \end{bmatrix}_t = \mathrm{i}\partial_x^2\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1 \\ \mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2 \end{bmatrix}, \tag{3.16b}$$

where Γ is a real diagonal matrix: $\Gamma=\mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_l)$. Here,

$$\widetilde{R} = \mathrm{i}\mathrm{e}^{-\mathrm{i}x\Gamma+\mathrm{i}t\Gamma^2}\left(\mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2\right)\left(\mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1\right)^{-1}\mathrm{e}^{2\mathrm{i}tB}, \tag{3.17a}$$

$$\widetilde{Q} = \mathrm{i}\mathrm{e}^{-2\mathrm{i}tB}\left[\left(\mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2\right)\left(\mathrm{e}^{-\mathrm{i}\mu x-2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1\right)^{-1} - \Gamma - 2\mu I_l\right]\mathrm{e}^{\mathrm{i}x\Gamma-\mathrm{i}t\Gamma^2}. \tag{3.17b}$$

The characteristic polynomial of the square matrix in (3.16a) can be written (up to an overall constant) as

$$\det\begin{bmatrix} (\lambda+2\mu)I_l & -I_l \\ B & \lambda I_l-\Gamma \end{bmatrix} = \det\left[(\lambda+2\mu)\left(\lambda I_l-\Gamma\right)+B\right]. \tag{3.18}$$

Indeed, the $2l\times 2l$ eigenvalue problem,

$$\begin{bmatrix} -2\mu I_l & I_l \\ -B & \Gamma \end{bmatrix}\begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix} = \lambda\begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}, \tag{3.19}$$

can be rewritten as an $l\times l$ nonstandard eigenvalue problem:

$$\left[(\lambda+2\mu)\left(\lambda I_l-\Gamma\right)+B\right]\boldsymbol{f} = \boldsymbol{0}, \tag{3.20}$$

where $\boldsymbol{g} = (\lambda + 2\mu)\boldsymbol{f}$.

Because of the complexity of the computation in the general case, we first consider the simpler special case where B is a diagonal matrix. We set

$$B = \mathrm{diag}(b_1, b_2, \ldots, b_l), \quad b_j > 0,$$

and assume that each diagonal element of the matrix on the right-hand side of (3.18) can be factored as

$$(\lambda + 2\mu)(\lambda - \gamma_j) + b_j = [\lambda - (\xi_j + \mathrm{i}\eta_j)]\,[\lambda - (\xi_j - \mathrm{i}\eta_j)]\,,$$

which is equivalent to

$$\xi_j \pm \mathrm{i}\eta_j + \frac{b_j}{\xi_j \pm \mathrm{i}\eta_j - \gamma_j} = -2\mu.$$

Thus, we obtain

$$2\xi_j - \gamma_j = -2\mu, \quad b_j = (\xi_j - \gamma_j)^2 + \eta_j^2.$$

We consider that γ_j, $\eta_j (> 0)$ and μ are free real parameters, which determine ξ_j and b_j as

$$\xi_j = \frac{\gamma_j}{2} - \mu, \quad b_j = \left(\frac{\gamma_j}{2} + \mu\right)^2 + \eta_j^2.$$

Thus, the solution of the linear problem (3.16) can be written as

$$\begin{aligned}
&\left(\mathrm{e}^{-\mathrm{i}\mu x - 2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1\right)_{jk} = c_{jk}\mathrm{e}^{\mathrm{i}(\xi_j+\mathrm{i}\eta_j)x - \mathrm{i}(\xi_j+\mathrm{i}\eta_j)^2 t} + d_{jk}\mathrm{e}^{\mathrm{i}(\xi_j-\mathrm{i}\eta_j)x - \mathrm{i}(\xi_j-\mathrm{i}\eta_j)^2 t} \\
&= \mathrm{e}^{\mathrm{i}\left(\frac{\gamma_j}{2}-\mu\right)x - \mathrm{i}\left[\left(\frac{\gamma_j}{2}-\mu\right)^2 - \eta_j^2\right]t}\left\{c_{jk}\mathrm{e}^{-\eta_j[x-(\gamma_j-2\mu)t]} + d_{jk}\mathrm{e}^{\eta_j[x-(\gamma_j-2\mu)t]}\right\},
\end{aligned}$$

$$\begin{aligned}
&\left(\mathrm{e}^{-\mathrm{i}\mu x - 2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_2\right)_{jk} = (\xi_j + \mathrm{i}\eta_j + 2\mu)c_{jk}\mathrm{e}^{\mathrm{i}(\xi_j+\mathrm{i}\eta_j)x - \mathrm{i}(\xi_j+\mathrm{i}\eta_j)^2 t} + (\xi_j - \mathrm{i}\eta_j + 2\mu)d_{jk}\mathrm{e}^{\mathrm{i}(\xi_j-\mathrm{i}\eta_j)x - \mathrm{i}(\xi_j-\mathrm{i}\eta_j)^2 t} \\
&= \mathrm{e}^{\mathrm{i}\left(\frac{\gamma_j}{2}-\mu\right)x - \mathrm{i}\left[\left(\frac{\gamma_j}{2}-\mu\right)^2 - \eta_j^2\right]t}\left\{\left(\frac{\gamma_j}{2} + \mu + \mathrm{i}\eta_j\right)c_{jk}\mathrm{e}^{-\eta_j[x-(\gamma_j-2\mu)t]} + \left(\frac{\gamma_j}{2} + \mu - \mathrm{i}\eta_j\right)d_{jk}\mathrm{e}^{\eta_j[x-(\gamma_j-2\mu)t]}\right\},
\end{aligned}$$

where c_{jk} and d_{jk} are arbitrary constants, $1 \le j, k \le l$. Using (3.17) and omitting the tilde, we arrive at a one-dark soliton solution of the nonreduced matrix NLS system (2.1) in the form:

$$\begin{aligned}
Q = &-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}x\Gamma - \mathrm{i}t\left(\frac{3}{4}\Gamma^2 + \mu\Gamma + \mathcal{H}^2 + \mu^2 I_l\right)}\left\{\frac{1}{2}\Gamma + \mu I_l - \mathrm{i}\mathcal{H}\left[\mathrm{e}^{-x\mathcal{H} + t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{C} - \mathrm{e}^{x\mathcal{H} - t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{D}\right]\right. \\
&\left.\times\left[\mathrm{e}^{-x\mathcal{H} + t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{C} + \mathrm{e}^{x\mathcal{H} - t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{D}\right]^{-1}\right\}\mathrm{e}^{\mathrm{i}\frac{1}{2}x\Gamma - \mathrm{i}t\left(\frac{3}{4}\Gamma^2 + \mu\Gamma + \mathcal{H}^2 + \mu^2 I_l\right)},
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
R = &\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}x\Gamma + \mathrm{i}t\left(\frac{3}{4}\Gamma^2 + \mu\Gamma + \mathcal{H}^2 + \mu^2 I_l\right)}\left\{\frac{1}{2}\Gamma + \mu I_l + \mathrm{i}\mathcal{H}\left[\mathrm{e}^{-x\mathcal{H} + t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{C} - \mathrm{e}^{x\mathcal{H} - t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{D}\right]\right. \\
&\left.\times\left[\mathrm{e}^{-x\mathcal{H} + t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{C} + \mathrm{e}^{x\mathcal{H} - t\mathcal{H}(\Gamma - 2\mu I_l)}\mathcal{D}\right]^{-1}\right\}\mathrm{e}^{-\mathrm{i}\frac{1}{2}x\Gamma + \mathrm{i}t\left(\frac{3}{4}\Gamma^2 + \mu\Gamma + \mathcal{H}^2 + \mu^2 I_l\right)}.
\end{aligned}$$

Here, $\mathcal{H} := \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_l)$, $\mathcal{C} := (c_{jk})_{1\le j,k\le l}$ and $\mathcal{D} := (d_{jk})_{1\le j,k\le l}$. The Hermitian conjugation reduction $R = Q^\dagger$ can be realized by imposing the condition that the matrix $\mathcal{C}^\dagger\mathcal{H}\mathcal{D}$ is Hermitian:

$$\left(\mathcal{C}^\dagger\mathcal{H}\mathcal{D}\right)^\dagger = \mathcal{C}^\dagger\mathcal{H}\mathcal{D}.$$

Because the above soliton solution is invariant under the change $\mathcal{C} \to \mathcal{C}U$, $\mathcal{D} \to \mathcal{D}U$ for any nonsingular $l \times l$ matrix U, we assume without loss of generality that $\mathcal{C}^\dagger\mathcal{H}\mathcal{D}$ is a real diagonal matrix. A simple way to satisfy this constraint is to set the two matrices $\mathcal{C}$ and $\mathcal{D}$ as

$$\mathcal{C} = V \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_l \end{bmatrix}, \quad \mathcal{D} = \mathcal{H}^{-1}\left(V^{-1}\right)^\dagger \begin{bmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_l \end{bmatrix},$$

where V is an $l \times l$ nonsingular matrix and $\alpha_j, \beta_j \in \mathbb{R}$ $(j = 1, 2, \ldots, l)$.

To summarize, (3.21) provides a special one-dark soliton solution of the matrix NLS equation (2.5) in the self-defocusing case $\sigma = +1$; Γ and $\mathcal{H}$ are real diagonal matrices, μ is a real parameter and $\mathcal{C}^\dagger\mathcal{H}\mathcal{D}$ is a real diagonal matrix. In the more special case where Γ is a scalar matrix, i.e. $\Gamma = \gamma I_l$, (3.21) describes a conventional dark soliton moving with the velocity $\gamma - 2\mu$; even this special one-soliton solution is more general than the already known one-soliton solution derived using the inverse scattering method [27, 31].

Remark. Another matrix NLS equation,

$$\mathrm{i}Q_t + Q_{xx} - 2QQ^*Q = O,$$

can be obtained by imposing the complex conjugation reduction $R = Q^*$ on the matrix NLS system (2.1). A one-dark soliton solution of this equation is given by (3.21), where $\mathcal{C}$ and $\mathcal{D}$ are real matrices without any constraint on $\mathcal{C}^T\mathcal{H}\mathcal{D}$.

In the more general case where the matrix B in the linear problem (3.16) is a non-diagonal Hermitian matrix, we rewrite (3.16) as

$$\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix}_x = \mathrm{i}\begin{bmatrix} -\mu I_l + \frac{1}{2}\Gamma & I_l \\ -B + \left(\mu I_l + \frac{1}{2}\Gamma\right)^2 & -\mu I_l + \frac{1}{2}\Gamma \end{bmatrix}\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix}, \tag{3.22a}$$

$$\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix}_t = \mathrm{i}\partial_x^2 \begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix}. \tag{3.22b}$$

Here, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are $l \times l$ matrices defined as

$$\boldsymbol{S}_1 := \mathrm{e}^{-\mathrm{i}\mu x - 2\mathrm{i}\mu^2 t}\,\boldsymbol{\Psi}_1, \qquad \boldsymbol{S}_2 := \mathrm{e}^{-\mathrm{i}\mu x - 2\mathrm{i}\mu^2 t}\left(\boldsymbol{\Psi}_2 - \mu\,\boldsymbol{\Psi}_1 - \frac{1}{2}\Gamma\,\boldsymbol{\Psi}_1\right).$$

In terms of these $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, formula (3.17) can be rewritten in the symmetric form:

$$\widetilde{R} = \mathrm{i}\mathrm{e}^{-\mathrm{i}x\Gamma + \mathrm{i}t\Gamma^2}\left(\mu I_l + \frac{1}{2}\Gamma + \boldsymbol{S}_2\boldsymbol{S}_1^{-1}\right)\mathrm{e}^{2\mathrm{i}tB},$$

$$\widetilde{Q} = \mathrm{i}\mathrm{e}^{-2\mathrm{i}tB}\left(-\mu I_l - \frac{1}{2}\Gamma + \boldsymbol{S}_2\boldsymbol{S}_1^{-1}\right)\mathrm{e}^{\mathrm{i}x\Gamma - \mathrm{i}t\Gamma^2}.$$

Note that this solution is invariant under the change $\boldsymbol{S}_j \to \boldsymbol{S}_j U$ $(j = 1, 2)$ for any nonsingular $l \times l$ constant matrix U. To realize the Hermitian conjugation reduction $\widetilde{R} = \widetilde{Q}^\dagger$, we need only choose the solution of the linear problem (3.22) as

$$\begin{aligned}\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix} = \exp\Bigg\{&\mathrm{i}x\begin{bmatrix} -\mu I_l + \frac{1}{2}\Gamma & I_l \\ -B + \left(\mu I_l + \frac{1}{2}\Gamma\right)^2 & -\mu I_l + \frac{1}{2}\Gamma \end{bmatrix} \\ &-\mathrm{i}t\begin{bmatrix} -\mu I_l + \frac{1}{2}\Gamma & I_l \\ -B + \left(\mu I_l + \frac{1}{2}\Gamma\right)^2 & -\mu I_l + \frac{1}{2}\Gamma \end{bmatrix}^2\Bigg\}\begin{bmatrix} I_l \\ \mathrm{i}J \end{bmatrix},\end{aligned} \tag{3.23}$$

where J is an $l \times l$ constant Hermitian matrix. Indeed, because

$$\begin{aligned}\begin{bmatrix} \boldsymbol{S}_2 \\ \boldsymbol{S}_1 \end{bmatrix} &= \begin{bmatrix} O & I_l \\ I_l & O \end{bmatrix}\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix} \\ &= \exp\Bigg\{\mathrm{i}x\begin{bmatrix} -\mu I_l + \frac{1}{2}\Gamma & -B + \left(\mu I_l + \frac{1}{2}\Gamma\right)^2 \\ I_l & -\mu I_l + \frac{1}{2}\Gamma \end{bmatrix} \\ &\qquad -\mathrm{i}t\begin{bmatrix} -\mu I_l + \frac{1}{2}\Gamma & -B + \left(\mu I_l + \frac{1}{2}\Gamma\right)^2 \\ I_l & -\mu I_l + \frac{1}{2}\Gamma \end{bmatrix}^2\Bigg\}\begin{bmatrix} O & I_l \\ I_l & O \end{bmatrix}\begin{bmatrix} I_l \\ \mathrm{i}J \end{bmatrix},\end{aligned}$$

we obtain the relation:

$$\begin{bmatrix} \boldsymbol{S}_2^\dagger & \boldsymbol{S}_1^\dagger \end{bmatrix}\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix} = \begin{bmatrix} -\mathrm{i}J & I_l \end{bmatrix}\begin{bmatrix} I_l \\ \mathrm{i}J \end{bmatrix} = O.$$

This implies that $\left(\boldsymbol{S}_2\boldsymbol{S}_1^{-1}\right)^\dagger = -\boldsymbol{S}_2\boldsymbol{S}_1^{-1}$, which indeed guarantees the relation $\widetilde{R} = \widetilde{Q}^\dagger$.

To summarize, a general one-dark soliton solution of the matrix NLS equation (2.5) in the self-defocusing case $\sigma = +1$ can be expressed, up to unitary transformations, as

$$Q = \mathrm{e}^{-2\mathrm{i}tB} \left(\mu I_l + \frac{1}{2}\Gamma - \boldsymbol{S}_2 \boldsymbol{S}_1^{-1} \right) \mathrm{e}^{\mathrm{i}x\Gamma - \mathrm{i}t\Gamma^2}. \tag{3.24}$$

Here, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are given using the matrix exponential as (3.23); B and J are Hermitian matrices, Γ is a real diagonal matrix and μ is a real parameter.

To express the above solution more explicitly without using the matrix exponential as in (3.23), we need to re-parametrize the matrix B. We briefly consider the simplest nontrivial case of a 2×2 Hermitian matrix B,

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12}^* & b_{22} \end{bmatrix}, \tag{3.25}$$

where $b_{11}, b_{22} \in \mathbb{R}$ and $b_{12} \in \mathbb{C} \setminus \{0\}$. Note that the eigenvalue problem (3.19) can be rewritten as

$$\begin{bmatrix} -\mu I_l + \frac{1}{2}\Gamma & I_l \\ -B + \left(\mu I_l + \frac{1}{2}\Gamma\right)^2 & -\mu I_l + \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} - \mu \boldsymbol{f} - \frac{1}{2}\Gamma \boldsymbol{f} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} - \mu \boldsymbol{f} - \frac{1}{2}\Gamma \boldsymbol{f} \end{bmatrix}.$$

Thus, this also implies the relation $\boldsymbol{g} = (\lambda + 2\mu)\boldsymbol{f}$ and the nonstandard eigenvalue problem (3.20). We assume that the relevant characteristic polynomial (3.18) can be factored as

$$\begin{aligned} &\det \begin{bmatrix} (\lambda + 2\mu)(\lambda - \gamma_1) + b_{11} & b_{12} \\ b_{12}^* & (\lambda + 2\mu)(\lambda - \gamma_2) + b_{22} \end{bmatrix} \\ &= \left[\lambda^2 + (2\mu - \gamma_1)\lambda + b_{11} - 2\mu\gamma_1\right] \left[\lambda^2 + (2\mu - \gamma_2)\lambda + b_{22} - 2\mu\gamma_2\right] - |b_{12}|^2 \\ &= \left[\lambda - (\xi_1 + \mathrm{i}\eta_1)\right] \left[\lambda - (\xi_1 - \mathrm{i}\eta_1)\right] \left[\lambda - (\xi_2 + \mathrm{i}\eta_2)\right] \left[\lambda - (\xi_2 - \mathrm{i}\eta_2)\right]. \end{aligned} \tag{3.26}$$

We consider the generic case of $\gamma_1 \neq \gamma_2$, because the degenerate case $\gamma_1 = \gamma_2$ does not lead to an essentially new solution (cf. (3.21)). Noting that (3.26) is an identity in λ, we obtain

$$4\mu - \gamma_1 - \gamma_2 = -2\xi_1 - 2\xi_2, \tag{3.27a}$$

$$(2\mu - \gamma_1)(2\mu - \gamma_2) + b_{11} - 2\mu\gamma_1 + b_{22} - 2\mu\gamma_2 = \xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 + 4\xi_1\xi_2, \tag{3.27b}$$

$$(2\mu - \gamma_2)(b_{11} - 2\mu\gamma_1) + (2\mu - \gamma_1)(b_{22} - 2\mu\gamma_2) = -2\xi_1\left(\xi_2^2 + \eta_2^2\right) - 2\xi_2\left(\xi_1^2 + \eta_1^2\right), \tag{3.27c}$$

$$(b_{11} - 2\mu\gamma_1)(b_{22} - 2\mu\gamma_2) - |b_{12}|^2 = \left(\xi_1^2 + \eta_1^2\right)\left(\xi_2^2 + \eta_2^2\right). \tag{3.27d}$$

From (3.27a), we can express 2μ as

$$2\mu = \frac{1}{2}(\gamma_1 + \gamma_2) - \xi_1 - \xi_2. \tag{3.28}$$

Then, (3.27b) can be rewritten as

$$(b_{11} - 2\mu\gamma_1) + (b_{22} - 2\mu\gamma_2) = \eta_1^2 + \eta_2^2 + 2\xi_1\xi_2 + \frac{1}{4}(\gamma_1 - \gamma_2)^2. \tag{3.29}$$

Thus, combining (3.27c) and (3.29) and then using (3.27d), we obtain

$$\begin{aligned}
&b_{11} - 2\mu\gamma_1 \\
&= \frac{(\xi_1 - \xi_2)\,(\eta_1^2 - \eta_2^2)}{\gamma_1 - \gamma_2} + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 + 2\xi_1\xi_2\right) + \frac{1}{4}(\gamma_1 - \gamma_2)(\xi_1 + \xi_2) + \frac{1}{8}(\gamma_1 - \gamma_2)^2, \\
&b_{22} - 2\mu\gamma_2 \\
&= -\frac{(\xi_1 - \xi_2)\,(\eta_1^2 - \eta_2^2)}{\gamma_1 - \gamma_2} + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 + 2\xi_1\xi_2\right) - \frac{1}{4}(\gamma_1 - \gamma_2)(\xi_1 + \xi_2) + \frac{1}{8}(\gamma_1 - \gamma_2)^2, \\
&|b_{12}|^2 = \left[\frac{1}{2}\left(\eta_1^2 + \eta_2^2 + 2\xi_1\xi_2\right) + \frac{1}{8}(\gamma_1 - \gamma_2)^2\right]^2 - \left[\frac{(\xi_1 - \xi_2)\,(\eta_1^2 - \eta_2^2)}{\gamma_1 - \gamma_2} + \frac{1}{4}(\gamma_1 - \gamma_2)(\xi_1 + \xi_2)\right]^2 \\
&\qquad - \left(\xi_1^2 + \eta_1^2\right)\left(\xi_2^2 + \eta_2^2\right).
\end{aligned} \tag{3.30}$$

Further using (3.28), the matrix B in (3.25) can be determined from ξ_j, η_j and γ_j $(j = 1, 2)$ through the relations

$$\begin{aligned}
b_{11} = {} & \frac{(\xi_1 - \xi_2)\,(\eta_1^2 - \eta_2^2)}{\gamma_1 - \gamma_2} + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 + 2\xi_1\xi_2\right) - \frac{1}{4}(3\gamma_1 + \gamma_2)(\xi_1 + \xi_2) \\
& + \frac{1}{8}\left(5\gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2\right),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
b_{22} = {} & -\frac{(\xi_1 - \xi_2)\,(\eta_1^2 - \eta_2^2)}{\gamma_1 - \gamma_2} + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 + 2\xi_1\xi_2\right) - \frac{1}{4}(\gamma_1 + 3\gamma_2)(\xi_1 + \xi_2) \\
& + \frac{1}{8}\left(\gamma_1^2 + 2\gamma_1\gamma_2 + 5\gamma_2^2\right),
\end{aligned} \tag{3.32}$$

and (3.30). Note that the right-hand side of (3.30) must be positive and we can freely choose $\arg b_{12}$.

From the above results, the solution (3.23) of the linear problem (3.22)

can be written explicitly as

$$\begin{bmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{bmatrix} = W \begin{bmatrix} \mathrm{e}^{\mathrm{i}(\xi_1+\mathrm{i}\eta_1)x-\mathrm{i}(\xi_1+\mathrm{i}\eta_1)^2 t} & & & \\ & \mathrm{e}^{\mathrm{i}(\xi_1-\mathrm{i}\eta_1)x-\mathrm{i}(\xi_1-\mathrm{i}\eta_1)^2 t} & & \\ & & \mathrm{e}^{\mathrm{i}(\xi_2+\mathrm{i}\eta_2)x-\mathrm{i}(\xi_2+\mathrm{i}\eta_2)^2 t} & \\ & & & \mathrm{e}^{\mathrm{i}(\xi_2-\mathrm{i}\eta_2)x-\mathrm{i}(\xi_2-\mathrm{i}\eta_2)^2 t} \end{bmatrix}$$
$$\times W^{-1} \begin{bmatrix} I_l \\ \mathrm{i}J \end{bmatrix}.$$

Here, the 4×4 constant matrix W is defined as

$$W := \begin{bmatrix} \boldsymbol{v}(\xi_1 + \mathrm{i}\eta_1) & \boldsymbol{v}(\xi_1 - \mathrm{i}\eta_1) & \boldsymbol{v}(\xi_2 + \mathrm{i}\eta_2) & \boldsymbol{v}(\xi_2 - \mathrm{i}\eta_2) \end{bmatrix},$$

using the four-component column vector $\boldsymbol{v}(\lambda)$ given as

$$\boldsymbol{v}(\lambda) := \begin{bmatrix} -b_{12} \\ (\lambda + 2\mu)(\lambda - \gamma_1) + b_{11} \\ -\left(\lambda + \mu - \frac{1}{2}\gamma_1\right) b_{12} \\ \left(\lambda + \mu - \frac{1}{2}\gamma_2\right) \left[(\lambda + 2\mu)(\lambda - \gamma_1) + b_{11}\right] \end{bmatrix}.$$

Incidentally, we can also compute the matrix exponential $\mathrm{e}^{-2\mathrm{i}tB}$ in (3.24) explicitly, because B is a 2×2 Hermitian matrix.

Thus, in contrast to the one-bright soliton solution of the self-focusing matrix NLS equation, the general one-dark soliton solution (3.24) in the self-defocusing case is hightly nontrivial and interesting. That is, it is far from the straightforward matrix generalization of the scalar dark-soliton solution as given by

$$Q(x,t) = U_1 \begin{bmatrix} f_1(x,t) & & & \\ & f_2(x,t) & & \\ & & \ddots & \\ & & & f_l(x,t) \end{bmatrix} U_2,$$

where U_1 and U_2 are constant unitary matrices and $f_j(x,t)$ are either a plane-wave solution or a scalar dark-soliton solution of the self-defocusing NLS equation.

4 Vector dark soliton with internal degrees of freedom

In this section, we apply the limiting case of the binary Bäcklund–Darboux transformation defined in Proposition 2.4 and construct a vector dark-soliton solution with internal degrees of freedom.

We consider the plane-wave solution (3.3) of the vector NLS system (3.1) in the generic case where the wavenumbers γ_j are pairwise distinct. Let us denote the roots of the characteristic equation (3.5) of the matrix $\boldsymbol{Z}(\mu)$ as $\{\lambda_1, \lambda_2, \ldots, \lambda_{m+1}\}$; thus, it can be factored as

$$\begin{aligned}\det(\lambda I - \boldsymbol{Z}) &= (\lambda + 2\mu)\prod_{j=1}^{m}(\lambda - \gamma_j) + \sum_{k=1}^{m} a_k b_k \prod_{\substack{j=1\\ j\neq k}}^{m}(\lambda - \gamma_j) \\ &= \prod_{j=1}^{m+1}(\lambda - \lambda_j).\end{aligned}$$

Noting that this is an identity in λ, we obtain $m+1$ relations:

$$2\mu = \sum_{j=1}^{m}\gamma_j - \sum_{j=1}^{m+1}\lambda_j, \quad a_k b_k = \frac{\displaystyle\prod_{j=1}^{m+1}(\lambda_j - \gamma_k)}{\displaystyle\prod_{\substack{j=1\\ j\neq k}}^{m}(\gamma_j - \gamma_k)}, \quad k = 1, 2, \ldots, m. \tag{4.1}$$

For simplicity, we consider the three-component case ($m = 3$) and assume that

$$\gamma_1 < \gamma_2 < \gamma_3 \quad \text{or} \quad \gamma_1 > \gamma_2 > \gamma_3 \tag{4.2a}$$

and

$$a_1 b_1 > 0, \quad a_2 b_2 < 0, \quad a_3 b_3 > 0. \tag{4.2b}$$

The more general case of $m\,(\geq 4)$ components will be touched upon later. Then, equation (3.6), which is equivalent to the characteristic equation (3.5), can have two pairs of complex conjugate roots for an appropriate choice of the parameters. Thus, we parametrize the eigenvalues of the matrix $\boldsymbol{Z}(\mu)$ as

$$\lambda_1 = \xi_1 + \mathrm{i}\eta_1, \quad \lambda_2 = \xi_1 - \mathrm{i}\eta_1, \quad \lambda_3 = \xi_2 + \mathrm{i}\eta_2, \quad \lambda_4 = \xi_2 - \mathrm{i}\eta_2,$$

where $\xi_j \in \mathbb{R}$ and $\eta_j > 0$. From (4.1) with $m = 3$, we have

$$2\mu = \gamma_1 + \gamma_2 + \gamma_3 - 2\xi_1 - 2\xi_2, \tag{4.3}$$

$$a_k b_k = \frac{[(\xi_1 - \gamma_k)^2 + \eta_1^2]\,[(\xi_2 - \gamma_k)^2 + \eta_2^2]}{\displaystyle\prod_{\substack{j=1\\ j\neq k}}^{3}(\gamma_j - \gamma_k)}, \quad k = 1, 2, 3. \tag{4.4}$$

Note that (4.4) is indeed consistent with (4.2).

By setting $m = 3$ and

$$b_1 = a_1^*, \quad b_2 = -a_2^*, \quad b_3 = a_3^*, \tag{4.5}$$

the plane-wave solution (3.3) of the vector NLS system (3.1) reduces to the plane-wave solution of the vector NLS equation (2.4) with $\Sigma = \mathrm{diag}(1, -1, 1)$; note that (4.2b) is satisfied. Thus, the vector NLS equation with one focusing and two defocusing components,

$$\mathrm{i}q_{j,t} + q_{j,xx} - 2\left(|q_1|^2 - |q_2|^2 + |q_3|^2\right) q_j = 0, \quad j = 1, 2, 3, \tag{4.6}$$

has the plane-wave solution [21]:

$$q_j(x,t) = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left[\gamma_j^2 + 2\left(|a_1|^2 - |a_2|^2 + |a_3|^2\right)\right]t}, \quad j = 1, 2, 3. \tag{4.7}$$

We employ this plane-wave solution as the seed solution and apply a reduced version of Proposition 2.4; this defines a Bäcklund–Darboux transformation maintaining the Hermitian conjugation reduction between the row and column vector variables in the Lax pair.

Proposition 4.1. Consider the $(m+1) \times (m+1)$ Lax-pair representation for the m-component vector NLS equation (2.4) as given by

$$\begin{bmatrix} \phi \\ \boldsymbol{\psi} \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\mu & \boldsymbol{q} \\ \Sigma \boldsymbol{q}^\dagger & \mathrm{i}\mu I_m \end{bmatrix} \begin{bmatrix} \phi \\ \boldsymbol{\psi} \end{bmatrix}, \tag{4.8a}$$

$$\begin{bmatrix} \phi \\ \boldsymbol{\psi} \end{bmatrix}_t = \begin{bmatrix} -2\mathrm{i}\mu^2 - \mathrm{i}\boldsymbol{q}\Sigma\boldsymbol{q}^\dagger & 2\mu\boldsymbol{q} + \mathrm{i}\boldsymbol{q}_x \\ 2\mu\Sigma\boldsymbol{q}^\dagger - \mathrm{i}\Sigma\boldsymbol{q}_x^\dagger & 2\mathrm{i}\mu^2 I_m + \mathrm{i}\Sigma\boldsymbol{q}^\dagger\boldsymbol{q} \end{bmatrix} \begin{bmatrix} \phi \\ \boldsymbol{\psi} \end{bmatrix}. \tag{4.8b}$$

Here, the spectral parameter is set at a real value μ, ϕ is a scalar function, $\boldsymbol{\psi}$ is an m-component column-vector function and $\Sigma := \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$ is a diagonal matrix with entries $\sigma_j = +1$ or -1. The quadratic form $|\phi|^2 - \boldsymbol{\psi}^\dagger\Sigma\boldsymbol{\psi}$ is independent of x and t. Thus, if we consider a linear eigenfunction decaying rapidly as $x \to -\infty$:

$$\lim_{x\to-\infty} \begin{bmatrix} \phi(x) \\ \boldsymbol{\psi}(x) \end{bmatrix} = \mathbf{0},$$

this quantity vanishes identically, i.e.

$$|\phi|^2 = \boldsymbol{\psi}^\dagger\Sigma\boldsymbol{\psi}.$$

Then, the new potential $\boldsymbol{q}'$ defined by the formula

$$\boldsymbol{q}' := \boldsymbol{q} - \frac{\phi}{c_0 + \displaystyle\int_{-\infty}^{x} |\phi(y)|^2 \,\mathrm{d}y} \boldsymbol{\psi}^\dagger\Sigma,$$

where c_0 is a real constant, provides a new solution of the vector NLS equation (2.4). A linear eigenfunction associated with the new potential $\boldsymbol{q}'$ at the spectral parameter equal to μ is given by

$$\frac{1}{c_0 + \displaystyle\int_{-\infty}^{x} |\phi(y)|^2 \,\mathrm{d}y} \begin{bmatrix} \phi \\ \boldsymbol{\psi} \end{bmatrix},$$

which defines a bound state for $c_0 > 0$ and a proper choice of ϕ and $\boldsymbol{\psi}$. Note that if $c_0 > 0$, we can set $c_0 = 1$ by rescaling ϕ and $\boldsymbol{\psi}$ in the above formulas.

In the three-component case ($m = 3$), we choose a linear eigenfunction of the Lax-pair representation (4.8) with $\boldsymbol{q}$ given by (4.7) as (cf. subsections 2.6 and 3.1)

$$\begin{bmatrix} \phi \\ \boldsymbol{\psi} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-2\mathrm{i}\left(|a_1|^2-|a_2|^2+|a_3|^2\right)t} & & & \\ & \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma_1 x+\mathrm{i}\gamma_1^2 t} & & \\ & & \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma_2 x+\mathrm{i}\gamma_2^2 t} & \\ & & & \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma_3 x+\mathrm{i}\gamma_3^2 t} \end{bmatrix}$$

$$\times\, \mathrm{e}^{\mathrm{i}\mu x+2\mathrm{i}\mu^2 t} \left\{ c_1 \mathrm{e}^{\mathrm{i}(\xi_1-\mathrm{i}\eta_1)x-\mathrm{i}(\xi_1-\mathrm{i}\eta_1)^2 t} \begin{bmatrix} 1 \\ -\frac{a_1^*}{\xi_1-\mathrm{i}\eta_1-\gamma_1} \\ \frac{a_2^*}{\xi_1-\mathrm{i}\eta_1-\gamma_2} \\ -\frac{a_3^*}{\xi_1-\mathrm{i}\eta_1-\gamma_3} \end{bmatrix} + c_2 \mathrm{e}^{\mathrm{i}(\xi_2-\mathrm{i}\eta_2)x-\mathrm{i}(\xi_2-\mathrm{i}\eta_2)^2 t} \begin{bmatrix} 1 \\ -\frac{a_2^*}{\xi_2-\mathrm{i}\eta_2-\gamma_1} \\ \frac{a_2^*}{\xi_2-\mathrm{i}\eta_2-\gamma_2} \\ -\frac{a_3^*}{\xi_2-\mathrm{i}\eta_2-\gamma_3} \end{bmatrix} \right\}.$$

Here, the parameters satisfy the relations (4.3)–(4.5); c_1 and c_2 are complex constants. Because $\eta_j > 0$, this linear eigenfunction decays exponentially as $x \to -\infty$ and the quadratic form $|\phi|^2 - \boldsymbol{\psi}^\dagger \Sigma \boldsymbol{\psi}$ is identically equal to zero. Then, we introduce a real function f as

$$\begin{aligned} f &:= c_0 + \int_{-\infty}^{x} |\phi(y)|^2 \,\mathrm{d}y \\ &= c_0 + \int_{-\infty}^{x} \left| c_1 \mathrm{e}^{\mathrm{i}(\xi_1-\mathrm{i}\eta_1)y-\mathrm{i}(\xi_1-\mathrm{i}\eta_1)^2 t} + c_2 \mathrm{e}^{\mathrm{i}(\xi_2-\mathrm{i}\eta_2)y-\mathrm{i}(\xi_2-\mathrm{i}\eta_2)^2 t} \right|^2 \mathrm{d}y \\ &= c_0 + \frac{|c_1|^2}{2\eta_1} \mathrm{e}^{2\eta_1(x-2\xi_1 t)} + \frac{|c_2|^2}{2\eta_2} \mathrm{e}^{2\eta_2(x-2\xi_2 t)} \\ &\qquad + \mathrm{e}^{\eta_1(x-2\xi_1 t)+\eta_2(x-2\xi_2 t)} \left[\frac{c_1 c_2^*}{\eta_1+\eta_2+\mathrm{i}(\xi_1-\xi_2)} \mathrm{e}^{\mathrm{i}(\xi_1-\xi_2)x-\mathrm{i}(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2)t} \right. \\ &\qquad\qquad \left. + \frac{c_1^* c_2}{\eta_1+\eta_2-\mathrm{i}(\xi_1-\xi_2)} \mathrm{e}^{-\mathrm{i}(\xi_1-\xi_2)x+\mathrm{i}(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2)t} \right]. \end{aligned} \tag{4.9}$$

By assuming $c_0 > 0$, f becomes positive definite. We also introduce complex functions g_j $(j = 1, 2, 3)$ as

$$
\begin{aligned}
g_j := & \left\{ c_1 \mathrm{e}^{\eta_1(x-2\xi_1 t)+\mathrm{i}\xi_1 x-\mathrm{i}(\xi_1^2-\eta_1^2)t} + c_2 \mathrm{e}^{\eta_2(x-2\xi_2 t)+\mathrm{i}\xi_2 x-\mathrm{i}(\xi_2^2-\eta_2^2)t} \right\} \\
& \times \left\{ \frac{c_1^*}{\xi_1 + \mathrm{i}\eta_1 - \gamma_j} \mathrm{e}^{\eta_1(x-2\xi_1 t)-\mathrm{i}\xi_1 x+\mathrm{i}(\xi_1^2-\eta_1^2)t} + \frac{c_2^*}{\xi_2 + \mathrm{i}\eta_2 - \gamma_j} \mathrm{e}^{\eta_2(x-2\xi_2 t)-\mathrm{i}\xi_2 x+\mathrm{i}(\xi_2^2-\eta_2^2)t} \right\} \\
= & \frac{|c_1|^2}{\xi_1 + \mathrm{i}\eta_1 - \gamma_j} \mathrm{e}^{2\eta_1(x-2\xi_1 t)} + \frac{|c_2|^2}{\xi_2 + \mathrm{i}\eta_2 - \gamma_j} \mathrm{e}^{2\eta_2(x-2\xi_2 t)} \\
& + \mathrm{e}^{\eta_1(x-2\xi_1 t)+\eta_2(x-2\xi_2 t)} \left[\frac{c_1 c_2^*}{\xi_2 + \mathrm{i}\eta_2 - \gamma_j} \mathrm{e}^{\mathrm{i}(\xi_1-\xi_2)x-\mathrm{i}(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2)t} \right. \\
& \qquad \left. + \frac{c_1^* c_2}{\xi_1 + \mathrm{i}\eta_1 - \gamma_j} \mathrm{e}^{-\mathrm{i}(\xi_1-\xi_2)x+\mathrm{i}(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2)t} \right].
\end{aligned}
\tag{4.10}
$$

Thus, with the aid of Proposition 4.1, we obtain a one-dark soliton solution of the three-component NLS equation (4.6) in the form:

$$
q_j(x,t) = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left[\gamma_j^2 + 2\left(|a_1|^2 - |a_2|^2 + |a_3|^2\right)\right]t} \left\{ 1 - \mathrm{i}\frac{g_j}{f} \right\}, \quad j = 1, 2, 3. \tag{4.11}
$$

Here, the positive function f and the complex functions g_j are given by (4.9) and (4.10), respectively; the amplitudes of the background plane waves are determined as (cf. (4.4) and (4.5))

$$
\begin{aligned}
|a_1|^2 &= \frac{\left[(\xi_1 - \gamma_1)^2 + \eta_1^2\right]\left[(\xi_2 - \gamma_1)^2 + \eta_2^2\right]}{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_1)}, \\
|a_2|^2 &= \frac{\left[(\xi_1 - \gamma_2)^2 + \eta_1^2\right]\left[(\xi_2 - \gamma_2)^2 + \eta_2^2\right]}{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)}, \\
|a_3|^2 &= \frac{\left[(\xi_1 - \gamma_3)^2 + \eta_1^2\right]\left[(\xi_2 - \gamma_3)^2 + \eta_2^2\right]}{(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)}.
\end{aligned}
$$

When $c_1 = 0$ or $c_2 = 0$, we have a conventional dark soliton, which has essentially a time-independent shape and velocity (cf. (3.10)). For nonzero values of c_1 and c_2, the vector dark soliton (4.11) entails the internal degrees of freedom and exhibits complicated behavior; in fact, the soliton's shape and velocity can change in time, so the final state is generally different from the initial state. This phenomenon can be called a spontaneous soliton mutation.

In a similar manner, we can obtain a one-dark soliton solution of the general m-component NLS equation (2.4) in the same form as (4.11):

$$q_j(x,t) = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left(\gamma_j^2 + 2\sum_{k=1}^m \sigma_k |a_k|^2\right)t} \left\{1 - \mathrm{i}\frac{g_j}{f}\right\}, \quad j = 1, 2, \ldots, m. \tag{4.12}$$

In the general m-component case, f and g_j are given as

$$\begin{aligned}
f &:= c_0 + \int_{-\infty}^{x} \left| \sum_{\alpha=1}^{m_1} c_\alpha \mathrm{e}^{\mathrm{i}(\xi_\alpha - \mathrm{i}\eta_\alpha)y - \mathrm{i}(\xi_\alpha - \mathrm{i}\eta_\alpha)^2 t} \right|^2 \mathrm{d}y \\
&= c_0 + \sum_{\alpha=1}^{m_1} \frac{|c_\alpha|^2}{2\eta_\alpha} \mathrm{e}^{2\eta_\alpha(x - 2\xi_\alpha t)} \\
&\quad + \sum_{1 \le \alpha < \beta \le m_1} \mathrm{e}^{\eta_\alpha(x-2\xi_\alpha t) + \eta_\beta(x - 2\xi_\beta t)} \left[\frac{c_\alpha c_\beta^*}{\eta_\alpha + \eta_\beta + \mathrm{i}(\xi_\alpha - \xi_\beta)} \mathrm{e}^{\mathrm{i}(\xi_\alpha - \xi_\beta)x - \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \right. \\
&\qquad \left. + \frac{c_\alpha^* c_\beta}{\eta_\alpha + \eta_\beta - \mathrm{i}(\xi_\alpha - \xi_\beta)} \mathrm{e}^{-\mathrm{i}(\xi_\alpha - \xi_\beta)x + \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \right],
\end{aligned}$$

$$\begin{aligned}
g_j &:= \left[\sum_{\alpha=1}^{m_1} c_\alpha \mathrm{e}^{\eta_\alpha(x - 2\xi_\alpha t) + \mathrm{i}\xi_\alpha x - \mathrm{i}(\xi_\alpha^2 - \eta_\alpha^2)t} \right] \left[\sum_{\beta=1}^{m_1} \frac{c_\beta^*}{\xi_\beta + \mathrm{i}\eta_\beta - \gamma_j} \mathrm{e}^{\eta_\beta(x - 2\xi_\beta t) - \mathrm{i}\xi_\beta x + \mathrm{i}(\xi_\beta^2 - \eta_\beta^2)t} \right] \\
&= \sum_{\alpha=1}^{m_1} \frac{|c_\alpha|^2}{\xi_\alpha + \mathrm{i}\eta_\alpha - \gamma_j} \mathrm{e}^{2\eta_\alpha(x - 2\xi_\alpha t)} \\
&\quad + \sum_{1 \le \alpha < \beta \le m_1} \mathrm{e}^{\eta_\alpha(x-2\xi_\alpha t) + \eta_\beta(x - 2\xi_\beta t)} \left[\frac{c_\alpha c_\beta^*}{\xi_\beta + \mathrm{i}\eta_\beta - \gamma_j} \mathrm{e}^{\mathrm{i}(\xi_\alpha - \xi_\beta)x - \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \right. \\
&\qquad \left. + \frac{c_\alpha^* c_\beta}{\xi_\alpha + \mathrm{i}\eta_\alpha - \gamma_j} \mathrm{e}^{-\mathrm{i}(\xi_\alpha - \xi_\beta)x + \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \right].
\end{aligned}$$

The sign σ_k in each nonlinear term in (2.4) and the amplitude $|a_k|$ of each background plane wave are determined through the relations:

$$\sigma_k |a_k|^2 = \frac{\prod_{i=1}^{m_1} \left[(\xi_i - \gamma_k)^2 + \eta_i^2 \right] \times \prod_{j=1}^{m_2} (\nu_j - \gamma_k)}{\prod_{\substack{j=1 \\ j \ne k}}^{m} (\gamma_j - \gamma_k)}, \quad k = 1, 2, \ldots, m.$$

Here, we assume that the roots $\{\lambda_1, \lambda_2, \ldots, \lambda_{m+1}\}$ of equation (3.5) with $b_k = \sigma_k a_k^*$ comprise m_1 pairs of complex conjugate roots $\{\xi_1 \pm \mathrm{i}\eta_1, \ldots, \xi_{m_1} \pm \mathrm{i}\eta_{m_1}\}$ ($\eta_j > 0$) and m_2 real roots $\{\nu_1, \ldots, \nu_{m_2}\}$, where $m = 2m_1 + m_2 - 1$.

5 Bright-soliton solutions on a plane-wave background

In this section, we apply the binary Bäcklund–Darboux transformation to the multicomponent NLS equations and obtain their bright-soliton solutions on a general plane-wave background. To simplify the derivation, we use a reduced version of Proposition 2.3, which defines the binary Bäcklund–Darboux transformation maintaining the Hermitian conjugation reduction between the two matrix dependent variables. In contrast to the construction of the dark-soliton solutions, the Bäcklund parameter μ for generating bright-soliton solutions is complex-valued, i.e., it has a nonzero imaginary part.

Proposition 5.1. Consider the $(l+m) \times (l+m)$ Lax-pair representation for the $l \times m$ matrix NLS equation (2.3) as given by (cf. (2.6) with (2.2))

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\mu I_l & Q \\ \Sigma Q^\dagger \Omega & \mathrm{i}\mu I_m \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{5.1}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = \begin{bmatrix} -2\mathrm{i}\mu^2 I_l - \mathrm{i}Q\Sigma Q^\dagger \Omega & 2\mu Q + \mathrm{i}Q_x \\ 2\mu\Sigma Q^\dagger \Omega - \mathrm{i}\Sigma Q_x^\dagger \Omega & 2\mathrm{i}\mu^2 I_m + \mathrm{i}\Sigma Q^\dagger \Omega Q \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{5.2}$$

Here, the spectral parameter ζ is fixed at a complex value μ, Ψ_1 is an $l \times l$ matrix and Ψ_2 is an $m \times l$ matrix; Σ and Ω are $m \times m$ and $l \times l$ diagonal matrices with diagonal entries equal to $+1$ or -1. Then, the new potential defined by the formula (see [46, 47] for the scalar case and [34, 38, 48, 49, 76] for the vector case),

$$Q' := Q + 2\mathrm{i}(\mu - \mu^*)\Psi_1 \left(\Psi_1^\dagger \Omega \Psi_1 - \Psi_2^\dagger \Sigma \Psi_2\right)^{-1} \Psi_2^\dagger \Sigma,$$

provides a new solution of the matrix NLS equation (2.3). A linear eigenfunction associated with the new potential Q' at the spectral parameter ζ set equal to μ^* is given by

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \left(\Psi_1^\dagger \Omega \Psi_1 - \Psi_2^\dagger \Sigma \Psi_2\right)^{-1}, \tag{5.3}$$

which defines a bound state for a proper choice of $\Psi_1(\mu)$ and $\Psi_2(\mu)$.[8] An adjoint linear eigenfunction,

$$\left(\Psi_1^\dagger \Omega \Psi_1 - \Psi_2^\dagger \Sigma \Psi_2\right)^{-1} \left[\, \Psi_1^\dagger \Omega \;\; -\Psi_2^\dagger \Sigma \,\right],$$

can provide another bound state at the spectral parameter ζ equal to μ.[9]

This proposition enables us to obtain a formal expression for the bright one-soliton solution on a plane-wave background using the matrix exponential of a constant non-diagonal matrix.

5.1 Matrix bright soliton

We first consider the matrix NLS equation (2.5) for an $l \times m$ matrix Q, which admits the general plane-wave solution (cf. (2.20)):

$$Q = \mathrm{e}^{-2\mathrm{i}t\sigma AA^\dagger} A \mathrm{e}^{\mathrm{i}x\Gamma - \mathrm{i}t\Gamma^2}.$$

Here, $\sigma = +1$ and $\sigma = -1$ correspond to the self-defocusing case and self-focusing case, respectively; Γ is an $m \times m$ real diagonal matrix and A is an $l \times m$ matrix such that $AA^\dagger$ is a real diagonal matrix. We employ this plane-wave solution as the seed solution and apply the binary Bäcklund–Darboux transformation defined in Proposition 5.1. Following the procedure described in subsection 2.6, we obtain a new solution of the matrix NLS equation (2.5) in the form:

$$\begin{aligned} Q &= \mathrm{e}^{-2\mathrm{i}t\sigma AA^\dagger} A \mathrm{e}^{\mathrm{i}x\Gamma - \mathrm{i}t\Gamma^2} + 2\mathrm{i}(\mu - \mu^*)\Psi_1 \left(\sigma \Psi_1^\dagger \Psi_1 - \Psi_2^\dagger \Psi_2\right)^{-1} \Psi_2^\dagger \\ &= \mathrm{e}^{-2\mathrm{i}t\sigma AA^\dagger} \left[A + 2(\mu - \mu^*)\, \boldsymbol{\Psi}_1 \left(\sigma\, \boldsymbol{\Psi}_1^\dagger \boldsymbol{\Psi}_1 - \boldsymbol{\Psi}_2^\dagger \boldsymbol{\Psi}_2\right)^{-1} \boldsymbol{\Psi}_2^\dagger\right] \mathrm{e}^{\mathrm{i}x\Gamma - \mathrm{i}t\Gamma^2}. \end{aligned} \tag{5.4}$$

Here, we omit the prime to distinguish the new solution from the old one. In the self-focusing case $\sigma = -1$, the new solution is regular for all x and t. The gauge-transformed linear eigenfunction used in (5.4) is given as (cf. (2.23)),

$$\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} = \mathrm{e}^{\mathrm{i}\mu x + 2\mathrm{i}\mu^2 t} \mathrm{e}^{\mathrm{i}x\boldsymbol{Z} - \mathrm{i}t\boldsymbol{Z}^2} \begin{bmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \end{bmatrix}, \qquad \boldsymbol{Z} := \begin{bmatrix} -2\mu I_l & A \\ -\sigma A^\dagger & \Gamma \end{bmatrix},$$

[8] In some cases, we need to consider a suitable linear combination of the columns in (5.3). Note that the spatial part of the Lax-pair representation $\Psi_x = U\Psi$ admits the gauge transformation $\Psi \to \mathrm{e}^{\mathrm{i}kx}\Psi$, $U \to U + \mathrm{i}kI$, where k is an arbitrary complex constant; this transformation is sometimes necessary to obtain a square-integrable eigenfunction.

[9] Thus, a multicomponent bright soliton is associated with a pair of bound states: a linear eigenfunction at $\zeta = \mu^*$ and an adjoint linear eigenfunction at $\zeta = \mu$. For the scalar NLS equation, the adjoint linear eigenfunction can be immediately rewritten as the usual (i.e., column-vector) linear eigenfunction at $\zeta = \mu$.

where $\mathscr{C}_1$ and $\mathscr{C}_2$ are $l \times l$ and $m \times l$ constant matrices, respectively. Note that the prefactor $\mathrm{e}^{\mathrm{i}\mu x + 2\mathrm{i}\mu^2 t}$ plays no role in formula (5.4). This expression is rather formal and appears to be not so useful in the present form for practical applications. Thus, we next consider the simpler case of the vector NLS equation ($l = 1$), particularly targeting on the two-component case ($m = 2$), and obtain a more explicit expression for the solution.

5.2 Vector bright soliton

A general plane-wave solution of the vector NLS equation (2.4) is given by setting $b_j = \sigma_j a_j^*$ in (3.3), i.e.

$$q_j(x,t) = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left(\gamma_j^2 + 2\sum_{k=1}^m \sigma_k |a_k|^2\right)t}, \quad j = 1, 2, \ldots, m. \tag{5.5}$$

Here, we assume that γ_j ($j = 1, 2, \ldots, m$) are pairwise distinct real constants. We recall the relations (4.1):

$$2\mu = \sum_{j=1}^{m} \gamma_j - \sum_{j=1}^{m+1} \lambda_j$$

and

$$\sigma_k |a_k|^2 = \frac{\displaystyle\prod_{j=1}^{m+1} (\lambda_j - \gamma_k)}{\displaystyle\prod_{\substack{j=1 \\ j \neq k}}^{m} (\gamma_j - \gamma_k)}, \quad k = 1, 2, \ldots, m, \tag{5.6}$$

where μ is a complex parameter. Thus, the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{m+1}\}$ of the matrix $\boldsymbol{Z}(\mu)$ must satisfy the condition that $\prod_{j=1}^{m+1} (\lambda_j - \gamma_k)$ is real-valued for all k. That is, if we set

$$\lambda_j = \xi_j + \mathrm{i}\eta_j, \quad j = 1, 2, \ldots, m+1$$

with real ξ_j and η_j, then we have

$$\prod_{j=1}^{m+1} [(\xi_j - \gamma_k) + \mathrm{i}\eta_j] \in \mathbb{R}, \quad k = 1, 2, \ldots, m.$$

This means that if the eigenvalues $\lambda_j = \xi_j + \mathrm{i}\eta_j$ ($j = 1, 2, \ldots, m+1$) are given, the wavenumbers γ_k ($k = 1, 2, \ldots, m$) are determined as the solutions of an

algebraic equation for γ,

$$\mathrm{Im}\left\{\prod_{j=1}^{m+1}\left[(\xi_j-\gamma)+\mathrm{i}\eta_j\right]\right\}=0. \tag{5.7}$$

This is a real-coefficient equation of degree m, which is much easier to solve than the characteristic equation for determining the eigenvalues $\lambda_j=\xi_j+\mathrm{i}\eta_j$ $(j=1,2,\ldots,m+1)$ (cf. (3.5)). Indeed, the latter is a complex-coefficient equation of degree $m+1$. This is why we consider that the eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_{m+1}\}$ are free complex parameters, from which the wavenumbers $\{\gamma_1,\gamma_2,\ldots,\gamma_m\}$ of the background plane waves are determined.

Then, by expressing the right-hand side of (5.6) only in terms of ξ_j and η_j $(j=1,2,\ldots,m+1)$, we can determine the sign σ_k in each cubic term in the vector NLS equation (2.4) and the amplitude $|a_k|$ of each background plane wave. It is very useful to consider geometrically how the argument of the complex function $\prod_{j=1}^{m+1}(\lambda_j-\gamma)$ changes as γ moves on the real axis from $-\infty$ to $+\infty$. For example, when all the eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_{m+1}\}$ lie in the upper-half (or lower-half) complex plane, we have $\sigma_k=-1$ for all k, which corresponds to the self-focusing case.

In the scalar NLS case $(m=1)$, (5.7) is a linear equation for determining γ_1. If $\eta_1+\eta_2\neq 0$, we have

$$\gamma_1=\frac{\xi_1\eta_2+\xi_2\eta_1}{\eta_1+\eta_2}, \tag{5.8}$$

which gives

$$\begin{aligned}\sigma_1|a_1|^2&=(\lambda_1-\gamma_1)(\lambda_2-\gamma_1)\\&=-\eta_1\eta_2\left[1+\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}\right)^2\right].\end{aligned}$$

Thus, $\sigma_1=-\mathrm{sgn}\,(\eta_1\eta_2)$. To obtain a regular solution of the NLS equation, we assume $\eta_1\eta_2>0$ and consider the self-focusing case $(\sigma_1=-1)$. Then, the matrix $\boldsymbol{Z}(\mu)$ takes the form (cf. (3.4)):

$$\boldsymbol{Z}=\begin{bmatrix}\lambda_1+\lambda_2-\gamma_1 & a_1\\ a_1^* & \gamma_1\end{bmatrix}.$$

We choose the eigenvectors corresponding to the eigenvalues $\lambda_j = \xi_j + \mathrm{i}\eta_j$ as

$$\boldsymbol{Z}\begin{bmatrix} a_1 \\ \eta_2\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}-\mathrm{i}\right)\end{bmatrix} = \lambda_1\begin{bmatrix} a_1 \\ \eta_2\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}-\mathrm{i}\right)\end{bmatrix},$$

$$\boldsymbol{Z}\begin{bmatrix} a_1 \\ -\eta_1\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}+\mathrm{i}\right)\end{bmatrix} = \lambda_2\begin{bmatrix} a_1 \\ -\eta_1\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}+\mathrm{i}\right)\end{bmatrix}.$$

Then, the matrix exponential in (2.23) can be computed explicitly to provide the gauge-transformed linear eigenfunction as

$$\begin{aligned}\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} &= c_1 \mathrm{e}^{\mathrm{i}x\frac{1}{2}(\gamma_1+\lambda_1-\lambda_2)+\mathrm{i}t\frac{1}{2}\left[\gamma_1^2-2\gamma_1(\lambda_1+\lambda_2)-\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2\right]}\begin{bmatrix} a_1 \\ \eta_2\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}-\mathrm{i}\right)\end{bmatrix} \\ &\quad + c_2 \mathrm{e}^{\mathrm{i}x\frac{1}{2}(\gamma_1-\lambda_1+\lambda_2)+\mathrm{i}t\frac{1}{2}\left[\gamma_1^2-2\gamma_1(\lambda_1+\lambda_2)+\lambda_1^2-\lambda_2^2+2\lambda_1\lambda_2\right]}\begin{bmatrix} a_1 \\ -\eta_1\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}+\mathrm{i}\right)\end{bmatrix},\end{aligned}$$

where c_1 and c_2 are arbitrary complex constants. Thus, by adapting formula (5.4) to the scalar NLS equation $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0$, we obtain the bright one-soliton solution on the plane-wave background as (cf. [6–8, 11])

$$q(x,t) = a_1 \mathrm{e}^{\mathrm{i}\gamma_1 x - \mathrm{i}\left(\gamma_1^2 - 2|a_1|^2\right)t}\frac{g}{f}. \tag{5.9}$$

Here, the positive function $f(x,t)$ and the complex function $g(x,t)$ are

$$\begin{aligned} f := {} & |c_1|^2\eta_2\left[(\xi_1-\xi_2)^2+(\eta_1+\eta_2)^2\right]\mathrm{e}^{-(\eta_1-\eta_2)x+2(\xi_1\eta_1-\xi_2\eta_2)t} \\ & + |c_2|^2\eta_1\left[(\xi_1-\xi_2)^2+(\eta_1+\eta_2)^2\right]\mathrm{e}^{(\eta_1-\eta_2)x-2(\xi_1\eta_1-\xi_2\eta_2)t} \\ & - 2\mathrm{i}c_1^*c_2\eta_1\eta_2\left[(\xi_1-\xi_2)+\mathrm{i}(\eta_1+\eta_2)\right]\mathrm{e}^{-\mathrm{i}(\xi_1-\xi_2)x+\mathrm{i}\left(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2\right)t} \\ & + 2\mathrm{i}c_1c_2^*\eta_1\eta_2\left[(\xi_1-\xi_2)-\mathrm{i}(\eta_1+\eta_2)\right]\mathrm{e}^{\mathrm{i}(\xi_1-\xi_2)x-\mathrm{i}\left(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2\right)t}\end{aligned}$$

and

$$\begin{aligned} g := {} & |c_1|^2\eta_2\left[(\xi_1-\xi_2)+\mathrm{i}(\eta_1+\eta_2)\right]^2\mathrm{e}^{-(\eta_1-\eta_2)x+2(\xi_1\eta_1-\xi_2\eta_2)t} \\ & + |c_2|^2\eta_1\left[(\xi_1-\xi_2)-\mathrm{i}(\eta_1+\eta_2)\right]^2\mathrm{e}^{(\eta_1-\eta_2)x-2(\xi_1\eta_1-\xi_2\eta_2)t} \\ & + 2\mathrm{i}c_1^*c_2\eta_2^2\left[(\xi_1-\xi_2)+\mathrm{i}(\eta_1+\eta_2)\right]\mathrm{e}^{-\mathrm{i}(\xi_1-\xi_2)x+\mathrm{i}\left(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2\right)t} \\ & - 2\mathrm{i}c_1c_2^*\eta_1^2\left[(\xi_1-\xi_2)-\mathrm{i}(\eta_1+\eta_2)\right]\mathrm{e}^{\mathrm{i}(\xi_1-\xi_2)x-\mathrm{i}\left(\xi_1^2-\xi_2^2-\eta_1^2+\eta_2^2\right)t},\end{aligned}$$

respectively. The amplitude and the wavenumber of the background plane wave are determined by

$$|a_1|^2 = \eta_1\eta_2\left[1+\left(\frac{\xi_1-\xi_2}{\eta_1+\eta_2}\right)^2\right]$$

and (5.8) respectively, while the complex phase of a_1 is arbitrary, which reflects the $U(1)$ symmetry of the scalar NLS equation. Note that f and g can be simplified by setting $c_2/c_1 =: \mathrm{e}^{\delta+\mathrm{i}\theta}$ and using real δ and θ as new parameters. Thinking more intuitively, we can set $c_1 = c_2 = 1$ by constant shifts of x and t. The additional condition $\eta_1 \neq \eta_2$ is necessary for (5.9) to provide a genuine soliton solution associated with a pair of bound states of the Lax pair. In fact, in the special case of $\eta_1 = \eta_2$ and $\xi_1 \neq \xi_2$, (5.9) describes an oscillating solution on the plane-wave background, which is localized in time t (cf. [10, 11]).

Next, we consider the two-component case ($m = 2$). Then, (5.7) becomes a quadratic equation for γ,

$$(\xi_1 - \gamma)(\xi_2 - \gamma)\eta_3 + (\xi_2 - \gamma)(\xi_3 - \gamma)\eta_1 + (\xi_3 - \gamma)(\xi_1 - \gamma)\eta_2 - \eta_1\eta_2\eta_3 = 0,$$

which has two real solutions γ_1 and γ_2 ($\gamma_1 \neq \gamma_2$). We consider only the generic case of $\eta_1 + \eta_2 + \eta_3 \neq 0$ when the above equation is indeed quadratic. Thus, the discriminant denoted by Δ must be positive:

$$\begin{aligned}\Delta := {} & [(\xi_1 + \xi_2)\eta_3 + (\xi_2 + \xi_3)\eta_1 + (\xi_3 + \xi_1)\eta_2]^2 \\ & - 4(\eta_1 + \eta_2 + \eta_3)(\xi_1\xi_2\eta_3 + \xi_2\xi_3\eta_1 + \xi_3\xi_1\eta_2 - \eta_1\eta_2\eta_3) > 0,\end{aligned}$$

and the wavenumbers γ_1 and γ_2 are given by

$$\gamma_1 = \frac{(\xi_1 + \xi_2)\eta_3 + (\xi_2 + \xi_3)\eta_1 + (\xi_3 + \xi_1)\eta_2 - \sqrt{\Delta}}{2(\eta_1 + \eta_2 + \eta_3)}, \tag{5.10a}$$

$$\gamma_2 = \frac{(\xi_1 + \xi_2)\eta_3 + (\xi_2 + \xi_3)\eta_1 + (\xi_3 + \xi_1)\eta_2 + \sqrt{\Delta}}{2(\eta_1 + \eta_2 + \eta_3)}. \tag{5.10b}$$

From (5.6), we have the relations:

$$\sigma_1|a_1|^2 = \frac{(\xi_1 - \gamma_1)(\xi_2 - \gamma_1)(\xi_3 - \gamma_1) - \eta_1\eta_2(\xi_3 - \gamma_1) - \eta_2\eta_3(\xi_1 - \gamma_1) - \eta_1\eta_3(\xi_2 - \gamma_1)}{\gamma_2 - \gamma_1}, \tag{5.11a}$$

$$\sigma_2|a_2|^2 = \frac{(\xi_1 - \gamma_2)(\xi_2 - \gamma_2)(\xi_3 - \gamma_2) - \eta_1\eta_2(\xi_3 - \gamma_2) - \eta_2\eta_3(\xi_1 - \gamma_2) - \eta_1\eta_3(\xi_2 - \gamma_2)}{\gamma_1 - \gamma_2}. \tag{5.11b}$$

As mentioned above, considering the range of the complex argument of $\prod_{j=1}^{3}(\lambda_j - \gamma)$ as a function of real γ, we can establish a correspondence between the configuration of $\{\lambda_1, \lambda_2, \lambda_3\}$ and the combination of the signs $\{\sigma_1, \sigma_2\}$. That is, (i) when $\eta_j > 0$ ($j = 1, 2, 3$), i.e., all the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$

lie in the upper-half plane, we have a self-focusing nonlinearity $\sigma_1 = \sigma_2 = -1$; (ii) when $\eta_1, \eta_2 > 0$, $-(\eta_1 + \eta_2) < \eta_3 < 0$ and $\Delta > 0$, we have a mixed focusing-defocusing nonlinearity $\{\sigma_1, \sigma_2\} = \{+1, -1\}$; (iii) when $\eta_1, \eta_2 > 0$ and $\eta_3 < -(\eta_1 + \eta_2)$, we have a self-defocusing nonlinearity $\sigma_1 = \sigma_2 = +1$; and so on.

When at least one of $\{\sigma_1, \sigma_2\}$ is -1, i.e., there exist focusing component(s) in the vector dependent variable, the associated eigenvalue problem is not self-adjoint (cf. subsection 2.2); thus, the bound-state eigenvalues can take complex values. In the self-defocusing case $\sigma_1 = \sigma_2 = +1$, the eigenvalue problem is self-adjoint, so the bound-state eigenvalues are restricted to be real; thus, we cannot construct a genuine bright-soliton solution by applying the binary Bäcklund–Darboux transformation used in this section. However, we can obtain a "soliton-like" solution in the self-defocusing case using a complex-valued Bäcklund parameter μ, not associated with a bound state.

The matrix $\boldsymbol{Z}(\mu)$ with $2\mu = \gamma_1 + \gamma_2 - (\lambda_1 + \lambda_2 + \lambda_3)$ takes the form (cf. (3.4)):

$$\boldsymbol{Z} = \begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 - \gamma_1 - \gamma_2 & a_1 & a_2 \\ -\sigma_1 a_1^* & \gamma_1 & 0 \\ -\sigma_2 a_2^* & 0 & \gamma_2 \end{bmatrix}.$$

We choose the eigenvectors corresponding to the eigenvalues $\lambda_j = \xi_j + \mathrm{i}\eta_j$ as

$$\boldsymbol{Z} \begin{bmatrix} 1 \\ -\frac{\sigma_1 a_1^*}{\lambda_j - \gamma_1} \\ -\frac{\sigma_2 a_2^*}{\lambda_j - \gamma_2} \end{bmatrix} = \lambda_j \begin{bmatrix} 1 \\ -\frac{\sigma_1 a_1^*}{\lambda_j - \gamma_1} \\ -\frac{\sigma_2 a_2^*}{\lambda_j - \gamma_2} \end{bmatrix}, \qquad j = 1, 2, 3.$$

Then, the matrix exponential in (2.23) can be computed explicitly to provide the gauge-transformed linear eigenfunction as

$$\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} = \mathrm{e}^{\mathrm{i}\mu x + 2\mathrm{i}\mu^2 t} \sum_{j=1}^{3} c_j \mathrm{e}^{\mathrm{i}\lambda_j x - \mathrm{i}\lambda_j^2 t} \begin{bmatrix} 1 \\ -\frac{\sigma_1 a_1^*}{\lambda_j - \gamma_1} \\ -\frac{\sigma_2 a_2^*}{\lambda_j - \gamma_2} \end{bmatrix},$$

where $\boldsymbol{\Psi}_1$ is a scalar and $\boldsymbol{\Psi}_2$ is a two-component column vector; c_j are arbitrary complex constants. Substituting this expression into a slightly modified version of formula (5.4), we obtain a desired solution of the two-component vector NLS equation [(2.4) with $m = 2$] as

$$q_j = a_j \mathrm{e}^{\mathrm{i}\gamma_j x - \mathrm{i}\left(\gamma_j^2 + 2\sum_{k=1}^{2} \sigma_k |a_k|^2\right) t} \left\{ 1 + 2\mathrm{i}\left(\eta_1 + \eta_2 + \eta_3\right) \frac{g_j}{f} \right\}, \qquad j = 1, 2. \tag{5.12}$$

Here, the real function $f(x,t)$ and the complex functions $g_j(x,t)$ are

$$
\begin{aligned}
f := & \sum_{\alpha=1}^{3} |c_\alpha|^2 \, \mathrm{e}^{-2\eta_\alpha(x-2\xi_\alpha t)} \left[1 - \frac{\sigma_1 |a_1|^2}{|\lambda_\alpha - \gamma_1|^2} - \frac{\sigma_2 |a_2|^2}{|\lambda_\alpha - \gamma_2|^2} \right] \\
& + \sum_{1 \le \alpha < \beta \le 3} \mathrm{e}^{-\eta_\alpha(x-2\xi_\alpha t) - \eta_\beta(x-2\xi_\beta t)} \\
& \quad \times \left\{ c_\alpha c_\beta^* \, \mathrm{e}^{\mathrm{i}(\xi_\alpha - \xi_\beta)x - \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \left[1 - \frac{\sigma_1 |a_1|^2}{(\lambda_\alpha - \gamma_1)\left(\lambda_\beta^* - \gamma_1\right)} - \frac{\sigma_2 |a_2|^2}{(\lambda_\alpha - \gamma_2)\left(\lambda_\beta^* - \gamma_2\right)} \right] \right. \\
& \qquad \left. + \, c_\alpha^* c_\beta \, \mathrm{e}^{-\mathrm{i}(\xi_\alpha - \xi_\beta)x + \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \left[1 - \frac{\sigma_1 |a_1|^2}{(\lambda_\alpha^* - \gamma_1)(\lambda_\beta - \gamma_1)} - \frac{\sigma_2 |a_2|^2}{(\lambda_\alpha^* - \gamma_2)(\lambda_\beta - \gamma_2)} \right] \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
g_j := & \sum_{\alpha=1}^{3} |c_\alpha|^2 \, \mathrm{e}^{-2\eta_\alpha(x-2\xi_\alpha t)} \frac{1}{\lambda_\alpha^* - \gamma_j} + \sum_{1 \le \alpha < \beta \le 3} \mathrm{e}^{-\eta_\alpha(x-2\xi_\alpha t) - \eta_\beta(x-2\xi_\beta t)} \\
& \times \left[c_\alpha c_\beta^* \, \mathrm{e}^{\mathrm{i}(\xi_\alpha - \xi_\beta)x - \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \frac{1}{\lambda_\beta^* - \gamma_j} + c_\alpha^* c_\beta \, \mathrm{e}^{-\mathrm{i}(\xi_\alpha - \xi_\beta)x + \mathrm{i}(\xi_\alpha^2 - \xi_\beta^2 - \eta_\alpha^2 + \eta_\beta^2)t} \frac{1}{\lambda_\alpha^* - \gamma_j} \right],
\end{aligned}
$$

respectively. The wavenumbers γ_j and the amplitudes $|a_j|$ of the background plane waves as well as the signs σ_j are determined through (5.10) and (5.11). Concerning σ_j, we briefly comment on the three cases (i)–(iii) mentioned above (cf. [22]).

(i) $\eta_j > 0$ $(j = 1, 2, 3)$: the self-focusing case $\sigma_1 = \sigma_2 = -1$. The function $f(x,t)$ is positive and the solution (5.12) is regular for all x and t. For this solution to be a genuine soliton solution, the special case $\eta_1 = \eta_2 = \eta_3$ should be excluded. If the additional condition $\eta_1 > \eta_2 + \eta_3$ is satisfied up to a re-numbering of $\{\lambda_1, \lambda_2, \lambda_3\}$, the linear eigenfunction given by (cf. Proposition 5.1 and (2.21))

$$
\begin{bmatrix} \mathrm{e}^{-2\mathrm{i}t \sum_{k=1}^{2} \sigma_k |a_k|^2} \, \boldsymbol{\Psi}_1 \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}x\Gamma + \mathrm{i}t\Gamma^2} \, \boldsymbol{\Psi}_2 \end{bmatrix} \left(\boldsymbol{\Psi}_1^* \boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2^\dagger \boldsymbol{\Psi}_2 \right)^{-1}
$$

decays rapidly as $x \to \pm\infty$ and provides a bound state; otherwise, we need to multiply it by $\mathrm{e}^{\mathrm{i}kx}$ with a suitable complex parameter $k \in \mathbb{C}$.

(ii) $\eta_1, \eta_2 > 0$, $-(\eta_1 + \eta_2) < \eta_3 < 0$ and $\Delta > 0$: the mixed focusing-defocusing case $\{\sigma_1, \sigma_2\} = \{+1, -1\}$. We need to choose the parameters appropriately so that $f(x,t) > 0$ or < 0 for all x and t.

(iii) $\eta_1, \eta_2 > 0$ and $\eta_3 < -(\eta_1 + \eta_2)$: the self-defocusing case $\sigma_1 = \sigma_2 = +1$. We set $c_3 = 0$ in the definitions of $f(x,t)$ and $g_j(x,t)$ so that (5.12) can be a regular solution without singularities. Indeed, for an appropriate choice of the parameters, it corresponds to the "soliton-like" solution obtained by Park and Shin in a more implicit form using Cardano's formula [34] (also see [49]).

For (ii) and (iii), the existence of the conserved densities $|q_j|^2 - \text{const.}$ $(j = 1, 2)$ [20, 21, 29] implies that the regularity of the solution is generally preserved under the time evolution.

6 Concluding remarks

In this paper, we have applied three types of Bäcklund–Darboux transformations to the multicomponent NLS equations with a self-focusing, self-defocusing or mixed focusing-defocusing nonlinearity. Using a general plane-wave solution as the seed solution, we constructed various interesting solutions such as the vector/matrix dark-soliton solutions and bright-soliton solutions on the plane-wave background. These solutions generally admit the internal degrees of freedom and provide highly nontrivial generalizations of the corresponding scalar NLS solitons. The main new feature of our approach is an appropriate re-parametrization of the parameters that appear in the application of the Bäcklund–Darboux transformations; this allows us to express the soliton solutions in more explicit and tractable forms than those reported in the literature (cf. [22, 34]). The price to pay is that unlike the usual naïve parametrization, not all of the amplitudes and the wavenumbers of the background plane waves remain independent free parameters; rather, they can depend on other parameters characterizing the soliton such as soliton's width and velocity. More precisely, a multicomponent soliton generally has a time-dependent and nonrecurrent shape due in part to the beating effect with the background plane waves; thus, it is not always meaningful to consider soliton's width and velocity.

There exist two major methods of obtaining exact solutions of integrable partial differential equations: the inverse scattering method [83] and the Hirota bilinear method [84]. However, these two methods are not effective for constructing general multicomponent solitons under the plane-wave boundary conditions. Indeed, explicit expressions for the vector/matrix NLS solitons with internal degrees of freedom are accompanied by a set of complicated constraints on the parameters, which is quite difficult to obtain and resolve using these methods. The method based on Bäcklund–Darboux transformations is more appropriate for this purpose. Interestingly, in contrast to the

common belief, even the most fundamental Bäcklund–Darboux transformation, called an elementary Bäcklund–Darboux transformation, can generate nontrivial dark-soliton solutions such as (3.21) and (3.24). A more elaborate transformation, the limiting case of the binary Bäcklund–Darboux transformation, is indispensable for obtaining vector dark-soliton solutions with internal degrees of freedom, such as (4.11) and (4.12). It would be interesting to consider a matrix generalization of the Hirota bilinear method, which allows us to obtain formal expressions of solutions involving the exponential of a non-diagonal matrix, such as (3.24) and (5.4).

In this paper, we focused on the construction of one-soliton solutions of the multicomponent NLS equations by applying a "one-step" Bäcklund–Darboux transformation. It is in principle possible to apply it iteratively at (generally) different values of the Bäcklund parameter; this defines a multifold Bäcklund–Darboux transformation, which can be used to obtain multicomponent multisoliton solutions under the plane-wave boundary conditions. However, to write them out explicitly, we need to consider and resolve a rather complicated system of algebraic equations satisfied by the parameters in the solutions. In practice, such explicit expresssions become too cumbersome even for the two-soliton solutions, so they are not useful compared with formal expressions involving a matrix exponential function.

Before closing this paper, we would like to remark that soliton solutions of the multicomponent NLS equations under simpler boundary conditions have been reported in the literature. The list of references on the multicomponent NLS solitons is too long (see, *e.g.*, [51]), so we only mention the original and most relevant ones. The bright N-soliton solution of the vector NLS equation under the vanishing boundary conditions was derived in [22] (also see [48, 85]) and the corresponding Hirota bilinear form and tau functions were presented in [86]. The two-component vector NLS equation was solved under special nonvanishing boundary conditions such that the two background plane waves have the same wavenumber and frequency (note the Galilean invariance) [24, 26] or one of them disappears [22, 34]; thus, the degenerate dark-soliton solutions (cf. [82]) and so-called dark-bright soliton solutions were obtained. In addition, these two classes of solutions can be related through the symmetry group of the two-component NLS equation, such as $SU(2)$ or $SU(1,1)$ [29, 34]. Surprisingly, most of the recent papers on the solutions of the vector NLS equation under nonvanishing boundary conditions did not refer to the pioneering work of Makhankov *et al.* in the 1980s (see [22] and references therein) or the subsequent contribution by Park and Shin [34, 49]. As far as we could check, such papers generally present no new solutions of the vector NLS equation.

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