Integrable discretization of the vector/matrix nonlinear Schrödinger equation and the associated Yang–Baxter map

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Abstract

The action of a Bäcklund–Darboux transformation on a spectral problem associated with a known integrable system can define a new discrete spectral problem. In this paper, we interpret a slightly generalized version of the binary Bäcklund–Darboux (or Zakharov–Shabat dressing) transformation for the nonlinear Schrödinger (NLS) hierarchy as a discrete spectral problem, wherein the two intermediate potentials appearing in the Darboux matrix are considered as a pair of new dependent variables. Then, we associate the discrete spectral problem with a suitable isospectral time-evolution equation, which forms the Lax-pair representation for a space-discrete NLS system. This formulation is valid for the most general case where the two dependent variables take values in (rectangular) matrices. In contrast to the matrix generalization of the Ablowitz–Ladik lattice, our discretization has a rational nonlinearity and admits a Hermitian conjugation reduction between the two dependent variables. Thus, a new proper space-discretization of the vector/matrix NLS equation is obtained; by changing the time part of the Lax pair, we also obtain an integrable space-discretization of the vector/matrix modified KdV (mKdV) equation. Because Bäcklund–Darboux transformations are permutable, we can increase the number of discrete independent variables in a multidimensionally consistent way. By solving the consistency condition on the two-dimensional lattice, we obtain a new Yang–Baxter map of the NLS type, which can be considered as a fully discrete analog of the principal chiral model for projection matrices.

Contents

1 Introduction

The problem of integrable discretization [1] has a close relationship with the theory of auto-Bäcklund transformations; this fact has been sporadically noticed for partial differential or differential-difference equations since the mid-1970s. Calogero and Degasperis [2, 3] and Chiu and Ladik [4] showed for some specific examples that a class of auto-Bäcklund transformations can be identified as discrete-time flows that belong to the same integrable hierarchy as the original continuous-time flows. Hirota [5] and Orfanidis [6, 7] showed for the sine-Gordon equation in light-cone coordinates that a one-parameter auto-Bäcklund transformation and the associated nonlinear superposition principle (Bianchi's permutability theorem) lead directly to its integrable discretizations. These results have culminated in the beautiful notion of Miwa variables (or Miwa shifts) in Sato theory [8, 9]. In rough terms, a one-parameter auto-Bäcklund transformation, called an elementary Bäcklund transformation, can be reinterpreted as a discrete-time flow, which can generate an infinite number of continuous-time flows through the Taylor series expansion in the Bäcklund (or time-step) parameter; the commutativity of two elementary Bäcklund transformations with (generally) different values of the Bäcklund parameter provides a fully discrete integrable system that admits a zero-curvature (or Lax-pair [10]) representation and contains information on the continuous-time flows. Such a unified point of view is so fascinating that one is tempted to believe that proper discretizations of a given integrable system can always be obtained from its elementary Bäcklund transformation or the associated nonlinear superposition formula. However, in fact, it is not that simple; we remark the following points.

- An integrable system appears as a member of an infinite hierarchy of commuting flows, which is actually a bi-infinite hierarchy comprising the positive and negative flows. For example, the sine-Gordon equation is the first negative flow of the integrable hierarchy, while its first nontrivial positive flow is (the potential form of) the modified KdV (mKdV) equation [11, 12]. It is not evident which particular flows in a bi-infinite hierarchy can be discretized by considering an elementary Bäcklund transformation and how to take the continuous limit.

- The idea to interpret an elementary Bäcklund transformation and the associated nonlinear superposition formula as defining discrete integrable systems is useful for integrable hierarchies with one scalar unknown. However, it is not the case for two (or more) component systems that admit a complex conjugation reduction between a pair of dependent variables, such as the nonlinear Schrödinger (NLS) system [11,12],

as well as their matrix generalizations that admit a Hermitian conjugation reduction between a pair of matrix dependent variables, such as the matrix NLS system [13]. Indeed, an elementary Bäcklund transformation generally does not maintain the complex/Hermitian conjugation reduction; thus, for a given integrable equation obtained as the result of the reduction, we cannot derive its proper discretization directly from an elementary Bäcklund transformation for the original nonreduced system (cf. [14–17]). Actually, we can consider a suitable composition of two (or more) elementary Bäcklund transformations [14, 18, 19] so that the composite (*e.g.*, binary) Bäcklund transformation can maintain the complex/Hermitian conjugation reduction; however, such a composite transformation involves either a square-root function with an indefinite sign [20, 21] or a nonlocal operation such as an indefinite integral [3], so it does not lead to discrete integrable systems that can be written in local form and define a unique time evolution (cf. (7.18) in [22] and (4.17) in [23]).

Thus, for each integrable system that admits the complex/Hermitian conjugation reduction, we need to construct its proper discretization on a case-by-case consideration and accumulate more knowledge on this subject.

In this paper, we construct a new proper space-discretization of the (generally rectangular) matrix generalization of the NLS system, namely, the matrix NLS system [13], which is an integrable system admitting the Hermitian conjugation reduction between the two matrix dependent variables and contains the vector NLS equation known as the Manakov model [24] as a special case. For the scalar NLS system, the proper and most natural space-discretization was found by Ablowitz and Ladik [25, 26] in the mid-1970s; however, its straightforward matrix generalization [27] (also see [28, 29] and references therein) does not admit a Hermitian conjugation reduction between the two matrix variables in local form, so it is not a proper space-discretization of the matrix NLS system. To obtain a proper space-discrete matrix NLS system, we follow the new approach introduced in our previous paper [30]. That is, we first reinterpret (a slightly generalized version of) a binary Bäcklund–Darboux transformation for the continuous matrix NLS hierarchy as a discrete spectral problem [16], where the two intermediate potentials appearing in the binary Bäcklund–Darboux transformation are considered as new dependent variables. Then, in view of the peculiar structure of the discrete spectral problem, we associate it with a suitable isospectral time-evolution equation to compose a Lax pair and derive an evolutionary lattice system from the compatibility condition called the zero-curvature equation. This lattice system involves some free parameters and, with a suitable choice

of the parameters, it admits the Hermitian conjugation reduction between the two dependent variables and provides a proper space-discretization of the matrix NLS system; thus, it can generate proper space-discrete analogs of various multicomponent NLS equations [13, 24, 31–36] obtained as reductions of the matrix NLS system.

The space-discrete NLS system derived in this paper has a rational nonlinearity, so it is intrinsically more complicated than the Ablowitz–Ladik discretization of the NLS system [25, 26] that has a polynomial nonlinearity. However, the rational nonlinearity can exhibit a saturation effect, so in some sense it is more "physical" than the polynomial nonlinearity.

The space-discrete matrix NLS system appears as a member of an infinite hierarchy of commuting flows. Other flows of this integrable hierarchy can be obtained by changing the temporal part of the Lax pair. In particular, we can derive a space-discrete analog of the matrix mKdV system, which admits the Hermitian conjugation (or matrix transposition) reduction between the two dependent variables. Thus, integrable space-discretizations of various multicomponent mKdV equations [19, 23, 31, 36–39] can be obtained through reductions.

We derive the space-discrete matrix NLS hierarchy from a (slightly generalized) binary Bäcklund–Darboux transformation for the continuous matrix NLS hierarchy by utilizing it as the underlying discrete spectral problem. Actually, we can consider an arbitrary number of Bäcklund–Darboux transformations with (generally) different values of the Bäcklund parameters and assign a new discrete independent variable to each Bäcklund–Darboux transformation; that is, a single application of each transformation is identified with a unit shift of the corresponding discrete independent variable. The consistency of such a multidimensional lattice is guaranteed by Bianchi's permutability theorem for two Bäcklund–Darboux transformations on each quadrilateral. In our approach, the dependent variables are the intermediate potentials that appear in each Bäcklund–Darboux transformation, so they are assigned to the edges of the lattice [40–44]. Then, the permutability condition for Bäcklund–Darboux transformations on a quadrilateral can be solved explicitly, providing a new Yang–Baxter map (to use Veselov's terminology [43, 45]); this is apparently related to the factorizability property of an N-soliton collision into pairwise collisions in the continuous matrix NLS hierarchy (cf. [46]). The permutability condition can be regarded as a matrix re-factorization problem, which directly provides the Lax representation for the Yang–Baxter map [40, 43, 44, 47]. We can identify this Yang–Baxter map as a fully discrete analog of the principal chiral model [48] restricted to the space of projection matrices.

This paper is organized as follows. In section 2, we introduce (a slightly generalized version of) the binary Bäcklund–Darboux transformation for the continuous matrix NLS hierarchy, consider it as a discrete spectral problem and associate it with a suitable isospectral time-evolution equation. Then, the compatibility condition for this Lax pair provides a lattice system involving some arbitrary parameters; with a suitable choice of the parameters, it provides a proper space-discrete analog of the matrix NLS system. By changing the choice of the parameters appropriately, we also obtain a proper space-discretization of the matrix mKdV system. In section 3, we solve Bianchi's permutability condition for the Bäcklund–Darboux transformations in such a way that a new Yang–Baxter map of the NLS type together with its Lax representation can be obtained; this considerably generalizes an analogous result of Goncharenko and Veselov [43,47]. Section 4 is devoted to concluding remarks.

In appendix A, we present some result obtained by the author in the late 1990s but left unpublished. We consider a Lax-pair representation for a vector analog of the sine-Gordon equation [49, 50] and describe how to discretize one of the two independent variables preserving the integrability; this is somewhat related to sections 2 and 3 of this paper.

2 A proper discretization of the matrix NLS hierarchy

In this section, we utilize a (slightly generalized) binary Bäcklund–Darboux transformation for the continuous matrix NLS hierarchy as a discrete spectral problem and derive proper space-discrete analogs of the matrix NLS system and the matrix mKdV system.

2.1 The continuous matrix NLS hierarchy

In 1974, Zakharov and Shabat [13] proposed the matrix generalization of the NLS system [11, 12]:

$$\begin{cases} \mathrm{i}Q_{t_2} + Q_{xx} - 2QRQ = O, \\ \mathrm{i}R_{t_2} - R_{xx} + 2RQR = O, \end{cases} \tag{2.1}$$

where the dependent variables Q and R are $l_1 \times l_2$ and $l_2 \times l_1$ (generally rectangular) matrices and the subscripts denote the partial differentiation. The symbol O on the right-hand side of the equations is used to stress that the dependent variables can take values in matrices. The matrix NLS system

(2.1), as well as related equations, has been studied intensively in recent years (see, *e.g.*, [29] and references cited therein).

The matrix NLS system (2.1) is obtained as the compatibility condition for the overdetermined linear system of partial differential equations [51,52]:

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I_{l_1} & Q \\ R & \mathrm{i}\zeta I_{l_2} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{2.2}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_{t_2} = \begin{bmatrix} -2\mathrm{i}\zeta^2 I_{l_1} - \mathrm{i}QR & 2\zeta Q + \mathrm{i}Q_x \\ 2\zeta R - \mathrm{i}R_x & 2\mathrm{i}\zeta^2 I_{l_2} + \mathrm{i}RQ \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{2.3}$$

Here, ζ is a constant spectral parameter, and I_{l_1} and I_{l_2} are the $l_1 \times l_1$ and $l_2 \times l_2$ unit matrices, respectively. In the following, we suppress the indices of the unit matrices, noting that the dependent variables can take values in (generally) rectangular matrices. Equations (2.2) and (2.3) comprise the Lax-pair representation [10] for the matrix NLS system (2.1). Any constant scalar matrix can be added to each Lax matrix, which does not affect the compatibility condition.

The matrix NLS system (2.1) is a positive flow in a bi-infinite hierarchy of mutually commuting flows, which are all associated with the same spectral problem (2.2). The next higher flow in the matrix NLS hierarchy is a matrix generalization [51,52] of the complex mKdV equation [11,12,53], which reads

$$\begin{cases} Q_{t_3} + Q_{xxx} - 3Q_xRQ - 3QRQ_x = O, \\ R_{t_3} + R_{xxx} - 3R_xQR - 3RQR_x = O. \end{cases} \tag{2.4}$$

The first negative flow in the matrix NLS hierarchy is a matrix analog of the complex sine-Gordon equation, which we rewrite in the form:[1]

$$\begin{cases} Q_{t_{-1}} + 2u(I + vu)^{-1} = O, \\ R_{t_{-1}} - 2v(I + uv)^{-1} = O, \\ u_x + 2\mathrm{i}ku = Q - uRu, \\ v_x - 2\mathrm{i}kv = -R + vQv. \end{cases} \tag{2.5}$$

As long as we consider the first negative flow (2.5) for a single fixed value of k, the free parameter k is nonessential and can be set equal to zero by applying

[1]The first negative flow of the scalar NLS hierarchy [11,12] admits a number of different expressions, which are related through simple transformations of dependent variables. Such different (but essentially equivalent) expressions are often called by different names in correspondence to different physical phenomena (or mathematical objects), such as self-induced transparency (or Maxwell–Bloch) equations [12,54,55] (also see §4.4.b of [56]), stimulated Raman scattering, complex sine-Gordon equation, Pohlmeyer–Lund–Regge system, reduced nonlinear σ-model, etc. There are too many references to mention here.

a point transformation involving $\mathrm{e}^{\pm 2\mathrm{i}kx}$; thus, it reduces to the simpler form:

$$\begin{cases} Q_{t_{-1}} + 2u(I+vu)^{-1} = O, \\ R_{t_{-1}} - 2v(I+uv)^{-1} = O, \\ u_x = Q - uRu, \\ v_x = -R + vQv. \end{cases} \tag{2.6}$$

However, it is often more convenient to leave k as a free parameter. Note that we can consider an arbitrary linear superposition of the first negative flow (2.5) for different values of k [55]. In the case of scalar dependent variables, we can rewrite (2.5) (or (2.6)) as a closed non-evolutionary system for (Q, R) or (u, v).

The time part of the Lax-pair representation (2.3) for (2.1) is now replaced with

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_{t_3} = \begin{bmatrix} -4\mathrm{i}\zeta^3 I - 2\mathrm{i}\zeta QR + Q_x R - QR_x & 4\zeta^2 Q + 2\mathrm{i}\zeta Q_x - Q_{xx} + 2QRQ \\ 4\zeta^2 R - 2\mathrm{i}\zeta R_x - R_{xx} + 2RQR & 4\mathrm{i}\zeta^3 I + 2\mathrm{i}\zeta RQ + R_x Q - RQ_x \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$

for (2.4) and

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_{t_{-1}} = \frac{\mathrm{i}}{\zeta - k} \underbrace{\left\{ \begin{bmatrix} I & \\ & O \end{bmatrix} + \begin{bmatrix} -I & u \\ v & I \end{bmatrix}^{-1} \right\}} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{2.7}$$

for (2.5), respectively. Here, the underscored part,

$$\begin{aligned} P &:= \begin{bmatrix} I & \\ & O \end{bmatrix} + \begin{bmatrix} -I & u \\ v & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} uv\,(I+uv)^{-1} & u\,(I+vu)^{-1} \\ v\,(I+uv)^{-1} & (I+vu)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} u \\ I \end{bmatrix} (I+vu)^{-1} \begin{bmatrix} v & I \end{bmatrix} \\ &= \begin{bmatrix} -I & u \\ v & I \end{bmatrix} \begin{bmatrix} O & \\ & I \end{bmatrix} \begin{bmatrix} -I & u \\ v & I \end{bmatrix}^{-1}, \end{aligned} \tag{2.8}$$

is a projection matrix, i.e., it satisfies the condition $P^2 = P$. Note, incidentally, that the free parameter k in (2.5) and (2.7) can be generalized to an arbitrary function of t_{-1}.

One of the most important properties of the matrix NLS hierarchy, which is crucial for its physical applications, is that it admits the Hermitian conjugation reduction between the pair of dependent variables Q and R [33] (also

see [31,32,57–59]); this reduction results in a self-focusing, self-defocusing or mixed focusing-defocusing nonlinearity, which is briefly summarized in §2.1 of our previous paper [60]. In particular, by imposing the reduction $R = -Q^\dagger$ on the matrix NLS system (2.1) where the dagger denotes the Hermitian conjugation, we obtain the self-focusing matrix NLS equation [13]:

$$\mathrm{i}Q_{t_2} + Q_{xx} + 2QQ^\dagger Q = O. \tag{2.9}$$

In the same manner, the reduction $R = -Q^\dagger$ simplifies (2.4) to the matrix complex mKdV equation:

$$Q_{t_3} + Q_{xxx} + 3Q_xQ^\dagger Q + 3QQ^\dagger Q_x = O. \tag{2.10}$$

Another important property of the matrix NLS hierarchy is that the odd-order flows admit the matrix transposition reduction between the pair of variables Q and R. In particular, the matrix mKdV equation [23,37,38],

$$Q_{t_3} + Q_{xxx} + 3Q_xQ^TQ + 3QQ^TQ_x = O, \tag{2.11}$$

is obtained by imposing the reduction $R = -Q^T$ on (2.4), where the superscript ^{T} denotes the transpose of a matrix. In the case of a vector dependent variable, (2.11) gives the vector mKdV equation [19,31]:

$$\boldsymbol{q}_{t_3} + \boldsymbol{q}_{xxx} + 3\langle \boldsymbol{q}, \boldsymbol{q}_x\rangle \boldsymbol{q} + 3\langle \boldsymbol{q}, \boldsymbol{q}\rangle \boldsymbol{q}_x = \boldsymbol{0}. \tag{2.12}$$

If $\boldsymbol{q}$ is a two-component real-valued vector, i.e., $\boldsymbol{q} = (q_1, q_2)$, $q_1, q_2 \in \mathbb{R}$, the vector mKdV equation (2.12) can be rewritten as a single complex-valued mKdV equation for $q := q_1 + \mathrm{i}q_2$; this is often referred to as the Sasa–Satsuma equation [61].

Any proper discretization of the matrix NLS hierarchy should retain the feasibility of such reductions. In short, a discrete analog of the matrix NLS hierarchy is physically meaningful only if it is integrable and admits the Hermitian conjugation (or matrix transposition) reduction similar to the continuous case. That is, it should be able to provide integrable discretizations for the reduced equations such as (2.9)–(2.12).

2.2 Bäcklund–Darboux transformation as a discrete spectral problem

To obtain a proper discrete analog of the matrix NLS hierarchy, we start with a slightly generalized version of the binary Bäcklund–Darboux transformation [16,62,63] for the continuous matrix NLS hierarchy as given by

$$\begin{bmatrix} \widetilde{\Psi}_1 \\ \widetilde{\Psi}_2 \end{bmatrix} = \left\{ \begin{bmatrix} (\zeta\delta + \alpha)I & \\ & (-\zeta\gamma + \beta)I \end{bmatrix} - (\alpha\gamma + \beta\delta) \begin{bmatrix} \gamma I & u \\ v & \delta I \end{bmatrix}^{-1} \right\} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{2.13}$$

Here, α, β, γ and δ are arbitrary parameters, except that they are required to satisfy the condition $\alpha\gamma+\beta\delta\neq 0$; the special case $\delta=-\gamma(\neq 0)$ corresponds to the conventional binary Bäcklund–Darboux transformation [63] or the Zakharov–Shabat dressing method [16, 62], up to an inessential overall factor. Unlike the usual formulation of the binary Bäcklund–Darboux transformation, we do not express the intermediate potentials u and v in terms of linear eigenfunctions of the original Lax-pair representation [14–16]. Alternatively, we will consider u and v as a pair of new dependent variables defined on a lattice, wherein the lattice index $n\in\mathbb{Z}$ is understood intuitively as the number of iterations of the Bäcklund–Darboux transformation. To make it more explicit, we rewrite (2.13) as a discrete spectral problem

$$\begin{bmatrix}\Psi_{1,n+1}\\ \Psi_{2,n+1}\end{bmatrix}=L_n\begin{bmatrix}\Psi_{1,n}\\ \Psi_{2,n}\end{bmatrix}, \tag{2.14}$$

where the Lax matrix L_n is given by

$$\begin{aligned}
L_n&=\begin{bmatrix}(\zeta\delta+\alpha)I & \\ & (-\zeta\gamma+\beta)I\end{bmatrix}-(\alpha\gamma+\beta\delta)\begin{bmatrix}\gamma I & u_n\\ v_n & \delta I\end{bmatrix}^{-1}\\
&=\begin{bmatrix}(\zeta\delta+\alpha)I & \\ & (-\zeta\gamma+\beta)I\end{bmatrix}\\
&\quad-(\alpha\gamma+\beta\delta)\begin{bmatrix}\delta(\gamma\delta I-u_nv_n)^{-1} & -(\gamma\delta I-u_nv_n)^{-1}u_n\\ -(\gamma\delta I-v_nu_n)^{-1}v_n & \gamma(\gamma\delta I-v_nu_n)^{-1}\end{bmatrix}\\
&=\begin{bmatrix}\delta I & -u_n\\ -v_n & \gamma I\end{bmatrix}\begin{bmatrix}(\zeta\gamma-\beta)I & \\ & -(\zeta\delta+\alpha)I\end{bmatrix}\begin{bmatrix}\gamma I & u_n\\ v_n & \delta I\end{bmatrix}^{-1}\\
&=\begin{bmatrix}\gamma I & u_n\\ v_n & \delta I\end{bmatrix}^{-1}\begin{bmatrix}(\zeta\gamma-\beta)I & \\ & -(\zeta\delta+\alpha)I\end{bmatrix}\begin{bmatrix}\delta I & -u_n\\ -v_n & \gamma I\end{bmatrix}.
\end{aligned} \tag{2.15}$$

Actually, the Bäcklund parameters α, β, γ and δ satisfying the condition $\alpha\gamma+\beta\delta\neq 0$ can be arbitrary functions of the discrete independent variable n, but for brevity we usually consider them as constants. Each of the four expressions in (2.15) has its own advantages.

2.3 Isospectral time evolution

To compose a Lax pair, we need to associate (2.14) with a suitable isospectral time-evolution equation,

$$\begin{bmatrix}\Psi_{1,n}\\ \Psi_{2,n}\end{bmatrix}_t=M_n\begin{bmatrix}\Psi_{1,n}\\ \Psi_{2,n}\end{bmatrix}. \tag{2.16}$$

Here, the spectral parameter ζ involved in L_n and M_n is isospectral, i.e., $\zeta_t = 0$. The compatibility condition for (2.14) and (2.16) is given by (a space-discrete version of) the zero-curvature equation [25, 26, 64, 65]

$$L_{n,t} + L_n M_n - M_{n+1} L_n = O, \tag{2.17}$$

where L_n and M_n comprise the Lax pair. For the Lax matrix L_n in (2.15), we use the first expression to compute $L_{n,t}$ with the aid of the formula $(X^{-1})_t = -X^{-1} X_t X^{-1}$ valid for a square matrix X so that the equations of motion for u_n and v_n can be obtained explicitly.

Let us first consider the original continuous spectral problem (2.2), which now reads

$$\begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I & Q_n \\ R_n & \mathrm{i}\zeta I \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}. \tag{2.18}$$

The Bäcklund–Darboux transformation (2.14) with (2.15) preserves the form of the spectral problem (2.18) invariant. In the general case where the Bäcklund parameters α, β, γ and δ depend on n ($\alpha \to \alpha_n$ etc.), the compatibility condition (i.e., (2.17) with t replaced by x) provides a system of differential-difference equations:

$$\begin{cases} \gamma_n Q_{n+1} + \delta_n Q_n + 2\mathrm{i}(\alpha_n\gamma_n + \beta_n\delta_n) u_n (\gamma_n\delta_n I - v_n u_n)^{-1} = O, \\ \delta_n R_{n+1} + \gamma_n R_n + 2\mathrm{i}(\alpha_n\gamma_n + \beta_n\delta_n) v_n (\gamma_n\delta_n I - u_n v_n)^{-1} = O, \\ (\alpha_n\gamma_n + \beta_n\delta_n) u_{n,x} + \alpha_n\gamma_n^2 Q_{n+1} - \beta_n\delta_n^2 Q_n - u_n(\alpha_n R_{n+1} - \beta_n R_n) u_n = O, \\ (\alpha_n\gamma_n + \beta_n\delta_n) v_{n,x} + \beta_n\delta_n^2 R_{n+1} - \alpha_n\gamma_n^2 R_n - v_n(\beta_n Q_{n+1} - \alpha_n Q_n) v_n = O. \end{cases} \tag{2.19}$$

When $\gamma_n\delta_n \neq 0$, the first two equations relate the new potentials Q_{n+1} and R_{n+1} to the old potentials Q_n and R_n in the spectral problem (2.18) through the intermediate potentials u_n and v_n (cf. [14]); in this case, (2.19) can be reformulated as Riccati equations for u_n and v_n:

$$\begin{cases} \delta_n u_{n,x} = 2\mathrm{i}\alpha_n u_n + \delta_n^2 Q_n - u_n R_n u_n, \\ \gamma_n v_{n,x} = 2\mathrm{i}\beta_n v_n + \gamma_n^2 R_n - v_n Q_n v_n, \\ -\gamma_n u_{n,x} = 2\mathrm{i}\beta_n u_n + \gamma_n^2 Q_{n+1} - u_n R_{n+1} u_n, \\ -\delta_n v_{n,x} = 2\mathrm{i}\alpha_n v_n + \delta_n^2 R_{n+1} - v_n Q_{n+1} v_n. \end{cases} \tag{2.20}$$

Thus, the intermediate potentials u_n and v_n can be expressed in terms of the linear eigenfunctions of the spectral problem (2.18) [14, 15], *e.g.*

$$v_n = \gamma_n \Psi_{2,n} \Psi_{1,n}^{-1}\big|_{\zeta=\frac{\beta_n}{\gamma_n}} = -\delta_n \Psi_{2,n+1} \Psi_{1,n+1}^{-1}\big|_{\zeta=-\frac{\alpha_n}{\delta_n}},$$

where $\Psi_{1,n}$ and $\Psi_{1,n+1}$ are square matrices. In the case of scalar dependent variables, we can eliminate Q_n and R_n from (2.20) to obtain a closed system for u_n and v_n.

With a suitable choice of the parameters, (2.19) can be considered as an integrable time-discretization of the matrix sine-Gordon system (2.6). Indeed, in the case of $\beta_n = -\alpha_n$ and $\delta_n = -\gamma_n$, (2.19) provides a (generally nonuniform) discrete-time analog of (2.6):

$$\begin{cases} Q_{n+1} - Q_n - 4\mathrm{i}\alpha_n u_n(\gamma_n^2 I + v_n u_n)^{-1} = O, \\ R_{n+1} - R_n + 4\mathrm{i}\alpha_n v_n(\gamma_n^2 I + u_n v_n)^{-1} = O, \\ 2\gamma_n u_{n,x} + \gamma_n^2(Q_{n+1} + Q_n) - u_n(R_{n+1} + R_n)u_n = O, \\ 2\gamma_n v_{n,x} - \gamma_n^2(R_{n+1} + R_n) + v_n(Q_{n+1} + Q_n)v_n = O. \end{cases} \tag{2.21}$$

To obtain proper space-discretizations of positive flows of the matrix NLS hierarchy, we consider a temporal Lax matrix M_n that matches well with the factorized form of the spatial Lax matrix L_n in (2.15). Note that the first expression in (2.15) implies the ζ-independence of $L_{n,t}$, so $L_n M_n - M_{n+1} L_n$ in (2.17) is also ζ-independent. A simple ansatz for M_n satisfying this condition automatically is

$$M_n = \begin{bmatrix} \gamma I & u_n \\ v_n & \delta I \end{bmatrix} \begin{bmatrix} \frac{1}{\zeta\gamma-\beta} F_n & \\ & \frac{1}{\zeta\delta+\alpha} G_n \end{bmatrix} \begin{bmatrix} \gamma I & u_{n-1} \\ v_{n-1} & \delta I \end{bmatrix} + \begin{bmatrix} cI & \\ & dI \end{bmatrix}, \tag{2.22}$$

where F_n and G_n are ζ-independent square matrices and c and d are ζ-independent parameters. The ζ-dependence of the main part of M_n in (2.22) implies that it can be considered as a linear superposition of the temporal Lax matrix in (2.7) at two particular values of k (up to the addition of a constant scalar matrix), i.e., it corresponds to the two "first" negative flows of the matrix NLS hierarchy. Substituting (2.15) and (2.22) into (2.17), we obtain recurrence relations for F_n and G_n; to satisfy them identically, we set

$$F_n = a\,(\alpha\gamma + \beta\delta)\left(\gamma^2 I + u_{n-1}v_n\right)^{-1}, \quad G_n = b\,(\alpha\gamma + \beta\delta)\left(\delta^2 I + v_{n-1}u_n\right)^{-1},$$

where a and b are arbitrary constants. In fact, a and b, as well as c and d in (2.22), can depend arbitrarily on the time variable t, but we do not consider it in this paper. With the above choice of F_n and G_n, (2.17) for (2.15) and

(2.22) provides an evolutionary system of differential-difference equations:

$$\begin{cases} u_{n,t} + a\left(\gamma\delta I - u_n v_n\right)\left(\gamma^2 I + u_{n-1}v_n\right)^{-1}\left(\gamma u_n + \delta u_{n-1}\right) \\ \quad + b\left(\gamma u_{n+1} + \delta u_n\right)\left(\delta^2 I + v_n u_{n+1}\right)^{-1}\left(\gamma\delta I - v_n u_n\right) - (c-d)u_n = O, \\ v_{n,t} - b\left(\gamma\delta I - v_n u_n\right)\left(\delta^2 I + v_{n-1}u_n\right)^{-1}\left(\delta v_n + \gamma v_{n-1}\right) \\ \quad - a\left(\delta v_{n+1} + \gamma v_n\right)\left(\gamma^2 I + u_n v_{n+1}\right)^{-1}\left(\gamma\delta I - u_n v_n\right) + (c-d)v_n = O, \end{cases} \tag{2.23}$$

which is independent of α and β. Note that the order of products in (2.23) can be changed using identities such as

$$\begin{aligned} &\left(\gamma\delta I - u_n v_n\right)\left(\gamma^2 I + u_{n-1}v_n\right)^{-1}\left(\gamma u_n + \delta u_{n-1}\right) \\ &= \delta u_n + \frac{\delta^2}{\gamma}u_{n-1} - \frac{1}{\gamma}\left(\gamma u_n + \delta u_{n-1}\right)v_n\left(\gamma^2 I + u_{n-1}v_n\right)^{-1}\left(\gamma u_n + \delta u_{n-1}\right) \\ &= \left(\gamma u_n + \delta u_{n-1}\right)\left(\gamma^2 I + v_n u_{n-1}\right)^{-1}\left(\gamma\delta I - v_n u_n\right). \end{aligned}$$

In the case of scalar u_n and v_n, (2.23) is related to known integrable systems such as the lattice Heisenberg ferromagnet model [66–69] through simple changes of dependent variables (also see (2.19)–(2.20) in [70] together with [71]); thus, it is not really a new integrable system. However, (2.23) in the general matrix case, its reductions and identification as a discrete analog of the matrix NLS (or mKdV) equation have not been reported in the literature.

Considering its relation to the space-discrete Kaup–Newell system (see Propositions 2.1 and 2.2 in [72] with a variable change $v_n \to v_n^{-1}$), we can show that the lattice system (2.23) possesses an ultralocal (but noncanonical) Poisson bracket. For scalar u_n and v_n, it can be written as $u_{n,t} = \{u_n, H\}$ and $v_{n,t} = \{v_n, H\}$ with the Hamiltonian and the Poisson bracket given by

$$H = \sum_n \left[a\log\left(\frac{\gamma^2 + u_{n-1}v_n}{\gamma\delta - u_n v_n}\right) + b\log\left(\frac{\delta^2 + u_{n+1}v_n}{\gamma\delta - u_n v_n}\right) + \frac{c-d}{\gamma\delta - u_n v_n}\right]$$

and

$$\{u_m, u_n\} = \{v_m, v_n\} = 0, \qquad \{u_m, v_n\} = \delta_{mn}\left(\gamma\delta - u_n v_n\right)^2,$$

respectively. Here, δ_{mn} is the Kronecker delta, which has nothing to do with the free parameter δ.

By rescaling the variables and parameters as $u_n \to \gamma u_n$, $v_n \to \gamma v_n$, $a \to \gamma a$,

$b \to b/\gamma^2$ and setting $c = b/\gamma$, we can rewrite (2.23) as

$$\begin{cases} u_{n,t} + a\left(\delta I - \gamma u_n v_n\right)\left(I + u_{n-1}v_n\right)^{-1}\left(\gamma u_n + \delta u_{n-1}\right) \\ \quad + \dfrac{b}{\gamma}\left(\gamma u_{n+1} + \delta u_n\right)\left(\delta^2 I + \gamma^2 v_n u_{n+1}\right)^{-1}\left(\delta I - \gamma v_n u_n\right) - \dfrac{b}{\gamma}u_n + d u_n = O, \\ v_{n,t} - \dfrac{b}{\gamma}\left(\delta I - \gamma v_n u_n\right)\left(\delta^2 I + \gamma^2 v_{n-1}u_n\right)^{-1}\left(\gamma v_{n-1} + \delta v_n\right) \\ \quad - a\left(\gamma v_n + \delta v_{n+1}\right)\left(I + u_n v_{n+1}\right)^{-1}\left(\delta I - \gamma u_n v_n\right) + \dfrac{b}{\gamma}v_n - d v_n = O. \end{cases} \tag{2.24}$$

Then, by taking the limit $\gamma \to 0$, (2.24) reduces to a known integrable space-discretization of the NLS system [27, 73]:

$$\begin{cases} u_{n,t} + a\delta^2\left(I + u_{n-1}v_n\right)^{-1}u_{n-1} + \dfrac{b}{\delta}\left(u_{n+1} - u_n v_n u_n\right) + d u_n = O, \\ v_{n,t} - \dfrac{b}{\delta}\left(v_{n-1} - v_n u_n v_n\right) - a\delta^2 v_{n+1}\left(I + u_n v_{n+1}\right)^{-1} - d v_n = O, \end{cases}$$

which is associated with an elementary auto-Bäcklund transformation for the continuous matrix NLS hierarchy [14–16] as outlined in [74]. This limiting case no longer admits the complex (or Hermitian) conjugation reduction between u_n and v_n, so it does not provide a physically meaningful discretization of the NLS system.

When the parameters satisfy the conditions $\delta = \gamma^*$, $b = -a^*$ and $d = c^*$, the lattice system (2.23) admits not only the complex conjugation reduction for square matrices u_n and v_n but also the Hermitian conjugation reduction for (generally) rectangular matrices u_n and v_n. In particular, in the case of $\delta = \gamma = 1$, $b = -a^*$ and $d = c^*$, the Hermitian conjugation reduction $v_n = -u_n^\dagger$ simplifies (2.23) to

$$\begin{aligned} u_{n,t} &- a^*\left(I + u_n u_n^\dagger\right)\left(I - u_{n+1}u_n^\dagger\right)^{-1}\left(u_{n+1} + u_n\right) \\ &+ a\left(I + u_n u_n^\dagger\right)\left(I - u_{n-1}u_n^\dagger\right)^{-1}\left(u_n + u_{n-1}\right) - (c - c^*)u_n = O. \end{aligned} \tag{2.25}$$

By further setting $a = -\mathrm{i}$ and $c = -2\mathrm{i}$, (2.25) provides a new integrable space-discretization of the matrix NLS equation (2.9),

$$\begin{aligned} \mathrm{i}u_{n,t} &+ \left(I + u_n u_n^\dagger\right)\left(I - u_{n+1}u_n^\dagger\right)^{-1}\left(u_{n+1} + u_n\right) \\ &+ \left(I + u_n u_n^\dagger\right)\left(I - u_{n-1}u_n^\dagger\right)^{-1}\left(u_n + u_{n-1}\right) - 4u_n = O, \end{aligned}$$

which can also be written as

$$\begin{aligned} \mathrm{i}u_{n,t} &+ \left(u_{n+1} + u_{n-1} - 2u_n\right) + \left(u_{n+1} + u_n\right)u_n^\dagger\left(I - u_{n+1}u_n^\dagger\right)^{-1}\left(u_{n+1} + u_n\right) \\ &+ \left(u_n + u_{n-1}\right)u_n^\dagger\left(I - u_{n-1}u_n^\dagger\right)^{-1}\left(u_n + u_{n-1}\right) = O. \end{aligned} \tag{2.26}$$

This equation is defined on the three lattice sites $n-1$, n, $n+1$, so it is simpler and appears to be more interesting than the space-discrete matrix NLS equation proposed in our previous paper [28] (see (4.1) therein), which depends on five lattice sites after imposing the Hermitian conjugation reduction. Considering further reductions of the matrix dependent variable u_n, we can obtain integrable space-discretizations of various multicomponent NLS equations [34–36]. In particular, when u_n is a row (or column) vector, (2.26) provides a new integrable space-discretization of the vector NLS equation often referred to as the Manakov model [24]. Equation (2.26) corresponds to the self-focusing case, but it is also possible to consider the self-defocusing case or a mixed focusing-defocusing case as in the continuous case [31–33, 57–59].

Note that a fully discrete matrix NLS equation was proposed by van der Linden, Nijhoff, Capel and Quispel in 1986 (see (4.17) in [23]). However, it involves a square-root function of a general non-diagonal matrix defined using the Taylor series expansion, so it is not an explicit and closed expression. In fact, even in the scalar case, their fully discrete NLS equation (see (7.18) in [22]) is an implicit equation, which does not define the time evolution uniquely.

In the case of $a=-1$ and $c=0$, (2.25) provides an integrable space-discretization of the matrix complex mKdV equation (2.10) given by

$$\begin{aligned} u_{n,t} &+ \left(I+u_n u_n^\dagger\right)\left(I-u_{n+1}u_n^\dagger\right)^{-1}(u_{n+1}+u_n) \\ &- \left(I+u_n u_n^\dagger\right)\left(I-u_{n-1}u_n^\dagger\right)^{-1}(u_n+u_{n-1}) = O. \end{aligned}$$

When the parameters satisfy the conditions $\delta=\gamma$, $b=-a$ and $c=d=0$, the lattice system (2.23) admits the matrix transposition reduction between u_n and v_n. In particular, by setting $\delta=\gamma=1$, $b=-a=1$, $c=d=0$ and $v_n=-u_n^T$, (2.23) reduces to an integrable space-discretization of the matrix mKdV equation (2.11):

$$\begin{aligned} u_{n,t} &+ \left(I+u_n u_n^T\right)\left(I-u_{n+1}u_n^T\right)^{-1}(u_{n+1}+u_n) \\ &- \left(I+u_n u_n^T\right)\left(I-u_{n-1}u_n^T\right)^{-1}(u_n+u_{n-1}) = O. \end{aligned}$$

In the vector case, this reads

$$\boldsymbol{u}_{n,t} + (1+\langle \boldsymbol{u}_n, \boldsymbol{u}_n\rangle)\left(\frac{\boldsymbol{u}_{n+1}+\boldsymbol{u}_n}{1-\langle \boldsymbol{u}_{n+1}, \boldsymbol{u}_n\rangle} - \frac{\boldsymbol{u}_n+\boldsymbol{u}_{n-1}}{1-\langle \boldsymbol{u}_n, \boldsymbol{u}_{n-1}\rangle}\right) = \boldsymbol{0}, \qquad (2.27)$$

which provides an integrable space-discrete analog of the vector mKdV equation (2.12). In the single-component (i.e., scalar) case, this can be rewritten

in a more familiar form (cf. (3.5) in [75]) by setting $u_n =: \tan w_n$. When $\boldsymbol{u}_n$ is a real two-component vector, i.e., $\boldsymbol{u}_n = (u_n^{(1)}, u_n^{(2)})$, $u_n^{(1)}, u_n^{(2)} \in \mathbb{R}$, we can introduce a complex-valued function $\psi_n := u_n^{(1)} + \mathrm{i}u_n^{(2)}$ to rewrite (2.27) as a single evolution equation for ψ_n. Then, we obtain an integrable space-discretization of the complex mKdV equation often referred to as the Sasa–Satsuma equation [61].

When $\delta = \pm\gamma$, $b = -a$ and $c = d = 0$, we can reduce the lattice system (2.23) to a single closed equation for u_n by setting v_n as a constant scalar matrix. In particular, by setting $\delta = \gamma$, $b = -a$, $c = d = 0$, $v_n = -\gamma^2 I$ and $a\gamma = -1$, we obtain a proper space-discretization of the matrix KdV equation [10] (cf. (2.4) with $R = \text{const.}$), which reads

$$\begin{aligned} &u_{n,t} + (u_{n+1} - u_{n-1}) + (u_{n+1} + u_n)\,(I - u_{n+1})^{-1}\,(u_{n+1} + u_n) \\ &\quad - (u_n + u_{n-1})\,(I - u_{n-1})^{-1}\,(u_n + u_{n-1}) = O. \end{aligned}$$

Alternatively, by setting $\delta = -\gamma$, $b = -a$, $c = d = 0$, $v_n = \gamma^2 I$ and $a\gamma = 1$, we obtain

$$\begin{aligned} &u_{n,t} + (u_{n+1} + u_{n-1} - 2u_n) - (u_{n+1} - u_n)\,(I + u_{n+1})^{-1}\,(u_{n+1} - u_n) \\ &\quad - (u_n - u_{n-1})\,(I + u_{n-1})^{-1}\,(u_n - u_{n-1}) = O, \end{aligned}$$

which is an integrable space-discretization of the equation

$$u_t + u_{xx} - 2u_x(I + u)^{-1}u_x = O$$

that can be linearized as

$$\left[(I + u)^{-1}\right]_t + \left[(I + u)^{-1}\right]_{xx} = O.$$

3 Yang–Baxter map

In this section, we utilize the permutability property of Bäcklund–Darboux transformations to construct a new Yang–Baxter map, which provides a fully discrete analog of the principal chiral model for projection matrices and admits the Hermitian conjugation reduction.

Let us consider two copies of the "same" Bäcklund–Darboux transformation with different sets of Bäcklund parameters and intermediate potentials and assign them to two lattice directions m and n, respectively. A forward shift in the n-direction is defined as in (2.14) with (2.15), i.e.

$$\Psi_{m,n+1} = L_{m,n}\Psi_{m,n} \tag{3.1}$$

with the Lax matrix $L_{m,n}$ given by

$$L_{m,n} = \begin{bmatrix} (\zeta\delta+\alpha)I & \\ & (-\zeta\gamma+\beta)I \end{bmatrix} - (\alpha\gamma+\beta\delta) \begin{bmatrix} \gamma I & u_{m,n} \\ v_{m,n} & \delta I \end{bmatrix}^{-1}. \quad (3.2)$$

Here, the condition $\alpha\gamma+\beta\delta \neq 0$ is assumed; for brevity, the n-dependence of the Bäcklund parameters α, β, γ and δ is suppressed. We are only interested in the general case of $\gamma\delta \neq 0$ and do not consider the limiting case of γ (or δ) $\to 0$ where the binary Bäcklund–Darboux transformation can reduce to an elementary Bäcklund–Darboux transformation.

Similarly, we define a forward shift in the m-direction as

$$\Psi_{m+1,n} = V_{m,n}\Psi_{m,n} \quad (3.3)$$

with the Lax matrix $V_{m,n}$ given by

$$V_{m,n} = \begin{bmatrix} (\zeta d+a)I & \\ & (-\zeta c+b)I \end{bmatrix} - (ac+bd) \begin{bmatrix} cI & q_{m,n} \\ r_{m,n} & dI \end{bmatrix}^{-1}. \quad (3.4)$$

Note that the parameters a, b, c and d used in this section have no direct relationship with those used in subsection 2.3. The condition $ac+bd \neq 0$ is assumed and the m-dependence of a, b, c and d is suppressed. In addition, we assume $cd \neq 0$ and do not consider the limiting case of c (or d) $\to 0$ where the binary Bäcklund–Darboux transformation can reduce to an elementary Bäcklund–Darboux transformation. Note that the Lax matrices in (3.2) and (3.4) are both $(l_1+l_2)\times(l_1+l_2)$ block matrices, i.e., they are partitioned in the same manner.

The compatibility condition for (3.1) and (3.3) is given by (a fully discrete version of) the zero-curvature equation [6, 16, 26, 76]

$$L_{m+1,n}V_{m,n} = V_{m,n+1}L_{m,n}. \quad (3.5)$$

The Lax matrices $L_{m,n}$ and $V_{m,n}$ are of the same form (cf. (3.2) and (3.4)), so this is a matrix re-factorization problem [43, 45, 47]. This re-factorization problem can be solved explicitly; that is, for the Lax matrices originating from a Bäcklund–Darboux transformation as given above, the two matrices on the left-hand side of (3.5) can uniquely determine the two matrices on the right-hand side and vice versa. Thus, this provides a Yang–Baxter map admitting the Lax (or zero-curvature) representation [40, 43, 44, 47].

With the aid of a gauge transformation

$$\Psi_{m,n} = \begin{bmatrix} \delta^n d^m I & \\ & (-\gamma)^n(-c)^m I \end{bmatrix} \Psi'_{m,n}, \quad (3.6)$$

we can reduce the general case of $\gamma\delta \neq 0$ and $cd \neq 0$ to the simpler case of $\delta = -\gamma = 1$ and $d = -c = 1$ after a minor redefinition of the parameters and the dependent variables. The Lax matrices in this case take the same form as the standard binary Bäcklund–Darboux transformation [63] or the Zakharov–Shabat dressing method [16,62] up to an inessential overall factor, so they admit a compact expression in terms of a projection matrix.

Indeed, up to an inessential m-independent overall factor (cf. (3.5)), we can express the Lax matrix $L_{m,n}$ in the case of $\delta = -\gamma = 1$ as

$$\begin{aligned} L_{m,n} &= \begin{bmatrix} (\zeta+\alpha)I & \\ & (\zeta+\beta)I \end{bmatrix} + (\alpha-\beta) \begin{bmatrix} -I & u_{m,n} \\ v_{m,n} & I \end{bmatrix}^{-1} \\ &\propto I + \frac{\alpha-\beta}{\zeta+\beta} P_{m,n}. \end{aligned} \tag{3.7}$$

Here, the projection matrix $P_{m,n}$ is defined as

$$\begin{aligned} P_{m,n} &:= \begin{bmatrix} I & \\ & O \end{bmatrix} + \begin{bmatrix} -I & u_{m,n} \\ v_{m,n} & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} u_{m,n}v_{m,n}\left(I+u_{m,n}v_{m,n}\right)^{-1} & u_{m,n}\left(I+v_{m,n}u_{m,n}\right)^{-1} \\ v_{m,n}\left(I+u_{m,n}v_{m,n}\right)^{-1} & \left(I+v_{m,n}u_{m,n}\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} u_{m,n} \\ I \end{bmatrix} \left(I+v_{m,n}u_{m,n}\right)^{-1} \begin{bmatrix} v_{m,n} & I \end{bmatrix} \\ &= \begin{bmatrix} -I & u_{m,n} \\ v_{m,n} & I \end{bmatrix} \begin{bmatrix} O & \\ & I \end{bmatrix} \begin{bmatrix} -I & u_{m,n} \\ v_{m,n} & I \end{bmatrix}^{-1}, \end{aligned} \tag{3.8}$$

which indeed satisfies $(P_{m,n})^2 = P_{m,n}$. Note that the correspondence between $P_{m,n}$ and $(u_{m,n}, v_{m,n})$ in (3.8) is one-to-one. Owing to the relation

$$\left(I + \frac{\alpha-\beta}{\zeta+\beta}P_{m,n}\right)\left(I + \frac{\beta-\alpha}{\zeta+\alpha}P_{m,n}\right) = I,$$

the inverse of the Lax matrix $L_{m,n}$ has the same form as $L_{m,n}$ with $\alpha \leftrightarrow \beta$, up to an inessential overall factor. Note, incidentally, that $\left[\pm(I-2P_{m,n})\right]^2 = I$. Thus, using the "spin matrix" $S_{m,n} := -I + 2P_{m,n}$, the Lax matrix $L_{m,n}$ can be rewritten in the form:

$$L_{m,n} \propto I + \frac{\alpha-\beta}{2\zeta+\alpha+\beta} S_{m,n}, \qquad (S_{m,n})^2 = I.$$

In the simplest 2×2 case, this is essentially the spatial Lax matrix for the lattice Heisenberg ferromagnet model [66,67] (see [69] and §3.4 of [77] for the matrix generalization).

Similarly, we can express the Lax matrix $V_{m,n}$ in the case of $d=-c=1$ as

$$\begin{aligned} V_{m,n} &= \begin{bmatrix} (\zeta+a)I & \\ & (\zeta+b)I \end{bmatrix} + (a-b)\begin{bmatrix} -I & q_{m,n} \\ r_{m,n} & I \end{bmatrix}^{-1} \\ &\propto I + \frac{a-b}{\zeta+b}\mathscr{P}_{m,n}. \end{aligned} \tag{3.9}$$

Here, the projection matrix $\mathscr{P}_{m,n}$ is defined as

$$\begin{aligned} \mathscr{P}_{m,n} &:= \begin{bmatrix} I & \\ & O \end{bmatrix} + \begin{bmatrix} -I & q_{m,n} \\ r_{m,n} & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -I & q_{m,n} \\ r_{m,n} & I \end{bmatrix} \begin{bmatrix} O & \\ & I \end{bmatrix} \begin{bmatrix} -I & q_{m,n} \\ r_{m,n} & I \end{bmatrix}^{-1}, \end{aligned} \tag{3.10}$$

which satisfies $(\mathscr{P}_{m,n})^2 = \mathscr{P}_{m,n}$.

Substituting (3.7) and (3.9) into the fully discrete zero-curvature equation (3.5), we have

$$\left(I + \frac{\alpha-\beta}{\zeta+\beta}P_{m+1,n}\right)\left(I + \frac{a-b}{\zeta+b}\mathscr{P}_{m,n}\right) = \left(I + \frac{a-b}{\zeta+b}\mathscr{P}_{m,n+1}\right)\left(I + \frac{\alpha-\beta}{\zeta+\beta}P_{m,n}\right), \tag{3.11}$$

or equivalently,

$$\left(I + \frac{b-a}{\zeta+a}\mathscr{P}_{m,n}\right)\left(I + \frac{\beta-\alpha}{\zeta+\alpha}P_{m+1,n}\right) = \left(I + \frac{\beta-\alpha}{\zeta+\alpha}P_{m,n}\right)\left(I + \frac{b-a}{\zeta+a}\mathscr{P}_{m,n+1}\right), \tag{3.12}$$

which generalizes the re-factorization problem studied by Goncharenko and Veselov [43,47]. This is an identity in the spectral parameter ζ, so at $O(1/\zeta)$ it provides the conservation law:

$$(\alpha-\beta)\left(P_{m+1,n} - P_{m,n}\right) = (a-b)\left(\mathscr{P}_{m,n+1} - \mathscr{P}_{m,n}\right).$$

Using this conservation law, we can eliminate one of $P_{m+1,n}$, $P_{m,n}$, $\mathscr{P}_{m,n+1}$ and $\mathscr{P}_{m,n}$ in the matrix re-factorization problem (3.11) (or (3.12)). Then, it can be solved explicitly so that the map $(P_{m,n}, \mathscr{P}_{m,n+1}) \mapsto (P_{m+1,n}, \mathscr{P}_{m,n})$ and the map $(P_{m,n}, \mathscr{P}_{m,n}) \mapsto (P_{m+1,n}, \mathscr{P}_{m,n+1})$ as well as their inverses can be expressed in closed form; such a map represents a similarity transformation that transforms a projection matrix into a projection matrix of the same

rank. For instance, the map $(P_{m,n}, \mathscr{P}_{m,n+1}) \mapsto (P_{m+1,n}, \mathscr{P}_{m,n})$ reads

$$P_{m+1,n} = \left(I - \frac{\alpha-\beta}{\alpha-b}P_{m,n} - \frac{a-b}{\alpha-b}\mathscr{P}_{m,n+1}\right) P_{m,n} \left(I - \frac{\alpha-\beta}{\alpha-b}P_{m,n} - \frac{a-b}{\alpha-b}\mathscr{P}_{m,n+1}\right)^{-1}, \tag{3.13a}$$

$$\mathscr{P}_{m,n} = \left(I - \frac{\alpha-\beta}{\alpha-b}P_{m,n} - \frac{a-b}{\alpha-b}\mathscr{P}_{m,n+1}\right) \mathscr{P}_{m,n+1} \left(I - \frac{\alpha-\beta}{\alpha-b}P_{m,n} - \frac{a-b}{\alpha-b}\mathscr{P}_{m,n+1}\right)^{-1}, \tag{3.13b}$$

which is a new Yang–Baxter map acting on the space of projection matrices and admits the Lax representation (3.12). If (3.13) is viewed as a system defined on the two-dimensional lattice, the parameters α and β (resp. a and b) can be arbitrary functions of the discrete independent variable n (resp. m). Thus, (3.13) provides a new fully discrete analog of (a nonautonomous extension of) the principal chiral model [48] restricted to the space of projection matrices ($P^2 = P,\ \mathscr{P}^2 = \mathscr{P}$):

$$\frac{\partial P}{\partial \eta} = \frac{f(\eta)}{\alpha(\xi) - a(\eta)} [P, \mathscr{P}], \tag{3.14a}$$

$$\frac{\partial \mathscr{P}}{\partial \xi} = \frac{g(\xi)}{\alpha(\xi) - a(\eta)} [P, \mathscr{P}]. \tag{3.14b}$$

Here, $[P, \mathscr{P}] := P\mathscr{P} - \mathscr{P}P$ is the commutator. When $f(\eta)$ and $g(\xi)$ are purely imaginary and $\alpha(\xi)$ and $a(\eta)$ are real, the principal chiral model (3.14) admits the Hermitian conjugation reduction $P^\dagger = P$, $\mathscr{P}^\dagger = \mathscr{P}$.

The Lax-pair representation for (3.14) is

$$\Psi_\xi = -\frac{g(\xi)}{\zeta - \alpha(\xi)} P\Psi, \qquad \Psi_\eta = -\frac{f(\eta)}{\zeta - a(\eta)} \mathscr{P}\Psi, \tag{3.15}$$

where ζ is a constant spectral parameter. Note that we can normalize $g(\xi)$ and $f(\eta)$ to nonzero constants using a point transformation of the form $\xi \to X(\xi)$, $\eta \to Y(\eta)$. This Lax-pair representation can be identified with (2.7) for two different values of k, so we can understand the principal chiral model (3.14) as the compatibility condition for two "first" negative flows of the matrix NLS hierarchy.

The nonautonomous principal chiral model (3.14) implies the relation

$$\frac{1}{f(\eta)}\frac{\partial P}{\partial \eta} - \frac{1}{g(\xi)}\frac{\partial \mathscr{P}}{\partial \xi} = O. \tag{3.16}$$

Thus, substituting the expression (cf. the Lax-pair representation (3.15))

$$P = \frac{\alpha(\xi)}{g(\xi)}\Phi_\xi\Phi^{-1}, \qquad \mathscr{P} = \frac{a(\eta)}{f(\eta)}\Phi_\eta\Phi^{-1}$$

into (3.16), we obtain the simpler form:

$$\alpha(\xi)\left(\Phi_\xi\Phi^{-1}\right)_\eta - a(\eta)\left(\Phi_\eta\Phi^{-1}\right)_\xi = O.$$

It remains an open question whether this model is related to a more familiar nonautonomous chiral model that appears in general relativity (see, *e.g.*, [78] and references therein).

Let us derive a more explicit component form of the Yang–Baxter map (3.13). In view of the representations of the projection matrices in (3.8) and (3.10), we can obtain from (3.13) the following relations:

$$\begin{aligned}
&\begin{bmatrix} -I & u_{m+1,n} \\ v_{m+1,n} & I \end{bmatrix}\begin{bmatrix} X_{m,n} & \\ & Y_{m,n} \end{bmatrix} \\
&= \left\{ I - \frac{\alpha-\beta}{\alpha-b}\begin{bmatrix} u_{m,n}v_{m,n}\left(I+u_{m,n}v_{m,n}\right)^{-1} & u_{m,n}\left(I+v_{m,n}u_{m,n}\right)^{-1} \\ v_{m,n}\left(I+u_{m,n}v_{m,n}\right)^{-1} & \left(I+v_{m,n}u_{m,n}\right)^{-1} \end{bmatrix} - \frac{a-b}{\alpha-b}\times \right. \\
&\left. \begin{bmatrix} q_{m,n+1}r_{m,n+1}\left(I+q_{m,n+1}r_{m,n+1}\right)^{-1} & q_{m,n+1}\left(I+r_{m,n+1}q_{m,n+1}\right)^{-1} \\ r_{m,n+1}\left(I+q_{m,n+1}r_{m,n+1}\right)^{-1} & \left(I+r_{m,n+1}q_{m,n+1}\right)^{-1} \end{bmatrix}\right\}\begin{bmatrix} -I & u_{m,n} \\ v_{m,n} & I \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&\begin{bmatrix} -I & q_{m,n} \\ r_{m,n} & I \end{bmatrix}\begin{bmatrix} Z_{m,n} & \\ & W_{m,n} \end{bmatrix} \\
&= \left\{ I - \frac{\alpha-\beta}{\alpha-b}\begin{bmatrix} u_{m,n}v_{m,n}\left(I+u_{m,n}v_{m,n}\right)^{-1} & u_{m,n}\left(I+v_{m,n}u_{m,n}\right)^{-1} \\ v_{m,n}\left(I+u_{m,n}v_{m,n}\right)^{-1} & \left(I+v_{m,n}u_{m,n}\right)^{-1} \end{bmatrix} - \frac{a-b}{\alpha-b}\times \right. \\
&\left. \begin{bmatrix} q_{m,n+1}r_{m,n+1}\left(I+q_{m,n+1}r_{m,n+1}\right)^{-1} & q_{m,n+1}\left(I+r_{m,n+1}q_{m,n+1}\right)^{-1} \\ r_{m,n+1}\left(I+q_{m,n+1}r_{m,n+1}\right)^{-1} & \left(I+r_{m,n+1}q_{m,n+1}\right)^{-1} \end{bmatrix}\right\}\begin{bmatrix} -I & q_{m,n+1} \\ r_{m,n+1} & I \end{bmatrix},
\end{aligned}$$

where $X_{m,n}$, $Y_{m,n}$, $Z_{m,n}$ and $W_{m,n}$ are some square matrices. By eliminating these unknown quantities, we can express the Yang–Baxter map in compo-

nent form as

$$
\begin{aligned}
u_{m+1,n} &= \left[u_{m,n} - \frac{a-b}{\beta-b} q_{m,n+1} \left(I + r_{m,n+1} q_{m,n+1}\right)^{-1} \left(I + r_{m,n+1} u_{m,n}\right)\right] \\
&\quad \times \left[I - \frac{a-b}{\beta-b} \left(I + r_{m,n+1} q_{m,n+1}\right)^{-1} \left(I + r_{m,n+1} u_{m,n}\right)\right]^{-1} \\
&= \left[(\beta-b)\, u_{m,n} \left(I + r_{m,n+1} u_{m,n}\right)^{-1} - (a-b)\, q_{m,n+1} \left(I + r_{m,n+1} q_{m,n+1}\right)^{-1}\right] \\
&\quad \times \left[(\beta-b) \left(I + r_{m,n+1} u_{m,n}\right)^{-1} - (a-b) \left(I + r_{m,n+1} q_{m,n+1}\right)^{-1}\right]^{-1} \\
&= \left[(\beta-b) \left(I + u_{m,n} r_{m,n+1}\right)^{-1} - (a-b) \left(I + q_{m,n+1} r_{m,n+1}\right)^{-1}\right]^{-1} \\
&\quad \times \left[(\beta-b) \left(I + u_{m,n} r_{m,n+1}\right)^{-1} u_{m,n} - (a-b) \left(I + q_{m,n+1} r_{m,n+1}\right)^{-1} q_{m,n+1}\right], \\
v_{m+1,n} &= \left[(\alpha-a)\, v_{m,n} \left(I + q_{m,n+1} v_{m,n}\right)^{-1} + (a-b)\, r_{m,n+1} \left(I + q_{m,n+1} r_{m,n+1}\right)^{-1}\right] \\
&\quad \times \left[(\alpha-a) \left(I + q_{m,n+1} v_{m,n}\right)^{-1} + (a-b) \left(I + q_{m,n+1} r_{m,n+1}\right)^{-1}\right]^{-1}, \\
q_{m,n} &= \left[(\alpha-a)\, q_{m,n+1} \left(I + v_{m,n} q_{m,n+1}\right)^{-1} - (\alpha-\beta)\, u_{m,n} \left(I + v_{m,n} u_{m,n}\right)^{-1}\right] \\
&\quad \times \left[(\alpha-a) \left(I + v_{m,n} q_{m,n+1}\right)^{-1} - (\alpha-\beta) \left(I + v_{m,n} u_{m,n}\right)^{-1}\right]^{-1} \\
&= \left[(\alpha-a) \left(I + q_{m,n+1} v_{m,n}\right)^{-1} - (\alpha-\beta) \left(I + u_{m,n} v_{m,n}\right)^{-1}\right]^{-1} \\
&\quad \times \left[(\alpha-a) \left(I + q_{m,n+1} v_{m,n}\right)^{-1} q_{m,n+1} - (\alpha-\beta) \left(I + u_{m,n} v_{m,n}\right)^{-1} u_{m,n}\right], \\
r_{m,n} &= \left[(\beta-b)\, r_{m,n+1} \left(I + u_{m,n} r_{m,n+1}\right)^{-1} + (\alpha-\beta)\, v_{m,n} \left(I + u_{m,n} v_{m,n}\right)^{-1}\right] \\
&\quad \times \left[(\beta-b) \left(I + u_{m,n} r_{m,n+1}\right)^{-1} + (\alpha-\beta) \left(I + u_{m,n} v_{m,n}\right)^{-1}\right]^{-1}.
\end{aligned}
$$

Note that each of the expressions for $u_{m+1,n}$, $v_{m+1,n}$, $q_{m,n}$ and $r_{m,n}$ can be rewritten in many different (but nontrivially equivalent) forms, as is illustrated for $u_{m+1,n}$. In fact, these expressions can be derived more directly from the definition of the standard binary Bäcklund–Darboux transformation [16,63] as well as its inverse, because the intermediate potentials in each projection matrix (*e.g.*, $u_{m,n}$ and $v_{m,n}$ in (3.8)) can be expressed explicitly in terms of the linear eigenfunctions. In short, a companion map (see [41, 44]) represents the composition law for two binary Bäcklund–Darboux transformations.

When $\beta = \alpha^*$ and $b = a^*$, we can impose either the complex conjugation reduction $v_{m,n} = \sigma u^*_{m,n}$, $r_{m,n+1} = \sigma q^*_{m,n+1}$, $v_{m+1,n} = \sigma u^*_{m+1,n}$, $r_{m,n} = \sigma q^*_{m,n}$ for square matrices or the Hermitian conjugation reduction ($* \to \dagger$) for (generally) rectangular matrices, where σ is a nonzero real constant; for simplicity, we do not discuss a more general reduction involving constant Hermitian

matrices corresponding to the mixed focusing-defocusing case (cf. [31–33,57–59]). In particular, by setting $\beta = \alpha^*$, $b = a^*$, $v_{m,n} = u_{m,n}^\dagger$, $r_{m,n+1} = q_{m,n+1}^\dagger$, $v_{m+1,n} = u_{m+1,n}^\dagger$, $r_{m,n} = q_{m,n}^\dagger$, we obtain the reduced form of the Yang–Baxter map $(u_{m,n}, q_{m,n+1}) \mapsto (u_{m+1,n}, q_{m,n})$:

$$\begin{aligned} u_{m+1,n} = {} & \left[(\alpha^* - a^*) \left(I + u_{m,n} q_{m,n+1}^\dagger \right)^{-1} - (a - a^*) \left(I + q_{m,n+1} q_{m,n+1}^\dagger \right)^{-1} \right]^{-1} \\ & \times \left[(\alpha^* - a^*) \left(I + u_{m,n} q_{m,n+1}^\dagger \right)^{-1} u_{m,n} - (a - a^*) \left(I + q_{m,n+1} q_{m,n+1}^\dagger \right)^{-1} q_{m,n+1} \right], \\ q_{m,n} = {} & \left[(\alpha - a) \left(I + q_{m,n+1} u_{m,n}^\dagger \right)^{-1} - (\alpha - \alpha^*) \left(I + u_{m,n} u_{m,n}^\dagger \right)^{-1} \right]^{-1} \\ & \times \left[(\alpha - a) \left(I + q_{m,n+1} u_{m,n}^\dagger \right)^{-1} q_{m,n+1} - (\alpha - \alpha^*) \left(I + u_{m,n} u_{m,n}^\dagger \right)^{-1} u_{m,n} \right]. \end{aligned}$$

Recalling that the parameter α (resp. a) can depend on the discrete independent variable n (resp. m), we obtain a nonautonomous system defined on the two-dimensional lattice:

$$\begin{aligned} & (\alpha_n^* - a_m^*) \left(I + u_{m,n} q_{m,n+1}^\dagger \right)^{-1} (u_{m+1,n} - u_{m,n}) \\ & = (a_m - a_m^*) \left(I + q_{m,n+1} q_{m,n+1}^\dagger \right)^{-1} (u_{m+1,n} - q_{m,n+1}), \\ & (\alpha_n - a_m) \left(I + q_{m,n+1} u_{m,n}^\dagger \right)^{-1} (q_{m,n+1} - q_{m,n}) \\ & = (\alpha_n - \alpha_n^*) \left(I + u_{m,n} u_{m,n}^\dagger \right)^{-1} (u_{m,n} - q_{m,n}). \end{aligned}$$

This is a fully discrete analog of the principal chiral model for Hermitian projection matrices in component form:

$$\mathrm{i} \frac{\alpha(\xi) - a(\eta)}{\mu(\eta)} \frac{\partial u}{\partial \eta} = \left(I + u q^\dagger \right) \left(I + q q^\dagger \right)^{-1} (u - q), \tag{3.17a}$$

$$\mathrm{i} \frac{\alpha(\xi) - a(\eta)}{\nu(\xi)} \frac{\partial q}{\partial \xi} = \left(I + q u^\dagger \right) \left(I + u u^\dagger \right)^{-1} (u - q). \tag{3.17b}$$

That is, (3.17) is obtained from the principal chiral model (3.14) by setting (cf. (2.8))

$$\begin{aligned} & f(\eta) = -\mathrm{i}\mu(\eta), \quad g(\xi) = -\mathrm{i}\nu(\xi), \\ & P = \begin{bmatrix} I & \\ & O \end{bmatrix} + \begin{bmatrix} -I & u \\ u^\dagger & I \end{bmatrix}^{-1}, \\ & \mathscr{P} = \begin{bmatrix} I & \\ & O \end{bmatrix} + \begin{bmatrix} -I & q \\ q^\dagger & I \end{bmatrix}^{-1}, \end{aligned}$$

where $\alpha(\xi),\ a(\eta),\ \mu(\eta),\ \nu(\xi) \in \mathbb{R}$.

4 Concluding remarks

Auto-Bäcklund transformations for a continuous integrable system provide a useful clue to obtaining its proper discretizations, but this still remains to be clarified for two or more component systems such as the NLS system and their various reductions. In this paper, following the new approach introduced in our previous paper [30], we obtained a new proper space-discretization of the matrix NLS system, which admits the Lax-pair representation and permits not only the complex conjugation reduction but also the Hermitian conjugation reduction between the two dependent variables as in the continuous case [13] (see (2.26)). This is in contrast to the matrix generalization of the Ablowitz–Ladik discretization of the NLS system [27], which permits only the complex conjugation reduction and not the Hermitian conjugation reduction in local form (see [28, 29] and references therein). Thus, our space-discrete matrix NLS system is more appropriate and physically meaningful; indeed, it can generate proper space-discrete analogs of various multicomponent NLS equations [24, 31–36] by considering reductions analogous to the continuous case. By changing the time part of the Lax-pair representation, we can also obtain proper discretizations of the matrix mKdV system and the matrix sine-Gordon system, which admit various interesting reductions such as the space-discrete vector mKdV equation (2.27).

In our approach, we reinterpret (a slightly generalized version of) the binary Bäcklund–Darboux transformation as a discrete spectral problem, wherein the two intermediate potentials appearing in the Darboux matrix are used as a pair of new dependent variables. Bianchi's permutability theorem for Bäcklund–Darboux transformations implies that the number of discrete independent variables can be increased consistently to define a multidimensional lattice. The consistency condition on an elementary quadrilateral can be solved explicitly, providing a new Yang–Baxter map of the NLS type, which can be regarded as a fully discrete analog of a reduced form of the principal chiral model. It would be interesting to investigate whether this result can be related to the recent work of Caudrelier and Q. C. Zhang [46].

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1997 on multicomponent generalizations of the sine-Gordon equation.

A Integrable discretization of the vector sine-Gordon equation

In this appendix, we present an integrable discretization of a vector analog of the sine-Gordon equation [49, 50], which was obtained by the author in the late 1990s but left unpublished.

We start from the continuous case and consider a Lax-pair representation of the following form (cf. (2.2) and (2.7)):

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I & -\frac{1}{2\sqrt{c-f}}\mathcal{Q}_x \\ \frac{1}{2\sqrt{c-f}}\mathcal{R}_x & \mathrm{i}\zeta I \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{A.1a}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_{t_{-1}} = \frac{\mathrm{i}}{\zeta} \begin{bmatrix} \sqrt{c-f}\, I & \mathcal{Q} \\ \mathcal{R} & -\sqrt{c-f}\, I \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{A.1b}$$

Here, c is a constant, f is a scalar function and $\mathcal{Q}$ and $\mathcal{R}$ are square matrices. The compatibility condition for this overdetermined linear system provides the following three equations:

$$\begin{cases} f_x I = (\mathcal{Q}\mathcal{R})_x = (\mathcal{R}\mathcal{Q})_x\,, \\ \left(\dfrac{1}{\sqrt{c-f}}\mathcal{Q}_x\right)_{t_{-1}} - 4\mathcal{Q} = O, \\ \left(\dfrac{1}{\sqrt{c-f}}\mathcal{R}_x\right)_{t_{-1}} - 4\mathcal{R} = O. \end{cases} \tag{A.2}$$

To satisfy the first equation, we restrict $\mathcal{Q}$ and $\mathcal{R}$ to the form:

$$\mathcal{Q} = q^{(1)} I + \sum_{j=1}^{2M-1} q^{(j+1)} e_j, \quad \mathcal{R} = q^{(1)} I - \sum_{j=1}^{2M-1} q^{(j+1)} e_j, \tag{A.3}$$

where the $2^{M-1} \times 2^{M-1}$ matrices $\{e_1, e_2, \ldots, e_{2M-1}\}$ are generators of the Clifford algebra; they are linearly independent and satisfy the anticommutation relations:

$$\{e_j, e_k\}_+ := e_j e_k + e_k e_j = -2\delta_{jk} I.$$

Thus, we have

$$\mathcal{Q}\mathcal{R} = \mathcal{R}\mathcal{Q} = \langle \boldsymbol{q}, \boldsymbol{q} \rangle I,$$

where $\boldsymbol{q} = (q^{(1)}, \ldots, q^{(2M)})$, so the first equation in (A.2) is satisfied by setting $f = \langle \boldsymbol{q}, \boldsymbol{q} \rangle$. Then, (A.2) reduces to the vector sine-Gordon equation [49, 50]:[2]

$$\left(\frac{1}{\sqrt{c - \langle \boldsymbol{q}, \boldsymbol{q} \rangle}} \boldsymbol{q}_x \right)_{t_{-1}} - 4\boldsymbol{q} = \boldsymbol{0}, \tag{A.4}$$

or equivalently,

$$\boldsymbol{q}_{xt_{-1}} + \frac{\langle \boldsymbol{q}, \boldsymbol{q}_{t_{-1}} \rangle}{c - \langle \boldsymbol{q}, \boldsymbol{q} \rangle} \boldsymbol{q}_x - 4\sqrt{c - \langle \boldsymbol{q}, \boldsymbol{q} \rangle}\, \boldsymbol{q} = \boldsymbol{0}.$$

By setting

$$-\frac{1}{2\sqrt{c - \langle \boldsymbol{q}, \boldsymbol{q} \rangle}} \boldsymbol{q}_x =: \boldsymbol{u},$$

which changes the spectral problem (A.1a) to a standard form, we obtain the potential form of the vector sine-Gordon equation (A.4):

$$\frac{1}{\sqrt{4c - \langle \boldsymbol{u}_{t_{-1}}, \boldsymbol{u}_{t_{-1}} \rangle}} \boldsymbol{u}_{t_{-1}x} = 2\boldsymbol{u}. \tag{A.5}$$

More information on multicomponent generalizations of the sine-Gordon equation can be found in [79, 80] and references therein.

Next, we consider a discrete-time analog of the Lax-pair representation (A.1) as given by

$$\begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I & \Delta \mathcal{Q}_n - \frac{1}{2\sqrt{c-f_n}} \mathcal{Q}_{n,x} \\ -\Delta \mathcal{R}_n + \frac{1}{2\sqrt{c-f_n}} \mathcal{R}_{n,x} & \mathrm{i}\zeta I \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix},$$

$$\begin{bmatrix} \Psi_{1,n+1} \\ \Psi_{2,n+1} \end{bmatrix} = \left\{ I + \frac{\mathrm{i}\Delta}{\zeta} \begin{bmatrix} \sqrt{c - f_n}\, I & \mathcal{Q}_n \\ \mathcal{R}_n & -\sqrt{c - f_n}\, I \end{bmatrix} \right\} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix},$$

where Δ is a time-step parameter. The compatibility condition provides

$$\begin{cases} f_{n,x} I = (\mathcal{Q}_n \mathcal{R}_n)_x = (\mathcal{R}_n \mathcal{Q}_n)_x, \\ \dfrac{1}{\Delta} \left(\dfrac{1}{\sqrt{c - f_{n+1}}} \mathcal{Q}_{n+1,x} - \dfrac{1}{\sqrt{c - f_n}} \mathcal{Q}_{n,x} \right) - 2\mathcal{Q}_{n+1} - 2\mathcal{Q}_n = O, \\ \dfrac{1}{\Delta} \left(\dfrac{1}{\sqrt{c - f_{n+1}}} \mathcal{R}_{n+1,x} - \dfrac{1}{\sqrt{c - f_n}} \mathcal{R}_{n,x} \right) - 2\mathcal{R}_{n+1} - 2\mathcal{R}_n = O. \end{cases} \tag{A.6}$$

[2] In constructing explicit smooth solutions, each square-root function in this appendix suffers from an intrinsic sign problem. Note that (A.4) in the scalar case reduces to the sine-Gordon equation by setting $c = 1$ and $\boldsymbol{q} = \sin p$, but a sign ambiguity arises in extracting the square root as $\sqrt{\cos^2 p} = \pm \cos p$. Such a sign ambiguity can be resolved by expressing the dependent variables in the Lax pair appropriately in terms of trigonometric functions, but then the equations of motion become complicated in the multicomponent case.

Then, by restricting $\mathcal{Q}_n$ and $\mathcal{R}_n$ to the same form as in (A.3) and setting $f_n = \langle \boldsymbol{q}_n, \boldsymbol{q}_n \rangle$, (A.6) simplifies to a proper discrete-time analog of the vector sine-Gordon equation (A.4):

$$\frac{1}{\Delta}\left(\frac{1}{\sqrt{c - \langle \boldsymbol{q}_{n+1}, \boldsymbol{q}_{n+1} \rangle}}\boldsymbol{q}_{n+1,x} - \frac{1}{\sqrt{c - \langle \boldsymbol{q}_n, \boldsymbol{q}_n \rangle}}\boldsymbol{q}_{n,x}\right) - 2\boldsymbol{q}_{n+1} - 2\boldsymbol{q}_n = \boldsymbol{0}. \tag{A.7}$$

By setting

$$\Delta \boldsymbol{q}_n - \frac{1}{2\sqrt{c - \langle \boldsymbol{q}_n, \boldsymbol{q}_n \rangle}}\boldsymbol{q}_{n,x} =: \boldsymbol{u}_n,$$

we obtain a time-discretization of the potential vector sine-Gordon equation (A.5) as

$$\frac{1}{\Delta}\left(\boldsymbol{u}_{n+1,x} - \boldsymbol{u}_{n,x}\right) = \sqrt{4c - \frac{1}{\Delta^2}\langle \boldsymbol{u}_{n+1} - \boldsymbol{u}_n, \boldsymbol{u}_{n+1} - \boldsymbol{u}_n \rangle}\left(\boldsymbol{u}_{n+1} + \boldsymbol{u}_n\right). \tag{A.8}$$

Essentially the same equation was obtained independently by Balakhnev and Meshkov [81, 82] as an auto-Bäcklund transformation for a vector analog of the mKdV equation.

References

[1] Y. B. Suris: *The Problem of Integrable Discretization: Hamiltonian Approach* (Birkhäuser, Basel, 2003).

[2] F. Calogero: *Bäcklund transformations and functional relation for solutions of nonlinear partial differential equations solvable via the inverse scattering method*, Lett. Nuovo Cimento **14** (1975) 537–543.

[3] F. Calogero and A. Degasperis: *Nonlinear evolution equations solvable by the inverse spectral transform. I*, Nuovo Cimento B **32** (1976) 201–242.

[4] S.-C. Chiu and J. F. Ladik: *Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique*, J. Math. Phys. **18** (1977) 690–700.

[5] R. Hirota: *Nonlinear partial difference equations III; Discrete sine-Gordon equation*, J. Phys. Soc. Jpn. **43** (1977) 2079–2086.

[6] S. J. Orfanidis: *Discrete sine-Gordon equations*, Phys. Rev. D **18** (1978) 3822–3827.

[7] S. J. Orfanidis: *Sine-Gordon equation and nonlinear σ model on a lattice*, Phys. Rev. D **18** (1978) 3828–3832.

[8] T. Miwa: *On Hirota's difference equations*, Proc. Japan Acad. A **58** (1982) 9–12.

[9] E. Date, M. Jimbo and T. Miwa: *Method for generating discrete soliton equations. I*, J. Phys. Soc. Jpn. **51** (1982) 4116–4124.

[10] P. D. Lax: *Integrals of nonlinear equations of evolution and solitary waves*, Commun. Pure Appl. Math. **21** (1968) 467–490.

[11] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: *Nonlinear-evolution equations of physical significance*, Phys. Rev. Lett. **31** (1973) 125–127.

[12] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: *The inverse scattering transform—Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974) 249–315.

[13] V. E. Zakharov and A. B. Shabat: *A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I*, Funct. Anal. Appl. **8** (1974) 226–235.

[14] B. G. Konopelchenko: *Elementary Bäcklund transformations, nonlinear superposition principle and solutions of the integrable equations*, Phys. Lett. A **87** (1982) 445–448.

[15] D. V. Chudnovsky and G. V. Chudnovsky: *Bäcklund transformation as a method of decomposition and reproduction of two-dimensional nonlinear systems*, Phys. Lett. A **87** (1982) 325–329.

[16] D. V. Chudnovsky and G. V. Chudnovsky: *Bäcklund transformations and lattice systems with G-gauge symmetries*, Phys. Lett. A **89** (1982) 117–122.

[17] E. Date, M. Jimbo and T. Miwa: *Method for generating discrete soliton equations. IV*, J. Phys. Soc. Jpn. **52** (1983) 761–765.

[18] B. G. Konopelchenko: *The group structure of Bäcklund transformations*, Phys. Lett. A **74** (1979) 189–192.

[19] B. G. Konopelchenko: *Transformation properties of the integrable evolution equations*, Phys. Lett. B **100** (1981) 254–260.

[20] H.-H. Chen: *General derivation of Bäcklund transformations from inverse scattering problems*, Phys. Rev. Lett. **33** (1974) 925–928.

[21] G. L. Lamb, Jr.: *Bäcklund transformations for certain nonlinear evolution equations*, J. Math. Phys. **15** (1974) 2157–2165.

[22] G. R. W. Quispel, F. W. Nijhoff, H. W. Capel and J. van der Linden: *Linear integral equations and nonlinear difference-difference equations*, Physica A **125** (1984) 344–380.

[23] J. van der Linden, F. W. Nijhoff, H. W. Capel and G. R. W. Quispel: *Linear integral equations and multicomponent nonlinear integrable systems I*, Physica A **137** (1986) 44–80.

[24] S. V. Manakov: *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Sov. Phys.–JETP **38** (1974) 248–253.

[25] M. J. Ablowitz and J. F. Ladik: *Nonlinear differential–difference equations and Fourier analysis*, J. Math. Phys. **17** (1976) 1011–1018.

[26] M. J. Ablowitz and J. F. Ladik: *A nonlinear difference scheme and inverse scattering*, Stud. Appl. Math. **55** (1976) 213–229.

[27] V. S. Gerdzhikov and M. I. Ivanov: *Hamiltonian structure of multicomponent nonlinear Schrödinger equations in difference form*, Theor. Math. Phys. **52** (1982) 676–685.

[28] T. Tsuchida: *Integrable discretizations of derivative nonlinear Schrödinger equations*, J. Phys. A: Math. Gen. **35** (2002) 7827–7847.

[29] A. Dimakis and F. Müller-Hoissen: *Solutions of matrix NLS systems and their discretizations: a unified treatment*, Inverse Probl. **26** (2010) 095007.

[30] T. Tsuchida: *On a new integrable discretization of the derivative nonlinear Schrödinger* (*Chen–Lee–Liu*) *equation*, arXiv:1501.01956 [nlin.SI] (2015).

[31] N. Yajima and M. Oikawa: *A class of exactly solvable nonlinear evolution equations*, Prog. Theor. Phys. **54** (1975) 1576–1577.

[32] V. G. Makhankov, N. V. Makhaldiani and O. K. Pashaev: *On the integrability and isotopic structure of the one-dimensional Hubbard model in the long wave approximation*, Phys. Lett. A **81** (1981) 161–164.

[33] V. G. Makhan'kov and O. K. Pashaev: *Nonlinear Schrödinger equation with noncompact isogroup*, Theor. Math. Phys. **53** (1982) 979–987.

[34] P. P. Kulish and E. K. Sklyanin: O(N)*-invariant nonlinear Schrödinger equation—A new completely integrable system*, Phys. Lett. A **84** (1981) 349–352.

[35] A. P. Fordy and P. P. Kulish: *Nonlinear Schrödinger equations and simple Lie algebras*, Commun. Math. Phys. **89** (1983) 427–443.

[36] S. I. Svinolupov: *Generalized Schrödinger equations and Jordan pairs*, Commun. Math. Phys. **143** (1992) 559–575.

[37] C. Athorne and A. Fordy: *Generalised KdV and MKdV equations associated with symmetric spaces*, J. Phys. A: Math. Gen. **20** (1987) 1377–1386.

[38] C. Athorne: *Some integrable equations associated with symmetric spaces*, Nonlinear Evolutions: Proceedings of the IVth Workshop on Nonlinear Evolution Equations and Dynamical Systems (World Scientific, 1988) edited by J. J. P. Leon, pp. 191–198.

[39] S. I. Svinolupov: *Jordan algebras and integrable systems*, Funct. Anal. Appl. **27** (1993) 257–265.

[40] Y. B. Suris and A. P. Veselov: *Lax matrices for Yang–Baxter maps*, J. Nonlinear Math. Phys. **10** Suppl. 2 (2003) 223–230.

[41] V. E. Adler, A. I. Bobenko and Y. B. Suris: *Geometry of Yang–Baxter maps: pencils of conics and quadrirational mappings*, Commun. Anal. Geom. **12** (2004) 967–1007.

[42] V. G. Papageorgiou, A. G. Tongas and A. P. Veselov: *Yang–Baxter maps and symmetries of integrable equations on quad-graphs*, J. Math. Phys. **47** (2006) 083502.

[43] A. Veselov: *Yang–Baxter maps: dynamical point of view*, Combinatorial Aspect of Integrable Systems, Mathematical Society of Japan Memoirs Vol. 17 (2007) pp. 145–167.

[44] A. I. Bobenko and Y. B. Suris: *Discrete Differential Geometry. Integrable Structure*, Graduate Studies in Mathematics, Vol. 98 (AMS, Providence, 2008).

[45] A. P. Veselov: *Yang–Baxter maps and integrable dynamics*, Phys. Lett. A **314** (2003) 214–221.

[46] V. Caudrelier and Q. C. Zhang: *Yang–Baxter and reflection maps from vector solitons with a boundary*, Nonlinearity **27** (2014) 1081–1103.

[47] V. M. Goncharenko and A. P. Veselov: *Yang–Baxter maps and matrix solitons*, New Trends in Integrability and Partial Solvability, NATO Science Series Vol. 132 (2004) pp. 191–197.

[48] V. E. Zakharov and A. V. Mikhaĭlov: *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Sov. Phys. JETP **47** (1978) 1017–1027.

[49] K. Pohlmeyer and K.-H. Rehren: *Reduction of the two-dimensional* $\mathrm{O}(n)$ *nonlinear* σ*-model*, J. Math. Phys. **20** (1979) 2628–2632.

[50] H. Eichenherr and K. Pohlmeyer: *Lax pairs for certain generalizations of the sine-Gordon equation*, Phys. Lett. B **89** (1979) 76–78.

[51] V. E. Zakharov: *The inverse scattering method*, "Solitons" edited by R. K. Bullough and P. J. Caudrey (Topics in Current Physics 17, Springer, Berlin, 1980) pp. 243–285.

[52] B. G. Konopelchenko: *On the structure of integrable evolution equations*, Phys. Lett. A **79** (1980) 39–43.

[53] R. Hirota: *Exact envelope-soliton solutions of a nonlinear wave equation*, J. Math. Phys. **14** (1973) 805–809.

[54] G. L. Lamb, Jr.: *Phase variation in coherent-optical-pulse propagation*, Phys. Rev. Lett. **31** (1973) 196–199.

[55] M. J. Ablowitz, D. J. Kaup and A. C. Newell: *Coherent pulse propagation, a dispersive, irreversible phenomenon*, J. Math. Phys. **15** (1974) 1852–1858.

[56] M. J. Ablowitz and H. Segur: *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[57] M. J. Ablowitz: *Lectures on the inverse scattering transform*, Stud. Appl. Math. **58** (1978) 17–94.

[58] A. C. Newell: *The general structure of integrable evolution equations*, Proc. R. Soc. Lond. A **365** (1979) 283–311.

[59] V. E. Zakharov and E. I. Schulman: *To the integrability of the system of two coupled nonlinear Schrödinger equations*, Physica D **4** (1982) 270–274.

[60] T. Tsuchida: *Exact solutions of multicomponent nonlinear Schrödinger equations under general plane-wave boundary conditions*, arXiv:1308.6623 [nlin.SI] (2013).

[61] N. Sasa and J. Satsuma: *New-type of soliton solutions for a higher-order nonlinear Schrödinger equation*, J. Phys. Soc. Jpn. **60** (1991) 409–417.

[62] V. E. Zakharov and A. B. Shabat: *Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II*, Funct. Anal. Appl. **13** (1979) 166–174.

[63] M. A. Sall': *Darboux transformations for non-Abelian and nonlocal equations of the Toda chain type*, Theor. Math. Phys. **53** (1982) 1092–1099.

[64] F. Kako and N. Mugibayashi: *Complete integrability of general nonlinear differential-difference equations solvable by the inverse method. II*, Prog. Theor. Phys. **61** (1979) 776–790.

[65] A. G. Izergin and V. E. Korepin: *A lattice model related to the nonlinear Schrödinger equation*, Sov. Phys. Dokl. **26** (1981) 653–654.

[66] E. K. Sklyanin: *Some algebraic structures connected with the Yang–Baxter equation*, Funct. Anal. Appl. **16** (1982) 263–270.

[67] Y. Ishimori: *An integrable classical spin chain*, J. Phys. Soc. Jpn. **51** (1982) 3417–3418.

[68] F. D. M. Haldane: *Excitation spectrum of a generalised Heisenberg ferromagnetic spin chain with arbitrary spin*, J. Phys. C: Solid State Phys. **15** (1982) L1309–1313.

[69] L. Faddeev: *Integrable models in* $(1+1)$*-dimensional quantum field theory*, Recent Advances in Field Theory and Statistical Mechanics (Les Houches Summer School, Session XXXIX, 1982) edited by J.-B. Zuber and R. Stora (North-Holland, Amsterdam, 1984) pp. 561–608.

[70] G. M. Pritula and V. E. Vekslerchik: *Toda–Heisenberg chain: interacting* σ*-fields in two dimensions*, J. Nonlinear Math. Phys. **18** (2011) 443–459.

[71] V. E. Adler and A. B. Shabat: *On the one class of hyperbolic systems*, SIGMA **2** (2006) 093.

[72] T. Tsuchida: *A refined and unified version of the inverse scattering method for the Ablowitz–Ladik lattice and derivative NLS lattices*, arXiv:1206.3210 [nlin.SI] (2012).

[73] I. Merola, O. Ragnisco and G.-Z. Tu: *A novel hierarchy of integrable lattices*, Inverse Probl. **10** (1994) 1315–1334.

[74] T. Tsuchida: *Comment on "Discretisations of constrained KP hierarchies"*, arXiv:1406.7324 [nlin.SI] (2014).

[75] R. Hirota: *Exact N-soliton solution of nonlinear lumped self-dual network equations*, J. Phys. Soc. Jpn. **35** (1973) 289–294.

[76] M. J. Ablowitz and J. F. Ladik: *On the solution of a class of nonlinear partial difference equations*, Stud. Appl. Math. **57** (1977) 1–12.

[77] T. Tsuchida: *A systematic method for constructing time discretizations of integrable lattice systems: local equations of motion*, J. Phys. A: Math. Theor. **43** (2010) 415202.

[78] A. Dimakis, N. Kanning and F. Müller-Hoissen: *The non-autonomous chiral model and the Ernst equation of general relativity in the bidifferential calculus framework*, SIGMA **7** (2011) 118.

[79] I. Bakas, Q-H. Park and H.-J. Shin: *Lagrangian formulation of symmetric space sine-Gordon models*, Phys. Lett. B **372** (1996) 45–52.

[80] S. C. Anco and T. Wolf: *Some symmetry classifications of hyperbolic vector evolution equations*, arXiv:nlin/0412015 [nlin.SI] (2004).

[81] M. Y. Balakhnev: *A class of integrable evolutionary vector equations*, Theor. Math. Phys. **142** (2005) 8–14.

[82] M. J. Balakhnev and A. G. Meshkov: *On a classification of integrable vectorial evolutionary equations*, J. Nonlinear Math. Phys. **15** (2008) 212–226.