Soliton solutions to integrable lattices of the derivative NLS type

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§1 Introduction

There are three well-known derivative nonlinear Schrödinger (DNLS) equations, that is,

(i) Kaup–Newell equation:

$$\begin{cases} \mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 r)_x = 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}(r^2 q)_x = 0. \end{cases} \tag{1}$$

(ii) Chen–Lee–Liu equation:

$$\begin{cases} \mathrm{i}q_t + q_{xx} - \mathrm{i}qrq_x = 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}rqr_x = 0. \end{cases} \tag{2}$$

(iii) Gerdjikov–Ivanov equation:

$$\begin{cases} \mathrm{i}q_t + q_{xx} + \mathrm{i}qr_x q + \frac{1}{2}q^3 r^2 = 0, \\ \mathrm{i}r_t - r_{xx} + \mathrm{i}rq_x r - \frac{1}{2}r^3 q^2 = 0. \end{cases} \tag{3}$$

Each of (1)–(3) admits the reduction of complex conjugation $r = \pm q^*$. Note that the sign $\pm$ is not essential (cf. $x \mapsto -x$).

If we apply the dependent variable transformation,

$$Q = q \exp\left(-2\mathrm{i}\delta \int^x qr \,\mathrm{d}x'\right),$$
$$R = r \exp\left(2\mathrm{i}\delta \int^x qr \,\mathrm{d}x'\right),$$

to the Gerdjikov–Ivanov equation (3), we obtain a one-parameter family of DNLS equations:

$$\begin{aligned} &\mathrm{i}Q_t + Q_{xx} + \mathrm{i}(4\delta+1)Q^2R_x + 4\mathrm{i}\delta QRQ_x \\ &\qquad + (\delta+1/2)(4\delta+1)Q^3R^2 = 0, \\ &\mathrm{i}R_t - R_{xx} + \mathrm{i}(4\delta+1)R^2Q_x + 4\mathrm{i}\delta RQR_x \\ &\qquad - (\delta+1/2)(4\delta+1)R^3Q^2 = 0. \end{aligned}$$

This coincides with

the Kaup–Newell equation (1) if $\delta = -\frac{1}{2}$,

the Chen–Lee–Liu equation (2) if $\delta = -\frac{1}{4}$,

and thus (1)–(3) are related by a change of variables.

Each of the three representative DNLS equations (1)–(3) has its own advantages;

(i) Kaup–Newell equation:

- Simple and local Poisson bracket:

$$\{q(x), r(y)\} = \delta'(x - y).$$

- The reduced equation

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0,$$

  has many physical applications.

(ii) Chen–Lee–Liu equation:

- Canonical Poisson bracket:

$$\{q(x), r(y)\} = \mathrm{i}\delta(x - y).$$

(iii) Gerdjikov–Ivanov equation (with a rescaling of variables):

$$\begin{cases} \mathrm{i}q_t + q_{xx} - 2qr_xq - 2q^3r^2 = 0, \\ \mathrm{i}r_t - r_{xx} - 2rq_xr + 2r^3q^2 = 0. \end{cases}$$

- Related to the (non-reduced) NLS equation,

$$\begin{cases} \mathrm{i}u_t + u_{xx} - 2u^2v = 0, \\ \mathrm{i}v_t - v_{xx} + 2v^2u = 0, \end{cases} \tag{4}$$

either by a Miura transformation,

$$u = q, \quad v = r_x + r^2q, \tag{5}$$

or, by another Miura transformation,

$$u = -q_x + q^2r, \qquad v = r.$$

§2 Inverse scattering method for DNLS

We can solve the Gerdjikov–Ivanov equation by the inverse scattering method associated with the Zakharov–Shabat eigenvalue problem.

As the Gerdjikov–Ivanov equation is embedded in the NLS equation in two different ways ($u = q$ or $v = r$), we need to study gauge transformations linking these two embeddings.

More precisely, we notice a chain of gauge-equivalent spectral problems,

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta & q \\ r_x + r^2 q & \mathrm{i}\zeta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 - r\Psi_1 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta + qr & q \\ 2\mathrm{i}\zeta r & \mathrm{i}\zeta - qr \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 - r\Psi_1 \end{bmatrix},$$

$$\begin{bmatrix} 2\mathrm{i}\zeta\Psi_1 \\ \Psi_2 - r\Psi_1 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta + qr & 2\mathrm{i}\zeta q \\ r & \mathrm{i}\zeta - qr \end{bmatrix} \begin{bmatrix} 2\mathrm{i}\zeta\Psi_1 \\ \Psi_2 - r\Psi_1 \end{bmatrix},$$

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta & -q_x + q^2 r \\ r & \mathrm{i}\zeta \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \qquad (7)$$

$$\Phi_1 := 2\mathrm{i}\zeta\Psi_1 - q(\Psi_2 - r\Psi_1),$$
$$\Phi_2 := \Psi_2 - r\Psi_1.$$

We assume rapidly decaying boundary conditions as $x \to \pm\infty$ and introduce the scattering data between Jost functions for (6) by

$$\phi(x,\zeta) = \bar{\psi}(x,\zeta)A(\zeta) + \psi(x,\zeta)2\mathrm{i}\zeta B(\zeta),$$
$$\bar{\phi}(x,\zeta) = \bar{\psi}(x,\zeta)\bar{B}(\zeta) - \psi(x,\zeta)\bar{A}(\zeta).$$

Then, the scattering data for the other Zakharov–Shabat eigenvalue problem (7) is given by

$$\phi'(x,\zeta) = \bar{\psi}'(x,\zeta)A(\zeta) + \psi'(x,\zeta)B(\zeta),$$
$$\bar{\phi}'(x,\zeta) = \bar{\psi}'(x,\zeta)2\mathrm{i}\zeta\bar{B}(\zeta) - \psi'(x,\zeta)\bar{A}(\zeta).$$

Therefore, the Gerdjikov–Ivanov equation can be solved using *two copies* of the inverse scattering formulae for the Zakharov–Shabat eigenvalue problem.

We have only to replace the function $F(x)$,

$$F(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{B(\xi)}{A(\xi)}\mathrm{e}^{2\mathrm{i}\xi x}\mathrm{d}\xi - \mathrm{i}\sum_{j=1}^{N} C_j \mathrm{e}^{2\mathrm{i}\zeta_j x},$$

in one set of Gel'fand–Levitan–Marchenko equations by $\partial_x F(x)$, and

$$\bar{F}(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\bar{B}(\xi)}{\bar{A}(\xi)}\mathrm{e}^{-2\mathrm{i}\xi x}\mathrm{d}\xi + \mathrm{i}\sum_{k=1}^{\bar{N}} \bar{C}_k \mathrm{e}^{-2\mathrm{i}\bar{\zeta}_k x},$$

in the other set of Gel'fand–Levitan–Marchenko equations by $-\partial_x \bar{F}(x)$.

The quantity

$$\exp\Big(\int_{-\infty}^{x} qr\,\mathrm{d}x'\Big),$$

can be computed without actually performing the integration (Kawata et al.). Thus, we also obtain an explicit formula for the solutions to the Kaup–Newell equation.

Note that the Kaup–Newell equation,

$$\begin{cases} \mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 r)_x = 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}(r^2 q)_x = 0, \end{cases}$$

can be written in the potential form:

$$\begin{cases} \mathrm{i}\psi_t + \psi_{xx} - \mathrm{i}\psi_x^2 \phi_x = 0, \\ \mathrm{i}\phi_t - \phi_{xx} - \mathrm{i}\phi_x^2 \psi_x = 0, \end{cases}$$

with $q = \psi_x$ and $r = \phi_x$. In fact, all elementary solutions to the Kaup–Newell equation can be rewritten as the x-derivative of elementary functions.

§3 Integrable discretizations

We discuss integrable space discretizations of DNLS equations that admit the reduction of complex conjugation like $r = \pm q^*$.

An integrable discrete version of the NLS equation (4) was proposed by Ablowitz and Ladik,

$$\begin{aligned} \mathrm{i}u_{n,t} + u_{n+1} + u_{n-1} - 2u_n \\ - u_n v_n (u_{n+1} + u_{n-1}) = 0, \end{aligned} \quad \text{(8a)}$$

$$\begin{aligned} \mathrm{i}v_{n,t} - v_{n+1} - v_{n-1} + 2v_n \\ + v_n u_n (v_{n+1} + v_{n-1}) = 0. \end{aligned} \quad \text{(8b)}$$

Remark. The reduction $v_n = \pm u_n^*$ changes (8) to a single equation,

$$\mathrm{i}u_{n,t} + u_{n+1} + u_{n-1} - 2u_n \mp |u_n|^2 (u_{n+1} + u_{n-1}) = 0.$$

We have shown that a natural discrete analogue of the Miura map (5),

$$u_n = q_n, \quad v_n = r_{n+1} - r_n + r_{n+1} r_n q_n,$$

connects the Ablowitz–Ladik lattice (8) with the following lattice:

$$\begin{aligned} &\mathrm{i} q_{n,t} + q_{n+1} + q_{n-1} - 2q_n \\ &\quad - q_n (r_{n+1} - r_n)(q_{n+1} + q_{n-1}) \\ &\quad - q_n^2 r_{n+1} r_n (q_{n+1} + q_{n-1}) = 0, \quad \text{(9a)} \\ &\mathrm{i} r_{n,t} - r_{n+1} - r_{n-1} + 2r_n \\ &\quad - r_n (q_n - q_{n-1})(r_{n+1} + r_{n-1}) \\ &\quad + r_n^2 q_n q_{n-1} (r_{n+1} + r_{n-1}) = 0. \quad \text{(9b)} \end{aligned}$$

This gives an integrable semi-discretization of the Gerdjikov–Ivanov equation and admits the reduction of complex conjugation,

$$r_n = \pm \mathrm{i} q^*_{n-\frac{1}{2}}.$$

We note that the conserved density of the Ablowitz–Ladik lattice (8),

$$\log(1-u_n v_n) = \log(1+q_n r_n)+\log(1-q_n r_{n+1}),$$

splits into two conserved densities of (9),

$$\log(1+q_n r_n), \quad \log(1-q_n r_{n+1}).$$

As a discrete analogue of the transformation,

$$Q = q\exp\left(-2\mathrm{i}\delta\int^x qr\,\mathrm{d}x'\right),$$
$$R = r\exp\left(2\mathrm{i}\delta\int^x qr\,\mathrm{d}x'\right),$$

we apply to the semi-discrete Gerdjikov–Ivanov equation (9) the following transformation:

$$Q_n = q_n \prod_{j=-\infty}^{n}\left(\frac{1-q_{j-1}r_j}{1+q_j r_j}\right)^{-2\delta},$$
$$R_n = r_n \prod_{j=-\infty}^{n}\left(\frac{1-q_{j-1}r_j}{1+q_{j-1}r_{j-1}}\right)^{2\delta}.$$

Then we obtain a one-parameter family of DNLS lattices, which admit the reduction,

$$r_n = \pm \mathrm{i} q^{\;*}_{n-\frac{1}{2}}.$$

In the same way as in the continuous case, we obtain a semi-discrete Chen–Lee–Liu equation by setting $\delta = -\frac{1}{4}$.

Next, setting $\delta = -\frac{1}{2}$ $(Q_n \to q_n,\ R_n \to r_n)$, we obtain the lattice,

$$\mathrm{i}q_{n,t} + \frac{q_{n+1}}{1 - q_{n+1}r_{n+1}} - \frac{q_n}{1 - q_n r_n} - \frac{q_n}{1 + q_n r_{n+1}} + \frac{q_{n-1}}{1 + q_{n-1}r_n} = 0, \tag{10a}$$

$$\mathrm{i}r_{n,t} - \frac{r_{n+1}}{1 + r_{n+1}q_n} + \frac{r_n}{1 + r_n q_{n-1}} + \frac{r_n}{1 - r_n q_n} - \frac{r_{n-1}}{1 - r_{n-1}q_{n-1}} = 0, \tag{10b}$$

which gives an integrable space discretization of the Kaup–Newell equation,

$$\begin{cases} \mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 r)_x = 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}(r^2 q)_x = 0. \end{cases}$$

The semi-discrete Kaup–Newell equation (10) can be written in a Hamiltonian form,

$$q_{n,t} = \{q_n, H\}, \quad r_{n,t} = \{r_n, H\},$$

where the Hamiltonian and the Poisson brackets are defined by

$$H = \sum_n \log\big[(1 - q_n r_n)(1 + q_n r_{n+1})\big],$$

$$\{q_n, q_m\} = \{r_n, r_m\} = 0,$$
$$\{q_n, r_m\} = \mathrm{i}(\delta_{n,m} - \delta_{n+1,m}).$$

This reduces to the local Hamiltonian structure of the Kaup–Newell equation in a continuous limit.

The semi-discrete Kaup–Newell equation is associated with the inverse scattering problem,

$$\Psi_{n+1} = L_n \Psi_n,$$

$$L_n = \begin{bmatrix} z + (\frac{1}{z} - z) q_n r_n & \frac{1}{z} q_n \\ (\frac{1}{z} - z) r_n & \frac{1}{z} \end{bmatrix}. \qquad (11)$$

Remark 1. By a rescaling,

$$z = \exp(-\Delta \cdot \mathrm{i}\zeta), \quad q_n \mapsto \Delta q_n, \quad r_n \mapsto -\frac{\mathrm{i}}{2} r_n,$$

the Lax matrix (11) has the asymptotic form

$$L_n = \begin{bmatrix} 1 - \Delta \cdot \mathrm{i}\zeta & \Delta \cdot q_n \\ \Delta \cdot \zeta r_n & 1 + \Delta \cdot \mathrm{i}\zeta \end{bmatrix} + O(\Delta^2),$$

which reduces to the Kaup–Newell eigenvalue problem,

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta & q \\ \zeta r & \mathrm{i}\zeta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix},$$

in the continuum limit ($\Delta \to 0$).

Remark 2. The L_n–matrix (11) can be factored into a product of two matrices:

$$\begin{bmatrix} z+(\frac{1}{z}-z)q_n r_n & \frac{1}{z}q_n \\ (\frac{1}{z}-z)r_n & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} z & q_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (\frac{1}{z}-z)r_n & \frac{1}{z} \end{bmatrix}.$$

This corresponds to the decomposition into a sum of two matrices in the continuous case:

$$\begin{bmatrix} -\mathrm{i}\zeta & q \\ \zeta r & \mathrm{i}\zeta \end{bmatrix} = \begin{bmatrix} -\mathrm{i}\zeta & q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \zeta r & \mathrm{i}\zeta \end{bmatrix}.$$

Remark 3. The zero-curvature condition

$$L_{n,t} + L_n M_n - M_{n+1} L_n = O, \quad L_n = L_n^{(2)} L_n^{(1)},$$

is decomposed into two conditions

$$L_{n,t}^{(1)} + L_n^{(1)} M_n - N_{n+\frac{1}{2}} L_n^{(1)} = O,$$
$$L_{n,t}^{(2)} + L_n^{(2)} N_{n+\frac{1}{2}} - M_{n+1} L_n^{(2)} = O.$$

The reduction of Hermitian conjugation

$$L_{n+\frac{1}{2}}^{(1)\,\dagger} \sim L_n^{(2)\,-1}, \qquad M_{n+\frac{1}{2}}^{\,\dagger} \sim -N_{n+\frac{1}{2}},$$

implies that the associated flows admit the reduction of complex conjugation $r_n \propto q^*_{n-\frac{1}{2}}$.

Remark 4. Through a gauge transformation $\Phi_n = g_n \Psi_n$, the L_n–matrix (11) is transformed to a new form,

$$L'_n = g_{n+1} L_n g_n^{-1}.$$

By properly choosing the gauge g_n, the matrix L'_n satisfies the fundamental r-matrix relation:

$$\{L'_n(\lambda) \overset{\otimes}{,} L'_m(\mu)\} = \delta_{n,m}[L'_n(\lambda) \otimes L'_m(\mu),\, r(\lambda, \mu)],$$

for a certain 4×4 matrix $r(\lambda, \mu)$. This guarantees that the semi-discrete Kaup–Newell hierarchy possesses a set of conserved quantities in involution,

$$\{\mathrm{I}_j, \mathrm{I}_k\} = 0, \quad j, k = 1, 2, \dots,$$

that is, the Liouville integrability.

§4 Inverse scattering for discrete DNLS

The inverse scattering method provides a solution formula for the Ablowitz–Ladik lattice, i.e.

$$u_n = K_1(n,n), \quad v_n = \bar{K}_2(n,n),$$

where $K_1(n,m)$ and $\bar{K}_2(n,m)$ solve the discrete Gel'fand–Levitan–Marchenko equations,

$$K_1(n,m) - \bar{F}(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} K_1(n,n+j) \\ \times F(n+j+k+1)\bar{F}(m+k+1) = 0, \quad m \geq n,$$

$$\bar{K}_2(n,m) + F(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \bar{K}_2(n,n+j) \\ \times \bar{F}(n+j+k+1)F(m+k+1) = 0, \quad m \geq n.$$

Here, $\bar{F}(m)$ and $F(m)$ are defined in terms of the scattering data as

$$\begin{aligned} \bar{F}(m) := \frac{1}{2\pi \mathrm{i}} \oint_{|\mu|=1} \bar{B}(\mu)\bar{A}(\mu)^{-1}\mu^{-m-2}\mathrm{d}\mu \\ + \sum_{j=1}^{\bar{N}} \bar{C}_j \bar{\mu}_j^{-m-2}, \\ F(m) := \frac{1}{2\pi \mathrm{i}} \oint_{|\mu|=1} B(\mu)A(\mu)^{-1}\mu^{m}\mathrm{d}\mu \\ - \sum_{j=1}^{N} C_j \mu_j^{m}, \end{aligned}$$

and solve the *linear* evolution equations,

$$\begin{cases} \mathrm{i}\dfrac{\partial \bar{F}(m)}{\partial t} + \bar{F}(m+1) + \bar{F}(m-1) - 2\bar{F}(m) = 0, \\ \mathrm{i}\dfrac{\partial F(m)}{\partial t} - F(m+1) - F(m-1) + 2F(m) = 0. \end{cases}$$

The semi-discrete Gerdjikov–Ivanov equation,

$$
\begin{aligned}
&\mathrm{i}q_{n,t} + q_{n+1} + q_{n-1} - 2q_n \\
&\qquad - q_n(r_{n+1} - r_n)(q_{n+1} + q_{n-1}) \\
&\qquad - q_n^2 r_{n+1} r_n (q_{n+1} + q_{n-1}) = 0, \\
&\mathrm{i}r_{n,t} - r_{n+1} - r_{n-1} + 2r_n \\
&\qquad - r_n(q_n - q_{n-1})(r_{n+1} + r_{n-1}) \\
&\qquad + r_n^2 q_n q_{n-1} (r_{n+1} + r_{n-1}) = 0,
\end{aligned}
$$

is related to the Ablowitz–Ladik lattice (8) by the Miura map

$$
u_n = q_n, \quad v_n = r_{n+1} - r_n + r_{n+1} r_n q_n,
$$

or, by another Miura map,

$$
u_n = q_{n-1} - q_n + q_{n-1} q_n r_n, \quad v_n = r_n.
$$

A gauge transformation that links these two Miura transformations is given as follows:

$$\begin{bmatrix} \Psi_{1,n+1} \\ \Psi_{2,n+1} \end{bmatrix} = \begin{bmatrix} z & q_n \\ r_{n+1} - r_n + r_{n+1} r_n q_n & \frac{1}{z} \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix},$$

$$\begin{bmatrix} \Phi_{1,n+1} \\ \Phi_{2,n+1} \end{bmatrix} = \begin{bmatrix} z & q_{n-1} - q_n + q_{n-1} q_n r_n \\ r_n & \frac{1}{z} \end{bmatrix} \begin{bmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{bmatrix},$$

$$\Phi_{1,n} = \Big(\frac{1}{z^2} - 1\Big)\Psi_{1,n} - \frac{q_{n-1}}{z(1 - r_n q_{n-1})} \times\Big(\Psi_{2,n} - \frac{1}{z} r_n \Psi_{1,n}\Big),$$

$$\Phi_{2,n} = \frac{1}{1 - r_n q_{n-1}}\Big(\Psi_{2,n} - \frac{1}{z} r_n \Psi_{1,n}\Big).$$

Considering the effect of the factor $1/z^2 - 1$ on the scattering data, we arrive at a solution formula for the semi-discrete Gerdjikov–Ivanov equation,

$$q_n = \kappa_1(n, n), \qquad r_n = \bar{\kappa}_2(n, n),$$

where $\kappa_1(n,m)$ and $\bar{\kappa}_2(n,m)$ satisfy modified discrete Gel'fand–Levitan–Marchenko equations,

$$\kappa_1(n,m) - \bar{F}(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\kappa_1(n,n+j)$$
$$\times \left[F(n+j+k+2) - F(n+j+k+1)\right]$$
$$\times \bar{F}(m+k+1) = 0, \qquad m \geq n,$$

$$\bar{\kappa}_2(n,m) + F(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\bar{\kappa}_2(n,n+j)$$
$$\times \left[\bar{F}(n+j+k) - \bar{F}(n+j+k+1)\right]$$
$$\times F(m+k+1) = 0, \qquad m \geq n.$$

After some manipulations, we also obtain a solution formula for the semi-discrete Kaup–Newell equation,

$$q_n = \chi_1(n,n) - \chi_1(n+1,n+1),$$
$$r_n = \bar{\chi}_2(n,n) - \bar{\chi}_2(n+1,n+1),$$

where $\chi_1(n,m)$ and $\bar{\chi}_2(n,m)$ satisfy

$$\begin{aligned}\chi_1(n,m) - \bar{f}(m) - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} &\\ \times \left[\chi_1(n,n+j) - \chi_1(n,n+j+1)\right] &\\ \times \left[f(n+j+k+2) - f(n+j+k+1)\right] &\\ \times \left[\bar{f}(m+k) - \bar{f}(m+k+1)\right] = 0, \quad m \geq n, &\end{aligned}$$

$$\begin{aligned}\bar{\chi}_2(n,m) + f(m) - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} &\\ \times \left[\bar{\chi}_2(n,n+j) - \bar{\chi}_2(n,n+j+1)\right] &\\ \times \left[\bar{f}(n+j+k) - \bar{f}(n+j+k+1)\right] &\\ \times \left[f(m+k) - f(m+k+1)\right] = 0, \quad m \geq n. &\end{aligned}$$

§5 Summary

- We have discussed integrable discretizations of DNLS equations, which admit the reduction of complex conjugation between two dependent variables.

- These DNLS lattices are integrable by the inverse scattering method. Specifically, they can be solved by considering two copies of the Ablowitz–Ladik eigenvalue problem and the gauge transformation linking them.

- Soliton solutions can be constructed straightforwardly by solving the *linear* summation (discrete integral) equations.