On a new integrable generalization of the Toda lattice and a discrete Yajima–Oikawa system

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Abstract

We propose a new integrable generalization of the Toda lattice wherein the original Flaschka–Manakov variables are coupled to newly introduced dependent variables; the general case wherein the additional dependent variables are vector-valued is considered. This generalization admits a Lax pair based on an extension of the Jacobi operator, an infinite number of conservation laws and, in a special case, a simple Hamiltonian structure. In fact, the second flow of this generalized Toda hierarchy reduces to the usual Toda lattice when the additional dependent variables vanish; the first flow of the hierarchy reduces to a long wave–short wave interaction model, known as the Yajima–Oikawa system, in a suitable continuous limit. This integrable discretization of the Yajima–Oikawa system is essentially different from the discrete Yajima–Oikawa system proposed in arXiv:1509.06996 (also see https://link.aps.org/doi/10.1103/PhysRevE.91.062902) and studied in arXiv:1804.10224. Two integrable discretizations of the nonlinear Schrödinger hierarchy, the Ablowitz–Ladik hierarchy and the Konopelchenko–Chudnovsky hierarchy, are contained in the generalized Toda hierarchy as special cases.

Contents

1 Introduction

The first and most prominent example of a discrete integrable system is the Toda lattice discovered half a century ago [1, 2]. The Newtonian equations of motion of the Toda lattice are given by

$$x_{n,tt} = \mathrm{e}^{x_{n+1}-x_n} - \mathrm{e}^{x_n - x_{n-1}}, \quad n \in \mathbb{Z}. \tag{1.1}$$

A simple but remarkable change of variables $u_n := \frac{1}{2}\mathrm{e}^{\frac{1}{2}(x_n - x_{n-1})}$, $w_n := \frac{1}{2}x_{n,t}$, called the Flaschka–Manakov variables [3–5], recast the Toda lattice (1.1) in a more convenient form:

$$u_{n,t} = u_n(w_n - w_{n-1}), \tag{1.2a}$$
$$w_{n,t} = 2\left(u_{n+1}^2 - u_n^2\right). \tag{1.2b}$$

Indeed, the Toda lattice in Flaschka–Manakov variables (1.2) is known to have much richer structure than the original Newtonian equations of motion (1.1) [6]. The complete integrability of the Toda lattice [3–5, 7] is based on the fact that (1.2) is equivalent to the compatibility condition for the overdetermined linear system, generally called the Lax pair [8]:

$$u_n\psi_{n-1} + w_n\psi_n + u_{n+1}\psi_{n+1} = \lambda\psi_n, \tag{1.3a}$$
$$\psi_{n,t} = u_{n+1}\psi_{n+1} - u_n\psi_{n-1}, \tag{1.3b}$$

where λ is a constant spectral parameter. In mathematical terms, the spatial part of the Lax pair (1.3a) is the eigenvalue problem for a symmetric tridiagonal matrix (Jacobi operator), so the Toda lattice defines an isospectral deformation of the Jacobi operator. In fact, (1.2) is the first nontrivial flow of an infinite hierarchy of isospectral flows associated with the spectral problem (1.3a).

In this paper, we consider an interesting extension of the Lax pair (1.3) to propose a nontrivial generalization of the Toda lattice hierarchy. This generalization provides a discrete analog of the generalization of the KdV hierarchy to a long wave–short wave interaction hierarchy, called the Yajima–Oikawa hierarchy [9, 10], in the continuous case. A space discretization of the Yajima–Oikawa system was already proposed in the recent paper [11] (also see [12]) and its Lax pair as well as the next higher symmetry was presented in [13]. The first flow of the generalized Toda hierarchy proposed in this paper provides, in a special case, a new integrable discretization of the Yajima–Oikawa system, which is essentially different from the discrete Yajima–Oikawa system studied in [11, 13] and has its own advantages; in particular, the discrete Yajima–Oikawa system in this paper possesses not

only a Lax pair and an infinite number of conservation laws but also a simple Hamiltonian structure, so the higher flows of the hierarchy can easily be constructed.

This paper is organized as follows. In section 2, we present an extension of the spectral problem (1.3a) to a two-component spectral problem for ψ_n and ϕ_n, which can also be rewritten as a nonlocal spectral problem for the single component ψ_n. Then, we associate two isospectral time-evolutionary systems for ψ_n and ϕ_n, and obtain from the compatibility conditions the first two isospectral flows of the generalized Toda hierarchy. We also show that the first flow can be reduced to the Yajima–Oikawa system in a suitable continuous limit. In section 3, we prove that in some special (or limiting) cases, the first and second flows of the generalized Toda hierarchy can be reduced to the elementary flows of two discrete nonlinear Schrödinger hierarchies, the Ablowitz–Ladik hierarchy [14] and the Konopelchenko–Chudnovsky hierarchy [15–17], or linear equations. In section 4, we demonstrate that the generalized Toda hierarchy possesses two infinite sets of conservation laws and, in a special case, a simple Hamiltonian structure; explicit expressions for the higher flows of the hierarchy can be constructed in a recursive manner. Section 5 is devoted to concluding remarks.

2 The generalized Toda hierarchy

2.1 Two-component spectral problem

As a generalization of the eigenvalue problem for the Jacobi operator in (1.3a), we consider the following two-component spectral problem for ψ_n and ϕ_n:

$$\begin{cases} \alpha u_n^{\gamma}\psi_{n-1} + \beta u_{n+1}^{\delta}\psi_{n+1} + w_n\psi_n + a_n\left(\gamma\phi_n + \delta\phi_{n+1}\right) = \lambda\psi_n, & \text{(2.1a)} \\ \phi_{n+1} - \phi_n = b_n\psi_n. & \text{(2.1b)} \end{cases}$$

Here, λ is a constant spectral parameter; α, β, γ and δ are arbitrary scalar constants except that they should satisfy the conditions $(\alpha\gamma, \beta\delta) \neq (0, 0)$ and $\gamma + \delta \neq 0$. When the additional dependent variables a_n and b_n vanish, we recover the original eigenvalue problem (1.3a) by setting $\alpha = \beta = \gamma = \delta = 1$. We consider the general case where u_n, w_n and ψ_n are scalar-valued functions and a_n, b_n and ϕ_n are vector-valued functions; that is, a_n is a row vector and b_n and ϕ_n are column vectors. Note that using (2.1b), we can express ϕ_n as a nonlocal function of ψ_n, so we can rewrite (2.1a) as a nonlocal spectral

problem for the single component ψ_n, *e.g.*,

$$\alpha u_n^\gamma \psi_{n-1} + \beta u_{n+1}^\delta \psi_{n+1} + (w_n + \delta a_n b_n)\, \psi_n + (\gamma + \delta)\, a_n \sum_{j=-\infty}^{n-1} b_j \psi_j = \lambda \psi_n,$$

or

$$\alpha u_n^\gamma \psi_{n-1} + \beta u_{n+1}^\delta \psi_{n+1} + w_n \psi_n + a_n \left(\gamma \sum_{j=-\infty}^{n-1} b_j \psi_j - \delta \sum_{j=n+1}^{\infty} b_j \psi_j \right) = \lambda \psi_n.$$

However, such nonlocal forms of the spectral problem are not convenient for later computations, so in this paper we focus on the original two-component form (2.1).

2.2 Isospectral flows

As a trivial isospectral deformation of the two-component spectral problem (2.1), we can consider the following:

$$\begin{cases} \psi_{n,t_0} = c\psi_n, & \text{(2.2a)} \\ \phi_{n,t_0} = d\phi_n. & \text{(2.2b)} \end{cases}$$

Here, c and d are arbitrary scalar constants. The compatibility conditions for the overdetermined linear systems (2.1) and (2.2) provide the trivial zeroth flow of the generalized Toda hierarchy:

$$\begin{cases} a_{n,t_0} - (c-d)\, a_n = 0, \\ b_{n,t_0} + (c-d)\, b_n = 0, \\ u_{n,t_0} = 0, \\ w_{n,t_0} = 0. \end{cases} \tag{2.3}$$

As a nontrivial isospectral deformation of the two-component spectral problem (2.1), we first consider the following:

$$\begin{cases} \psi_{n,t_1} = \alpha u_n^\gamma \psi_{n-1} + \beta u_{n+1}^\delta \psi_{n+1} + w_n \psi_n, & \text{(2.4a)} \\ \phi_{n,t_1} = -\alpha u_n^\gamma b_n \psi_{n-1} + \beta u_n^\delta b_{n-1} \psi_n. & \text{(2.4b)} \end{cases}$$

The compatibility conditions for the overdetermined linear systems (2.1) and (2.4) provide the first flow of the generalized Toda hierarchy:

$$\begin{cases} a_{n,t_1} - \alpha u_n^\gamma a_{n-1} - \beta u_{n+1}^\delta a_{n+1} - w_n a_n = 0, \\ b_{n,t_1} + \beta u_n^\delta b_{n-1} + \alpha u_{n+1}^\gamma b_{n+1} + b_n w_n = 0, \\ u_{n,t_1} + u_n \left(a_{n-1} b_{n-1} - a_n b_n \right) = 0, \\ w_{n,t_1} + \alpha\delta \left(u_n^\gamma a_{n-1} b_n - u_{n+1}^\gamma a_n b_{n+1} \right) + \beta\gamma \left(u_n^\delta a_n b_{n-1} - u_{n+1}^\delta a_{n+1} b_n \right) = 0. \end{cases} \tag{2.5}$$

Next, as another nontrivial isospectral deformation of the two-component spectral problem (2.1), we consider the following:

$$\begin{cases} \psi_{n,t_2} = -\alpha\gamma u_n^\gamma \psi_{n-1} + \beta\delta u_{n+1}^\delta \psi_{n+1} + \gamma\delta a_n b_n \psi_n, & (2.6a) \\ \phi_{n,t_2} = \alpha\gamma u_n^\gamma b_n \psi_{n-1} + \beta\delta u_n^\delta b_{n-1} \psi_n. & (2.6b) \end{cases}$$

The compatibility conditions for the overdetermined linear systems (2.1) and (2.6) provide the second flow of the generalized Toda hierarchy:

$$\begin{cases} a_{n,t_2} + \alpha\gamma u_n^\gamma a_{n-1} - \beta\delta u_{n+1}^\delta a_{n+1} - \gamma\delta a_n b_n a_n = 0, \\ b_{n,t_2} + \beta\delta u_n^\delta b_{n-1} - \alpha\gamma u_{n+1}^\gamma b_{n+1} + \gamma\delta b_n a_n b_n = 0, \\ u_{n,t_2} + u_n \left(w_{n-1} - w_n\right) - \left(\gamma - \delta\right) u_n \left(a_{n-1} b_{n-1} - a_n b_n\right) = 0, \\ w_{n,t_2} + \alpha\beta\left(\gamma + \delta\right)\left(u_n^{\gamma+\delta} - u_{n+1}^{\gamma+\delta}\right) - \alpha\gamma\delta\left(u_n^\gamma a_{n-1} b_n - u_{n+1}^\gamma a_n b_{n+1}\right) \\ \quad + \beta\gamma\delta\left(u_n^\delta a_n b_{n-1} - u_{n+1}^\delta a_{n+1} b_n\right) = 0. \end{cases} \tag{2.7}$$

Clearly, (2.7) is a generalization of the Toda lattice in Flaschka–Manakov variables (1.2), wherein a_n and b_n are newly added dependent variables. Note that the second flow (2.7) can be simplified by a change of variables $w_n - (\gamma - \delta)\, a_n b_n =: \widehat{w}_n$.

Remark. We call (2.5) and (2.7) the first flow and the second flow, respectively, just for convenience. In fact, one can call any linear combination of (2.5) and (2.7) the first (or second) flow.

Remark. We required the condition $(\alpha\gamma, \beta\delta) \neq (0, 0)$ for the spectral problem (2.1) to depend on the variable u_n. However, the above computation is valid even if this condition is removed; that is, in the special case $(\alpha\gamma, \beta\delta) = (0, 0)$, we obtain the isospectral flows (2.3), (2.5) and (2.7) without the equation of motion for u_n.

2.3 Gauge transformation

By applying the gauge transformation:

$$\psi_n = \widehat{\psi}_n \prod_{j=-\infty}^{n} u_j^\gamma, \quad a_n = \widehat{a}_n \prod_{j=-\infty}^{n} u_j^\gamma, \quad b_n = \widehat{b}_n \prod_{j=-\infty}^{n} u_j^{-\gamma}, \tag{2.8}$$

to the two-component spectral problem (2.1), we obtain

$$\begin{cases} \alpha\widehat{\psi}_{n-1} + \beta u_{n+1}^{\gamma+\delta} \widehat{\psi}_{n+1} + \left(w_n - \gamma \widehat{a}_n \widehat{b}_n\right) \widehat{\psi}_n + \left(\gamma + \delta\right) \widehat{a}_n \phi_{n+1} = \lambda \widehat{\psi}_n, & (2.9a) \\ \phi_{n+1} - \phi_n = \widehat{b}_n \widehat{\psi}_n. & (2.9b) \end{cases}$$

Thus, the condition $\gamma+\delta\neq 0$ is indeed crucial for the Lax pair to be unfake; otherwise, u_n is gauged away and the spectral problem does not involve the other dependent variables in a truly meaningful manner. By a redefinition of the dependent variables,

$$ku_n^{\gamma+\delta} =: \widetilde{u}_n, \quad w_n - \gamma\widehat{a}_n\widehat{b}_n =: \widetilde{w}_n, \quad (\gamma+\delta)\,\widehat{a}_n =: \widetilde{a}_n,$$

we can fully remove the parameters γ and δ from the spectral problem. Moreover, for nonzero values of α and β, we can apply a simple gauge transformation:

$$\widehat{\psi}_n =: \alpha^n\widetilde{\psi}_n, \quad \widetilde{a}_n =: \alpha^n\widetilde{\widetilde{a}}_n, \quad \widehat{b}_n =: \alpha^{-n}\widetilde{b}_n;$$

then, by choosing the constant k as $k=\alpha\beta$, we can remove the remaining parameters α and β from the spectral problem, which results in the representative case $\alpha=\beta=1$, $\gamma=0$ and $\delta=1$ in (2.1). Alternatively, we can consider the symmetric case $\alpha=\beta=\gamma=\delta=1$ as the representative of the spectral problem (2.1).

However, in the following, we mainly consider the original general form (2.1), because it encompasses some non-generic (say, $\alpha=0$) or limiting (say, $\gamma+\delta\to 0$) cases.

2.4 Symmetry properties

The spectral problem (2.1) and the isospectral flows (2.5) and (2.7) have two important symmetry properties:

(i) The spectral problem (2.1), the first flow (2.5) and the second flow (2.7) are invariant under the following transformation:
$\alpha\leftrightarrow\beta,\ \ \gamma\leftrightarrow\delta,\ \ \psi_n\to\psi_{-n},\ \ \phi_n\to\phi_{-n+1},\ \ w_n\to w_{-n},\ \ u_n\to u_{-n+1},$
$a_n\to a_{-n},\ \ b_n\to -b_{-n},\ \ t_2\to -t_2.$

(ii) The first flow (2.5) and the second flow (2.7) are invariant under the following transformation:
$\alpha\leftrightarrow\beta,\ \ \gamma\leftrightarrow\delta,\ \ a_n\to\pm b_n^T$ and $b_n\to\mp a_n^T$ (double sign in same order; the superscript ^{T} denotes the transpose of a vector), $\ t_1\to -t_1$.

2.5 Complex conjugation reduction

Reductions of an integrable system often result in more interesting systems than the original system from the point of view of physical or mathematical applications. For the generalized Toda hierarchy, we can impose a complex (or Hermitian) conjugation reduction on the additional dependent variables a_n and b_n, so the number of dependent variables can be diminished.

We first rescale the time variable in the first flow (2.5) as

$$\partial_{t_1} =: \mathrm{i}\partial_\tau.$$

Then, in the simple case of scalar u_n, w_n, a_n and b_n under the parametric conditions

$$\beta = \alpha^*, \quad \gamma = \delta \in \mathbb{R},$$

we can impose the complex conjugation reduction:

$$b_n = \mathrm{i}\sigma a_n^*, \quad u_n^* = u_n, \quad w_n^* = w_n,$$

where σ is an arbitrary real constant. For simplicity, we set $\alpha = \beta = \gamma = \delta = 1$ and $\sigma = 1$. Then, this reduction simplifies the first flow (2.5) to

$$\begin{cases} \mathrm{i}a_{n,\tau} = u_{n+1}a_{n+1} + w_n a_n + u_n a_{n-1}, \\ u_{n,\tau} = u_n \left(|a_n|^2 - |a_{n-1}|^2\right), \\ w_{n,\tau} = u_{n+1}\left(a_n a_{n+1}^* + a_{n+1}a_n^*\right) - u_n\left(a_{n-1}a_n^* + a_n a_{n-1}^*\right), \end{cases} \tag{2.10}$$

and the second flow (2.7) to

$$\begin{cases} a_{n,t_2} = u_{n+1}a_{n+1} - u_n a_{n-1} + \mathrm{i}\,|a_n|^2 a_n, \\ u_{n,t_2} = u_n\left(w_n - w_{n-1}\right), \\ w_{n,t_2} = 2\left(u_{n+1}^2 - u_n^2\right) - \mathrm{i}u_{n+1}\left(a_n a_{n+1}^* - a_{n+1}a_n^*\right) + \mathrm{i}u_n\left(a_{n-1}a_n^* - a_n a_{n-1}^*\right), \end{cases} \tag{2.11}$$

respectively, where $a_n \in \mathbb{C}$ and $u_n, w_n \in \mathbb{R}$. Note that (2.11) reduces to the Toda lattice in Flaschka–Manakov variables (1.2) by setting $a_n = 0$.

In the general case of scalar u_n and w_n, row-vector a_n and column-vector b_n, the generalized Toda hierarchy under the parametric conditions

$$\beta = \alpha^*, \quad \gamma = \delta \in \mathbb{R},$$

admits the Hermitian conjugation reduction:

$$b_n = \mathrm{i}\Sigma a_n^\dagger, \quad u_n^* = u_n, \quad w_n^* = w_n,$$

where Σ is an arbitrary constant Hermitian matrix. With the aid of linear transformations acting on the vector components of a_n, we can recast Σ in the canonical form: $\Sigma = \mathrm{diag}\,(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$. Moreover, if we exclude the uninteresting case of triangular (i.e., not truly coupled) systems, the diagonal elements of Σ must be either $+1$ or -1. Note that the first example of an integrable system with a cubic nonlinearity of mixed signs is

the vector nonlinear Schrödinger equation with both focusing and defocusing components [18–21].

In the simplest case of $\alpha = \beta = \gamma = \delta = 1$ and $\Sigma = I$, this reduction simplifies the first flow (2.5) (with $\partial_{t_1} = \mathrm{i}\partial_\tau$) to

$$\begin{cases} \mathrm{i}a_{n,\tau} = u_{n+1}a_{n+1} + w_n a_n + u_n a_{n-1}, \\ u_{n,\tau} = u_n \left(\langle a_n, a_n^* \rangle - \langle a_{n-1}, a_{n-1}^* \rangle \right), \\ w_{n,\tau} = u_{n+1} \left(\langle a_n, a_{n+1}^* \rangle + \langle a_{n+1}, a_n^* \rangle \right) - u_n \left(\langle a_{n-1}, a_n^* \rangle + \langle a_n, a_{n-1}^* \rangle \right), \end{cases} \tag{2.12}$$

and the second flow (2.7) to

$$\begin{cases} a_{n,t_2} = u_{n+1}a_{n+1} - u_n a_{n-1} + \mathrm{i}\langle a_n, a_n^* \rangle a_n, \\ u_{n,t_2} = u_n \left(w_n - w_{n-1} \right), \\ w_{n,t_2} = 2\left(u_{n+1}^2 - u_n^2 \right) - \mathrm{i}u_{n+1} \left(\langle a_n, a_{n+1}^* \rangle - \langle a_{n+1}, a_n^* \rangle \right) + \mathrm{i}u_n \left(\langle a_{n-1}, a_n^* \rangle - \langle a_n, a_{n-1}^* \rangle \right), \end{cases} \tag{2.13}$$

respectively, where $a_n \in \mathbb{C}^M$ and $u_n, w_n \in \mathbb{R}$.

2.6 Continuous limit to the Yajima–Oikawa system

The first flow of the generalized Toda hierarchy can be reduced to the long wave–short wave interaction model, called the Yajima–Oikawa system [9], in a suitable continuous limit. To see this, we first rescale the dependent variable a_n in (2.10) as $a_n \to \Delta^{\frac{1}{2}} a_n$, where Δ is a lattice parameter, to obtain

$$\begin{cases} \mathrm{i}a_{n,\tau} = u_{n+1}a_{n+1} + w_n a_n + u_n a_{n-1}, \\ u_{n,\tau} = \Delta u_n \left(|a_n|^2 - |a_{n-1}|^2 \right), \\ w_{n,\tau} = \Delta \left[u_{n+1} \left(a_n a_{n+1}^* + a_{n+1} a_n^* \right) - u_n \left(a_{n-1} a_n^* + a_n a_{n-1}^* \right) \right]. \end{cases} \tag{2.14}$$

Alternatively, (2.14) can be directly obtained from (2.5) in the scalar case by setting $\partial_{t_1} = \mathrm{i}\partial_\tau$, $\alpha = \beta = \gamma = \delta = 1$, $b_n = \mathrm{i}\Delta a_n^*$, $u_n^* = u_n$ and $w_n^* = w_n$. Then, by further setting

$$a_n = a(n\Delta, \tau), \quad u_n = \frac{1}{\Delta^2} + \frac{1}{2}u(n\Delta, \tau), \quad w_n = -\frac{2}{\Delta^2} + w(n\Delta, \tau),$$

and taking the continuous limit $\Delta \to 0$, (2.14) reduces to

$$\begin{cases} \mathrm{i}a_\tau = a_{xx} + (u + w)\, a, \\ u_\tau = w_\tau = 2\left(|a|^2 \right)_x, \end{cases}$$

where $x := n\Delta$. This is indeed the Yajima–Oikawa system [9] for the pair of dependent variables $(a, u+w)$, up to a trivial rescaling and Galilean transformation. The Lax-pair representation for this Yajima–Oikawa system can be obtained from (2.1) and (2.4) by taking the same continuous limit and thus is given by

$$\begin{cases} \psi_{xx} + (u+w)\,\psi + 2a\phi = \lambda\psi, \\ \phi_x = \mathrm{i}a^*\psi, \\ \mathrm{i}\psi_\tau = \psi_{xx} + (u+w)\,\psi, \\ \phi_\tau = a^*\psi_x - a_x^*\psi. \end{cases}$$

In the same manner, (2.12) can be reduced to the vector generalization of the Yajima–Oikawa system studied in [22–25].

3 Special cases: discrete nonlinear Schrödinger hierarchies and linearization

3.1 Reductions

A salient feature of the generalized Toda hierarchy is that one can equate the original Flaschka–Manakov variables u_n and w_n to some functions of the newly introduced variables a_n and b_n. This can be checked by a direct calculation for the simpler case of $\gamma = 0$ or $\delta = 0$. In view of the symmetry property (i) with respect to the space reflection as described in subsection 2.4, we consider the case $\gamma = 0$.

In the case $\gamma = 0$, the two-component spectral problem (2.1) reads

$$\begin{cases} \alpha\psi_{n-1} + \beta u_{n+1}^{\delta}\psi_{n+1} + w_n\psi_n + \delta a_n\phi_{n+1} = \lambda\psi_n, & \text{(3.1a)} \\ \phi_{n+1} - \phi_n = b_n\psi_n, & \text{(3.1b)} \end{cases}$$

while the first flow (2.5) and the second flow (2.7) read

$$\begin{cases} a_{n,t_1} - \alpha a_{n-1} - \beta u_{n+1}^{\delta} a_{n+1} - w_n a_n = 0, \\ b_{n,t_1} + \beta u_n^{\delta} b_{n-1} + \alpha b_{n+1} + b_n w_n = 0, \\ u_{n,t_1} + u_n\left(a_{n-1}b_{n-1} - a_n b_n\right) = 0, \\ w_{n,t_1} + \alpha\delta\left(a_{n-1}b_n - a_n b_{n+1}\right) = 0, \end{cases} \tag{3.2}$$

and

$$\begin{cases} a_{n,t_2} - \beta\delta u_{n+1}^{\delta} a_{n+1} = 0, \\ b_{n,t_2} + \beta\delta u_n^{\delta} b_{n-1} = 0, \\ u_{n,t_2} + u_n \left(w_{n-1} - w_n\right) + \delta u_n \left(a_{n-1}b_{n-1} - a_n b_n\right) = 0, \\ w_{n,t_2} + \alpha\beta\delta \left(u_n^{\delta} - u_{n+1}^{\delta}\right) = 0, \end{cases} \tag{3.3}$$

respectively.

A direct calculation shows that the following proposition holds true.

Proposition 3.1. *Both* (3.2) *and* (3.3)*, as well as any linear combination of them generated by* $\partial_T := \mu\partial_{t_1} + \nu\partial_{t_2}$*, admit the reduction that* u_n^{δ} *and* w_n *can be expressed as power series in* α *of the form:*

$$\begin{aligned} \beta u_n^{\delta} &= k + \delta a_{n-1}b_n + \alpha\frac{\delta}{k}a_{n-2}\left(I + \frac{\delta}{k}b_n a_{n-1}\right)b_{n+1} \\ &\quad + \alpha^2 u_n^{(2)} + \alpha^3 u_n^{(3)} + \cdots, \end{aligned} \tag{3.4a}$$

$$\begin{aligned} w_n &= \alpha\frac{\delta}{k}a_{n-1}b_{n+1} + \alpha^2\frac{\delta}{k^2}a_{n-2}\left(I + \frac{\delta}{k}b_n a_{n-1}\right)\left(I + \frac{\delta}{k}b_{n+1}a_n\right)b_{n+2} \\ &\quad + \alpha^3 w_n^{(3)} + \alpha^4 w_n^{(4)} + \cdots. \end{aligned} \tag{3.4b}$$

Here, k *is an arbitrary nonzero constant and* I *is the identity matrix of the same size as* $b_n a_{n-1}$*.*

We conjecture that the power series expansion (3.4) is common to all the isospectral flows associated with the spectral problem (3.1) and all the coefficients in the power series are local functions of a_n and b_n.

Let us recall that the gauge transformation (2.8) changes the spectral problem (2.1) to the spectral problem (2.9). Thus, by comparing (3.1) with (2.9), we find that the replacement:

$$\delta \to \gamma + \delta, \quad a_n \to a_n \prod_{j=-\infty}^{n} u_j^{-\gamma}, \quad b_n \to b_n \prod_{j=-\infty}^{n} u_j^{\gamma}, \quad w_n \to w_n - \gamma a_n b_n,$$

converts the reduction (3.4) in the special case $\gamma = 0$ to the reduction in the general case $\gamma \neq 0$.

Proposition 3.2. *The first flow* (2.5) *and the second flow* (2.7)*, as well as any linear combination of them generated by* $\partial_T := \mu\partial_{t_1} + \nu\partial_{t_2}$*, admit the reduction that* u_n *and* w_n *are determined implicitly by power series in* α *of*

the form:

$$\beta u_n^{\gamma+\delta} = k + (\gamma+\delta)\, u_n^{\gamma} a_{n-1} b_n + \alpha \frac{\gamma+\delta}{k} u_{n-1}^{\gamma} u_n^{\gamma} u_{n+1}^{\gamma} a_{n-2} \left(I + \frac{\gamma+\delta}{k} u_n^{\gamma} b_n a_{n-1} \right) b_{n+1}$$
$$+ \alpha^2 u_n^{(2)} + \alpha^3 u_n^{(3)} + \cdots ,$$

$$w_n = \gamma a_n b_n + \alpha \frac{\gamma+\delta}{k} u_n^{\gamma} u_{n+1}^{\gamma} a_{n-1} b_{n+1}$$
$$+ \alpha^2 \frac{\gamma+\delta}{k^2} u_{n-1}^{\gamma} u_n^{\gamma} u_{n+1}^{\gamma} u_{n+2}^{\gamma} a_{n-2} \left(I + \frac{\gamma+\delta}{k} u_n^{\gamma} b_n a_{n-1} \right) \left(I + \frac{\gamma+\delta}{k} u_{n+1}^{\gamma} b_{n+1} a_n \right) b_{n+2}$$
$$+ \alpha^3 w_n^{(3)} + \alpha^4 w_n^{(4)} + \cdots .$$

Here, k is an arbitrary nonzero constant and I is the identity matrix of the same size as $b_n a_{n-1}$.

With the aid of the symmetry property (i) or (ii) described in subsection 2.4, we also obtain a similar reduction that expresses u_n and w_n as power series in β whose coefficients are functions of a_n and b_n.

3.2 Ablowitz–Ladik hierarchy

In the non-generic case of $\alpha = \gamma = 0$ or $\beta = \delta = 0$, the form of the first flow (2.5) (or the second flow (2.7)) implies that we can set $w_n = 0$ in the two-component spectral problem (2.1) (cf. Proposition 3.1). In view of the symmetry property (i) described in subsection 2.4, we consider only the case $\alpha = \gamma = 0$. Then, by setting $\alpha = \gamma = 0$ and $w_n = 0$, the first flow (2.5) reduces to

$$\begin{cases} a_{n,t_1} - \beta u_{n+1}^{\delta} a_{n+1} = 0, \\ b_{n,t_1} + \beta u_n^{\delta} b_{n-1} = 0, \\ u_{n,t_1} + u_n \left(a_{n-1} b_{n-1} - a_n b_n \right) = 0, \end{cases} \tag{3.5}$$

and the second flow (2.7) reduces to

$$\begin{cases} a_{n,t_2} - \beta\delta u_{n+1}^{\delta} a_{n+1} = 0, \\ b_{n,t_2} + \beta\delta u_n^{\delta} b_{n-1} = 0, \\ u_{n,t_2} + \delta u_n \left(a_{n-1} b_{n-1} - a_n b_n \right) = 0, \end{cases} \tag{3.6}$$

respectively. We recall the condition $\beta\delta \neq 0$ for the spectral problem (2.1) with $\alpha = \gamma = 0$ and $w_n = 0$ (and consequently (3.5) and (3.6)) to depend on u_n. Then, (3.6) is equivalent to (3.5) up to a rescaling of the time variable,

so we consider only (3.5). Note that (3.5) with $\beta = -1$ and $\delta = 1$ coincides with (4.19) or (4.21) in [26]. Because (3.5) implies the relation:

$$\left(\beta u_n^\delta - \delta a_{n-1} b_n\right)_{t_1} = 0,$$

we can set

$$\beta u_n^\delta = k_n + \delta a_{n-1} b_n,$$

where k_n is independent of time t_1. Thus, the three-component system (3.5) is simplified to the two-component system:

$$\begin{cases} a_{n-1,t_1} - \left(k_n + \delta a_{n-1} b_n\right) a_n = 0, \\ b_{n,t_1} + \left(k_n + \delta a_{n-1} b_n\right) b_{n-1} = 0. \end{cases}$$

By a rescaling of a_n and b_n, we can normalize δ to 1. Moreover, for nonzero values of k_n, k_n can be fixed at 1 by the simple transformation

$$a_{n-1} = \left(\prod_j^{n-1} \frac{1}{k_j}\right) \boldsymbol{a}_n, \quad b_n = \left(\prod_j^{n} k_j\right) \boldsymbol{b}_n.$$

Here, $\boldsymbol{a}_n$ is a row vector and $\boldsymbol{b}_n$ is a column vector. Thus, we finally obtain (the straightforward vector generalization [27] of) an elementary flow of the Ablowitz–Ladik hierarchy [14]:

$$\begin{cases} \boldsymbol{a}_{n,t_1} - \left(1 + \boldsymbol{a}_n \boldsymbol{b}_n\right) \boldsymbol{a}_{n+1} = 0, \\ \boldsymbol{b}_{n,t_1} + \left(1 + \boldsymbol{a}_n \boldsymbol{b}_n\right) \boldsymbol{b}_{n-1} = 0. \end{cases} \tag{3.7}$$

The derivation given above implies that the Lax-pair representation for the Ablowitz–Ladik flow (3.7) is given by the two-component spectral problem

$$\begin{cases} \left(1 + \boldsymbol{a}_{n+1} \boldsymbol{b}_{n+1}\right) \psi_{n+1} + \boldsymbol{a}_{n+1} \phi_{n+1} = \lambda \psi_n, & \text{(3.8a)} \\ \phi_{n+1} - \phi_n = \boldsymbol{b}_n \psi_n, & \text{(3.8b)} \end{cases}$$

and the isospectral time-evolutionary system

$$\begin{cases} \psi_{n,t_1} = \left(1 + \boldsymbol{a}_{n+1} \boldsymbol{b}_{n+1}\right) \psi_{n+1}, \\ \phi_{n,t_1} = \left(1 + \boldsymbol{a}_n \boldsymbol{b}_n\right) \boldsymbol{b}_{n-1} \psi_n, \end{cases}$$

where ψ_n is a scalar and ϕ_n is a column vector.

To obtain the other elementary flow of the Ablowitz–Ladik hierarchy, we set $\gamma = 0$ and consider a suitable linear combination of the first flow (3.2) and the second flow (3.3) in the case $\alpha \sim 0$. Indeed, by setting

$$\partial_T := \frac{1}{\alpha\delta} \left(\partial_{t_2} - \delta \partial_{t_1}\right),$$

and imposing the reduction stated in Proposition 3.1,

$$\beta u_n^\delta = k + \delta a_{n-1} b_n + O(\alpha),$$
$$\frac{1}{\alpha} w_n = \frac{\delta}{k} a_{n-1} b_{n+1} + O(\alpha),$$

we have

$$\begin{cases} a_{n,T} + a_{n-1} + \left(\frac{\delta}{k} a_{n-1} b_{n+1} + O(\alpha) \right) a_n = 0, \\ b_{n,T} - b_{n+1} - b_n \left(\frac{\delta}{k} a_{n-1} b_{n+1} + O(\alpha) \right) = 0. \end{cases}$$

Thus, by taking the limit $\alpha \to 0$, setting $k = \delta = 1$ and writing $a_{n-1} = \boldsymbol{a}_n$ and $b_n = \boldsymbol{b}_n$, we obtain (the vector generalization [27] of) the other elementary flow of the Ablowitz–Ladik hierarchy [14]:

$$\begin{cases} \boldsymbol{a}_{n,T} + \boldsymbol{a}_{n-1} + \boldsymbol{a}_{n-1} \boldsymbol{b}_n \boldsymbol{a}_n = 0, \\ \boldsymbol{b}_{n,T} - \boldsymbol{b}_{n+1} - \boldsymbol{b}_n \boldsymbol{a}_n \boldsymbol{b}_{n+1} = 0. \end{cases} \tag{3.9}$$

The Lax-pair representation for (3.9) is given by the two-component spectral problem (3.8) and the isospectral time-evolutionary system

$$\begin{cases} \psi_{n,T} = -\boldsymbol{a}_n \boldsymbol{b}_{n+1} \psi_n - \psi_{n-1}, \\ \phi_{n,T} = \boldsymbol{b}_n \psi_{n-1}. \end{cases}$$

3.3 Konopelchenko–Chudnovsky hierarchy

Let us first recall the remark at the end of subsection 2.2. In the special case $\alpha = \delta = 0$, the first flow (2.5) (without the equation of motion for u_n) reads [26, 28]

$$\begin{cases} a_{n,t_1} - \beta a_{n+1} - w_n a_n = 0, \\ b_{n,t_1} + \beta b_{n-1} + b_n w_n = 0, \\ w_{n,t_1} + \beta\gamma \left(a_n b_{n-1} - a_{n+1} b_n \right) = 0, \end{cases} \tag{3.10}$$

which implies the relation:

$$\left(w_n - \gamma a_n b_n \right)_{t_1} = 0.$$

Thus, we can set

$$w_n = \mu_n + \gamma a_n b_n,$$

where μ_n is an arbitrary function of n, independent of time t_1. Then, the three-component system (3.10) is simplified to the two-component system:

$$\begin{cases} a_{n,t_1} - \beta a_{n+1} - (\mu_n + \gamma a_n b_n)\, a_n = 0, \\ b_{n,t_1} + \beta b_{n-1} + b_n\, (\mu_n + \gamma a_n b_n) = 0. \end{cases} \tag{3.11}$$

If we interpret t_1 as the continuous spatial variable x, (3.11) can be identified with an elementary auto-Bäcklund transformation for the continuous nonlinear Schrödinger hierarchy studied by Konopelchenko [15] and D. V. and G. V. Chudnovsky [16,17]; in this context, the n-dependence of μ_n is essential and not removable by any simple transformation. In the simplest case $\mu_n = 0$, (3.11) provides an elementary flow [27] of an integrable discrete nonlinear Schrödinger hierarchy, which we call the Konopelchenko–Chudnovsky hierarchy [15–17].

To obtain the other elementary flow [27,29] of the Konopelchenko–Chudnovsky hierarchy, we consider the second flow (2.7) in the case $\delta = 0$, $\alpha \sim 0$. The second flow (2.7) with $\delta = 0$ reads

$$\begin{cases} a_{n,t_2} + \alpha\gamma u_n^\gamma a_{n-1} = 0, \\ b_{n,t_2} - \alpha\gamma u_{n+1}^\gamma b_{n+1} = 0, \\ u_{n,t_2} + u_n\,(w_{n-1} - w_n) - \gamma u_n\,(a_{n-1}b_{n-1} - a_n b_n) = 0, \\ w_{n,t_2} + \alpha\beta\gamma\left(u_n^\gamma - u_{n+1}^\gamma\right) = 0. \end{cases} \tag{3.12}$$

We rescale the time derivative as

$$\partial_{t_{-1}} := \frac{1}{\alpha\gamma}\partial_{t_2},$$

and consider the reduction guaranteed by Proposition 3.2:

$$\begin{aligned} u_n^\gamma &= \frac{k}{\beta - \gamma a_{n-1} b_n} + O(\alpha), \\ w_n &= \gamma a_n b_n + O(\alpha). \end{aligned}$$

Thus, in the limit $\alpha \to 0$, (3.12) under this reduction reduces to the other elementary flow [27, 29] of the Konopelchenko–Chudnovsky hierarchy:

$$\begin{cases} a_{n,t_{-1}} + \dfrac{k}{\beta - \gamma a_{n-1} b_n} a_{n-1} = 0, \\ b_{n,t_{-1}} - \dfrac{k}{\beta - \gamma a_n b_{n+1}} b_{n+1} = 0. \end{cases}$$

3.4 Exceptional case: linearization

For the spectral problem (2.1) to involve the dependent variables in a meaningful manner, we assumed the condition $\gamma+\delta\neq 0$ (see subsection 2.3). Thus, the case $\gamma+\delta=0$ is exceptional and excluded from our consideration. In fact, the first flow (2.5) and the second flow (2.7) of the generalized Toda hierarchy in this exceptional case turn out to be linearizable.

The first flow (2.5) with $\delta=-\gamma$ is

$$\begin{cases} a_{n,t_1}-\alpha u_n^{\gamma}a_{n-1}-\beta u_{n+1}^{-\gamma}a_{n+1}-w_na_n=0,\\ b_{n,t_1}+\beta u_n^{-\gamma}b_{n-1}+\alpha u_{n+1}^{\gamma}b_{n+1}+b_nw_n=0,\\ u_{n,t_1}+u_n\left(a_{n-1}b_{n-1}-a_nb_n\right)=0,\\ w_{n,t_1}-\alpha\gamma\left(u_n^{\gamma}a_{n-1}b_n-u_{n+1}^{\gamma}a_nb_{n+1}\right)+\beta\gamma\left(u_n^{-\gamma}a_nb_{n-1}-u_{n+1}^{-\gamma}a_{n+1}b_n\right)=0. \end{cases} \tag{3.13}$$

From (3.13), we obtain the relations:

$$\left\{\exp\left[\int\left(\lim_{n\to-\infty}a_nb_n\right)\mathrm{d}t_1\right]\prod_{j=-\infty}^{n}u_j\right\}_{t_1}=a_nb_n\left\{\exp\left[\int\left(\lim_{n\to-\infty}a_nb_n\right)\mathrm{d}t_1\right]\prod_{j=-\infty}^{n}u_j\right\},$$

and

$$\left(w_n-\gamma a_nb_n\right)_{t_1}=0.$$

Thus, the new dependent variables defined as

$$\mathcal{A}_n:=\left\{\exp\left[\int\left(\lim_{n\to-\infty}a_nb_n\right)\mathrm{d}t_1\right]\prod_{j=-\infty}^{n}u_j\right\}^{-\gamma}a_n,$$

$$\mathcal{B}_n:=b_n\left\{\exp\left[\int\left(\lim_{n\to-\infty}a_nb_n\right)\mathrm{d}t_1\right]\prod_{j=-\infty}^{n}u_j\right\}^{\gamma},$$

satisfy the linear equations with n-dependent (but t_1-independent) coefficients:

$$\begin{cases} \mathcal{A}_{n,t_1}-\alpha\mathcal{A}_{n-1}-\beta\mathcal{A}_{n+1}-\left(w_n-\gamma a_nb_n\right)\mathcal{A}_n=0,\\ \mathcal{B}_{n,t_1}+\beta\mathcal{B}_{n-1}+\alpha\mathcal{B}_{n+1}+\mathcal{B}_n\left(w_n-\gamma a_nb_n\right)=0. \end{cases}$$

The second flow (2.7) with $\delta=-\gamma$:

$$\begin{cases} a_{n,t_2}+\alpha\gamma u_n^{\gamma}a_{n-1}+\beta\gamma u_{n+1}^{-\gamma}a_{n+1}+\gamma^2a_nb_na_n=0,\\ b_{n,t_2}-\beta\gamma u_n^{-\gamma}b_{n-1}-\alpha\gamma u_{n+1}^{\gamma}b_{n+1}-\gamma^2b_na_nb_n=0,\\ u_{n,t_2}+u_n\left(w_{n-1}-w_n\right)-2\gamma u_n\left(a_{n-1}b_{n-1}-a_nb_n\right)=0,\\ w_{n,t_2}+\alpha\gamma^2\left(u_n^{\gamma}a_{n-1}b_n-u_{n+1}^{\gamma}a_nb_{n+1}\right)-\beta\gamma^2\left(u_n^{-\gamma}a_nb_{n-1}-u_{n+1}^{-\gamma}a_{n+1}b_n\right)=0, \end{cases}$$

can be linearized in a similar manner.

4 Hamiltonian structure and Conservation laws

In the special case $\gamma = \delta$, the generalized Toda hierarchy possesses a local Hamiltonian structure; the trivial zeroth flow (2.3), the first flow (2.5) and the second flow (2.7) with $\gamma = \delta$ can be written in the Hamiltonian form:

$$a_{n,t_l}^{(j)} = \left\{a_n^{(j)}, H_l\right\}, \quad b_{n,t_l}^{(j)} = \left\{b_n^{(j)}, H_l\right\}, \quad u_{n,t_l} = \left\{u_n, H_l\right\}, \quad w_{n,t_l} = \left\{w_n, H_l\right\}, \quad l = 0, 1, 2,$$

where $a_n^{(j)}$ is the jth component of the row vector a_n and $b_n^{(j)}$ is the jth component of the column vector b_n. The set of nonvanishing Poisson brackets and Hamiltonians are given by

$$\left\{a_n^{(j)}, b_n^{(k)}\right\} = \frac{1}{\kappa}\delta_{jk}, \quad \{u_n, w_n\} = \frac{1}{\kappa}u_n, \quad \{u_n, w_{n-1}\} = -\frac{1}{\kappa}u_n, \tag{4.1}$$

and

$$H_0 = \kappa\,(c-d)\sum_n a_n b_n, \tag{4.2a}$$

$$H_1 = \kappa\sum_n \left(w_n a_n b_n + \alpha u_n^\gamma a_{n-1} b_n + \beta u_n^\gamma a_n b_{n-1}\right), \tag{4.2b}$$

$$H_2 = \kappa\sum_n \left[\frac{1}{2}w_n^2 + \frac{1}{2}\gamma^2\left(a_n b_n\right)^2 - \alpha\gamma u_n^\gamma a_{n-1} b_n + \beta\gamma u_n^\gamma a_n b_{n-1} + \alpha\beta u_n^{2\gamma}\right], \tag{4.2c}$$

respectively. Here, κ is an arbitrary nonzero constant and δ_{jk} is the Kronecker delta; $\sum_n \log u_n$ and $\sum_n w_n$ are Casimir functions. Note that (4.1) (especially with $\kappa = 4$) is a natural extension of the canonical Poisson bracket relations for the Toda lattice written in Flaschka–Manakov variables [3–5].

We present a recursive method for constructing two infinite sets of conservation laws of the generalized Toda hierarchy; in the special case $\gamma = \delta$, the method allows us to generate the higher flows of the hierarchy with the aid of the Poisson bracket relations (4.1).

First, we set

$$\frac{\psi_n}{\psi_{n+1}} =: \frac{\beta}{\lambda}u_{n+1}^\delta J_n, \quad \frac{\phi_{n+1}}{\psi_n} =: K_n, \tag{4.3}$$

and rewrite the two-component spectral problem (2.1) as the relations for J_n and K_n:

$$\begin{cases} J_n = 1 + \dfrac{1}{\lambda}\left(w_n - \gamma a_n b_n\right) J_n + \dfrac{\gamma+\delta}{\lambda} a_n K_n J_n + \dfrac{\alpha\beta}{\lambda^2} u_n^{\gamma+\delta} J_{n-1} J_n, & (4.4a) \\ K_n = b_n + \dfrac{\beta}{\lambda} u_n^\delta K_{n-1} J_{n-1}. & (4.4b) \end{cases}$$

Here, the condition $\beta \neq 0$ is assumed to derive (4.4a), but this assumption is not essential and can in fact be removed. Using the relations (4.4) recursively, we can express J_n and K_n as power series in $1/\lambda$:

$$J_n = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} J_n^{(j)}, \qquad K_n = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} K_n^{(j)}.$$

More specifically, we have the following power series expansions:

$$\begin{aligned}
J_n = 1 &+ \frac{1}{\lambda}\left(w_n + \delta a_n b_n\right) + \frac{1}{\lambda^2}\left[\left(w_n + \delta a_n b_n\right)^2 + (\gamma+\delta)\,\beta u_n^{\delta} a_n b_{n-1} + \alpha\beta u_n^{\gamma+\delta}\right] \\
&+ \frac{1}{\lambda^3}\Big\{\left(w_n + \delta a_n b_n\right)\left[\left(w_n + \delta a_n b_n\right)^2 + (\gamma+\delta)\,\beta u_n^{\delta} a_n b_{n-1} + \alpha\beta u_n^{\gamma+\delta}\right] \\
&+ (\gamma+\delta)\,\beta u_n^{\delta} a_n b_{n-1}\left(w_n + \delta a_n b_n\right) \\
&+ (\gamma+\delta)\,\beta u_n^{\delta}\left[\beta u_{n-1}^{\delta} a_n b_{n-2} + a_n b_{n-1}\left(w_{n-1} + \delta a_{n-1} b_{n-1}\right)\right] \\
&+ \alpha\beta u_n^{\gamma+\delta}\left(w_{n-1} + \delta a_{n-1} b_{n-1} + w_n + \delta a_n b_n\right)\Big\} \\
&+ O\left(\frac{1}{\lambda^4}\right), \\
K_n = b_n &+ \frac{1}{\lambda}\beta u_n^{\delta} b_{n-1} + \frac{1}{\lambda^2}\beta u_n^{\delta}\left[\beta u_{n-1}^{\delta} b_{n-2} + b_{n-1}\left(w_{n-1} + \delta a_{n-1} b_{n-1}\right)\right] \\
&+ O\left(\frac{1}{\lambda^3}\right).
\end{aligned}$$

We consider the identity [30, 31]:

$$\left[\log\left(\frac{\psi_n}{\psi_{n+1}}\right)\right]_{t_j} + \boldsymbol{\Delta}_n^{+}\left[\frac{\psi_{n,t_j}}{\psi_n}\right] = 0, \qquad j = 0, 1, 2, \tag{4.5}$$

where $\boldsymbol{\Delta}_n^{+}$ is the forward difference operator ($\boldsymbol{\Delta}_n^{+} f_n := f_{n+1} - f_n$); (4.5) can be rewritten with the aid of (4.3) and (2.2a), (2.4a) or (2.6a) as the conservation laws:

$$\left[\delta \log u_{n+1} + \log J_n\right]_{t_0} = 0, \tag{4.6a}$$

$$\left[\delta \log u_{n+1} + \log J_n\right]_{t_1} + \boldsymbol{\Delta}_n^{+}\left[\frac{\alpha\beta}{\lambda} u_n^{\gamma+\delta} J_{n-1} + \frac{\lambda}{J_n} + w_n\right] = 0, \tag{4.6b}$$

$$\left[\delta \log u_{n+1} + \log J_n\right]_{t_2} + \boldsymbol{\Delta}_n^{+}\left[-\frac{\alpha\beta\gamma}{\lambda} u_n^{\gamma+\delta} J_{n-1} + \frac{\delta\lambda}{J_n} + \gamma\delta a_n b_n\right] = 0. \tag{4.6c}$$

Substituting the power series expansion for J_n into (4.6) and equating the coefficients of different powers of $1/\lambda$ on the left-hand side to zero, we obtain

an infinite set of conservation laws for the first three flows of the generalized Toda hierarchy. Note that $\delta \log u_{n+1} + \log J_n$ is a generating function of the conserved densities; the first four conserved densities obtained in this manner are

$$
\begin{aligned}
I_n^{(0)} &= \log u_n, \\
I_n^{(1)} &= w_n + \delta a_n b_n, \\
I_n^{(2)} &= \frac{1}{2}\left(w_n + \delta a_n b_n\right)^2 + (\gamma+\delta)\,\beta u_n^{\delta} a_n b_{n-1} + \alpha\beta u_n^{\gamma+\delta}, \\
I_n^{(3)} &= \left[(\gamma+\delta)\,\beta u_n^{\delta} a_n b_{n-1} + \alpha\beta u_n^{\gamma+\delta}\right]\left(w_n + \delta a_n b_n + w_{n-1} + \delta a_{n-1} b_{n-1}\right) \\
&\quad + (\gamma+\delta)\,\beta^2 u_n^{\delta} u_{n-1}^{\delta} a_n b_{n-2} + \frac{1}{3}\left(w_n + \delta a_n b_n\right)^3 .
\end{aligned}
$$

Second, we set

$$
\frac{\psi_n}{\psi_{n-1}} =: \frac{\alpha}{\lambda} u_n^{\gamma} \mathcal{J}_n, \qquad \frac{\phi_n}{\psi_n} =: \mathcal{K}_n, \tag{4.7}
$$

and rewrite the two-component spectral problem (2.1) as the relations for $\mathcal{J}_n$ and $\mathcal{K}_n$:

$$
\begin{cases}
\mathcal{J}_n = 1 + \dfrac{1}{\lambda}\left(w_n + \delta a_n b_n\right)\mathcal{J}_n + \dfrac{\gamma+\delta}{\lambda} a_n \mathcal{K}_n \mathcal{J}_n + \dfrac{\alpha\beta}{\lambda^2} u_{n+1}^{\gamma+\delta} \mathcal{J}_{n+1} \mathcal{J}_n, & \text{(4.8a)} \\
\mathcal{K}_n = -b_n + \dfrac{\alpha}{\lambda} u_{n+1}^{\gamma} \mathcal{K}_{n+1} \mathcal{J}_{n+1}. & \text{(4.8b)}
\end{cases}
$$

Here, the condition $\alpha \neq 0$ is assumed to derive (4.8a), but this assumption is not essential and can in fact be removed. Using the relations (4.8) recursively, we can express $\mathcal{J}_n$ and $\mathcal{K}_n$ as power series in $1/\lambda$:

$$
\mathcal{J}_n = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} \mathcal{J}_n^{(j)}, \qquad \mathcal{K}_n = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} \mathcal{K}_n^{(j)}.
$$

More specifically, we have the following power series expansions:

$$
\begin{aligned}
\mathcal{J}_n = 1 &+ \frac{1}{\lambda}\left(w_n - \gamma a_n b_n\right) + \frac{1}{\lambda^2}\left[\left(w_n - \gamma a_n b_n\right)^2 - (\gamma+\delta)\,\alpha u_{n+1}^{\gamma} a_n b_{n+1} + \alpha\beta u_{n+1}^{\gamma+\delta}\right] \\
&+ \frac{1}{\lambda^3}\Big\{\left(w_n - \gamma a_n b_n\right)\left[\left(w_n - \gamma a_n b_n\right)^2 - (\gamma+\delta)\,\alpha u_{n+1}^{\gamma} a_n b_{n+1} + \alpha\beta u_{n+1}^{\gamma+\delta}\right] \\
&- (\gamma+\delta)\,\alpha u_{n+1}^{\gamma} a_n b_{n+1}\left(w_n - \gamma a_n b_n\right) \\
&- (\gamma+\delta)\,\alpha u_{n+1}^{\gamma}\left[\alpha u_{n+2}^{\gamma} a_n b_{n+2} + a_n b_{n+1}\left(w_{n+1} - \gamma a_{n+1} b_{n+1}\right)\right] \\
&+ \alpha\beta u_{n+1}^{\gamma+\delta}\left(w_{n+1} - \gamma a_{n+1} b_{n+1} + w_n - \gamma a_n b_n\right)\Big\} \\
&+ O\left(\frac{1}{\lambda^4}\right), \\
\mathcal{K}_n = -b_n &- \frac{1}{\lambda}\alpha u_{n+1}^{\gamma} b_{n+1} - \frac{1}{\lambda^2}\alpha u_{n+1}^{\gamma}\left[\alpha u_{n+2}^{\gamma} b_{n+2} + b_{n+1}\left(w_{n+1} - \gamma a_{n+1} b_{n+1}\right)\right] \\
&+ O\left(\frac{1}{\lambda^3}\right).
\end{aligned}
$$

We consider the identity [30, 31]:

$$
\left[\log\left(\frac{\psi_n}{\psi_{n-1}}\right)\right]_{t_j} - \boldsymbol{\Delta}_n^{+}\left[\frac{\psi_{n-1,t_j}}{\psi_{n-1}}\right] = 0, \qquad j = 0, 1, 2, \tag{4.9}
$$

where $\boldsymbol{\Delta}_n^{+}$ is the forward difference operator; (4.9) can be rewritten with the aid of (4.7) and (2.2a), (2.4a) or (2.6a) as the conservation laws:

$$
\left[\gamma \log u_n + \log \mathcal{J}_n\right]_{t_0} = 0, \tag{4.10a}
$$

$$
\left[\gamma \log u_n + \log \mathcal{J}_n\right]_{t_1} - \boldsymbol{\Delta}_n^{+}\left[\frac{\lambda}{\mathcal{J}_{n-1}} + \frac{\alpha\beta}{\lambda} u_n^{\gamma+\delta}\mathcal{J}_n + w_{n-1}\right] = 0, \tag{4.10b}
$$

$$
\left[\gamma \log u_n + \log \mathcal{J}_n\right]_{t_2} - \boldsymbol{\Delta}_n^{+}\left[-\frac{\gamma\lambda}{\mathcal{J}_{n-1}} + \frac{\alpha\beta\delta}{\lambda} u_n^{\gamma+\delta}\mathcal{J}_n + \gamma\delta a_{n-1} b_{n-1}\right] = 0. \tag{4.10c}
$$

Substituting the power series expansion for $\mathcal{J}_n$ into (4.10) and equating the coefficients of different powers of $1/\lambda$ on the left-hand side to zero, we obtain an infinite set of conservation laws for the first three flows of the generalized Toda hierarchy. Note that $\gamma \log u_n + \log \mathcal{J}_n$ is a generating function of the conserved densities; the first four conserved densities obtained in this manner

are

$$\mathcal{I}_n^{(0)} = \log u_n,$$

$$\mathcal{I}_n^{(1)} = w_n - \gamma a_n b_n,$$

$$\mathcal{I}_n^{(2)} = \frac{1}{2}\left(w_n - \gamma a_n b_n\right)^2 - (\gamma+\delta)\,\alpha u_{n+1}^{\gamma} a_n b_{n+1} + \alpha\beta u_{n+1}^{\gamma+\delta},$$

$$\begin{aligned}\mathcal{I}_n^{(3)} = &\left[-(\gamma+\delta)\,\alpha u_{n+1}^{\gamma} a_n b_{n+1} + \alpha\beta u_{n+1}^{\gamma+\delta}\right]\left(w_n - \gamma a_n b_n + w_{n+1} - \gamma a_{n+1} b_{n+1}\right)\\ &- (\gamma+\delta)\,\alpha^2 u_{n+1}^{\gamma} u_{n+2}^{\gamma} a_n b_{n+2} + \frac{1}{3}\left(w_n - \gamma a_n b_n\right)^3.\end{aligned}$$

Thus, two infinite sets of conservation laws for the generalized Toda hierarchy can be obtained. We consider linear combinations of the conserved densities as

$$\frac{1}{\gamma+\delta}\left(I_n^{(1)} - \mathcal{I}_n^{(1)}\right) = a_n b_n, \tag{4.11a}$$

$$\frac{1}{\gamma+\delta}\left(\gamma I_n^{(1)} + \delta \mathcal{I}_n^{(1)}\right) = w_n,$$

$$\frac{1}{\gamma+\delta}\left(I_n^{(2)} - \mathcal{I}_n^{(2)}\right) \equiv w_n a_n b_n - \frac{1}{2}(\gamma-\delta)\left(a_n b_n\right)^2 + \alpha u_n^{\gamma} a_{n-1} b_n + \beta u_n^{\delta} a_n b_{n-1}, \tag{4.11b}$$

$$\frac{1}{\gamma+\delta}\left(\gamma I_n^{(2)} + \delta \mathcal{I}_n^{(2)}\right) \equiv \frac{1}{2} w_n^2 + \frac{1}{2}\gamma\delta\left(a_n b_n\right)^2 - \alpha\delta u_n^{\gamma} a_{n-1} b_n + \beta\gamma u_n^{\delta} a_n b_{n-1} + \alpha\beta u_n^{\gamma+\delta}, \tag{4.11c}$$

$$\begin{aligned}\frac{1}{\gamma+\delta}\left(I_n^{(3)} - \mathcal{I}_n^{(3)}\right) \equiv\ & w_n^2 a_n b_n - (\gamma-\delta)\, w_n \left(a_n b_n\right)^2 + \frac{1}{3}\left(\gamma^2 - \gamma\delta + \delta^2\right)\left(a_n b_n\right)^3\\ &+ \alpha u_{n+1}^{\gamma} a_n b_{n+1}\left(w_n - \gamma a_n b_n + w_{n+1} - \gamma a_{n+1} b_{n+1}\right)\\ &+ \beta u_n^{\delta} a_n b_{n-1}\left(w_{n-1} + \delta a_{n-1} b_{n-1} + w_n + \delta a_n b_n\right)\\ &+ \alpha^2 u_{n+1}^{\gamma} u_{n+2}^{\gamma} a_n b_{n+2} + \beta^2 u_n^{\delta} u_{n-1}^{\delta} a_n b_{n-2} + \alpha\beta u_n^{\gamma+\delta}\left(a_{n-1} b_{n-1} + a_n b_n\right),\end{aligned} \tag{4.11d}$$

$$\begin{aligned}\frac{1}{\gamma+\delta}\left(\gamma I_n^{(3)} + \delta \mathcal{I}_n^{(3)}\right) \equiv\ & \frac{1}{3} w_n^3 + \gamma\delta w_n \left(a_n b_n\right)^2 - \frac{1}{3}\gamma\delta(\gamma-\delta)\left(a_n b_n\right)^3\\ &- \alpha\delta u_{n+1}^{\gamma} a_n b_{n+1}\left(w_n - \gamma a_n b_n + w_{n+1} - \gamma a_{n+1} b_{n+1}\right)\\ &+ \beta\gamma u_n^{\delta} a_n b_{n-1}\left(w_{n-1} + \delta a_{n-1} b_{n-1} + w_n + \delta a_n b_n\right)\\ &- \alpha^2\delta u_{n+1}^{\gamma} u_{n+2}^{\gamma} a_n b_{n+2} + \beta^2\gamma u_n^{\delta} u_{n-1}^{\delta} a_n b_{n-2} + \alpha\beta u_n^{\gamma+\delta}\left(w_{n-1} + w_n\right).\end{aligned} \tag{4.11e}$$

Here, the symbol '$\equiv$' indicates equivalence up to a total difference, i.e., $f_n \equiv g_n$ means that there exists a local function h_n such that $f_n - g_n = \mathbf{\Delta}_n^+ h_n$; thus, $f_n \equiv g_n$ implies $\sum_n f_n = \sum_n g_n$ under appropriate boundary conditions.

In the special case $\gamma = \delta$, (4.11a), (4.11b) and (4.11c) correspond to the Hamiltonian densities in (4.2), and (4.11d) and (4.11e) with the aid of the Poisson bracket relations (4.1) generate the third and fourth flows of the generalized Toda hierarchy. The third and fourth flows in the case $\gamma \neq \delta$ can be obtained by applying a transformation like the one considered in subsection 2.3.

5 Concluding remarks

In this paper, we proposed a new integrable generalization of the Toda lattice hierarchy, which can be regarded as a discrete analog of the generalization of the KdV hierarchy to the Yajima–Oikawa hierarchy [9, 10]. The generalized Toda hierarchy admits a Lax-pair representation and two infinite sets of conservation laws. The spatial part of the Lax-pair representation for the generalized Toda hierarchy is given by (2.1), which provides an interesting extension of the eigenvalue problem for the Jacobi operator. The generalized Toda hierarchy involves arbitrary parameters α, β, γ and δ and newly introduced dependent variables a_n and b_n; with a suitable choice of the parameters, the generalized Toda hierarchy possesses a simple Hamiltonian structure (cf. the Poisson bracket relations (4.1)) and a_n and b_n can be related by the complex conjugation reduction. Then, the first flow of the generalized Toda hierarchy (cf. (2.10)) can be reduced to the Yajima–Oikawa system in a continuous limit, while the second flow (cf. (2.11)) provides a generalization of the Toda lattice in Flaschka–Manakov variables (1.2); it is also possible to consider the general case where the newly introduced dependent variables are vector-valued functions (cf. (2.12) and (2.13)). Note that the generalized Toda hierarchy is different from, and apparently has no relation to, the discrete Yajima–Oikawa hierarchy recently studied in [11, 13] (also see [12] for a $(2+1)$-dimensional version [23, 32, 33] of the discrete Yajima–Oikawa system).

For some special choices of the parameters, (a suitable linear combination of) the first and second flows of the generalized Toda hierarchy can be reduced to the elementary flows of two discrete nonlinear Schrödinger hierarchies: the Ablowitz–Ladik hierarchy [14] and the Konopelchenko–Chudnovsky hierarchy [15–17]. For another special choice of the parameters, the first and second flows of the generalized Toda hierarchy can be linearized by a change of vari-

ables.

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