Thesis

Study of Multi-Component Soliton Equations Based on the Inverse Scattering Method

Takayuki Tsuchida

Received December 1999, in final form February 2000

Department of Physics, Graduate School of Science,
University of Tokyo,
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033,
Japan

Acknowledgments

I would like to express my special gratitude to my supervisor, Professor Miki Wadati, for valuable instruction and critical reading of the manuscript.

I sincerely thank Professor Harvey Segur, Department of Applied Mathematics, University of Colorado, Boulder, USA, for his kind hospitality during my stay in Boulder.

Thanks are due to Prof. K. Sogo, Prof. T. Yajima, Dr. Y. Ohta, Dr. N. Sasa, Dr. M. Hisakado, Dr. K. Nakayama, Dr. M. Shiroishi, Dr. H. Ujino, Dr. T. Tsurumi, Y. Fujii, R. Inoue and all the other members of Wadati Group for helpful suggestions and encouragements.

I wish to appreciate a Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists.

Contents

Chapter 1

Introduction

In recent years, the theory of solitons has been studied extensively as one of the most attractive fields in mathematical physics. Soliton phenomena are encountered in various branches of physics such as fluid dynamics, plasma physics, nonlinear optics, condensed matter physics and biophysics, and are known to be very fundamental. The history of solitons dates back to the observation by Scott-Russell in 1834. He observed that a heap of water in a narrow channel propagated at a constant speed without changing its form. In 1895, Korteweg and de Vries derived an equation (now known as the celebrated KdV equation) which describes shallow water waves. The KdV equation was shown to possess a solitary wave solution and thus the observation of Scott Russel was explained theoretically. However, since the KdV equation is a nonlinear partial differential equation, it was very difficult to investigate behaviors of solutions until the innovation of computers. Zabusky and Kruskal [106] studied the KdV equation by a numerical computation and discovered that solitary waves interact elastically with others, i.e. collisions do not change their profiles. In this sense, each solitary wave can be regarded as a particle and was named "soliton". In 1967, Gardner, Greene, Kruskal and Miura (GGKM) [30, 31] introduced a new method by which they solved the KdV equation analytically for the first time. An important step in their method is to express the KdV equation as the compatibility condition of an eigenvalue problem and a time evolution of the eigenfunction. The eigenvalue problem is of the form of Schrödinger equation for the potential given by the dynamical variable of the KdV equation. They first solved the scattering problem of the Schrödinger equation for the initial condition at $t = 0$ of the KdV equation and obtained the scattering data. The time evolution of the eigenfunction determined time dependence of the scattering data. Lastly, solving the inverse problem of the Schrödinger scattering problem, GGKM reconstructed the solution of the KdV equation at any time t from the scattering data at time t. The obtained solutions explicitly show that solitons in the KdV equation interact elastically with others [93]. Their method consists of the processes of solving the direct problem and the inverse problem of scattering and thus has been called the inverse scattering method (ISM). The discovery of the ISM opened the door to theoretical and analytical studies of completely integrable systems, which have become one of the most active fields in modern mathematical physics. After the success of GGKM, the ISM has been developed and sophisticated by a number of authors. Such studies have been enlarging the class of equations which have multi-soliton solutions (for short, soliton equations). Lax reformulated the work of GGKM and constructed a hierarchy of the KdV equation including higher-order KdV equations [57]. Now, we call a pair of two operators, which yields a soliton equation as the compatibility condition, the Lax pair. Zakharov and Shabat [108, 109] introduced a Lax pair of 2×2 matrix operators and solved the nonlinear Schrödinger (NLS) equation through a generalization of the ISM by GGKM. After another success of the ISM for the modified KdV (mKdV) equation [98, 99], Ablowitz, Kaup, Newell and Segur (AKNS) [4] unified a class of ISM-solvable equations by employing various time evolutions of the eigenfunction. Their formulation is called the AKNS formulation. Takhtajan [82] found that an extension of eigenvalue problem gives a solution of the Heisenberg ferromagnet (HF) equation. Kaup and Newell [47, 48] introduced a novel class of eigenvalue problem and solved a derivative nonlinear Schrödinger (DNLS) equation and the massive Thirring model. The massive Thirring model was also solved by Kuznetsov and Mikhailov [56]. Wadati, Konno and Ichikawa (WKI) [96] proposed a modified version of the Takhtajan's eigenvalue problem and derived a new hierarchy of soliton equations. We remark that almost all of the above-mentioned extensions of the ISM assume the form of Lax pairs expressed in terms of 2×2 matrices. The impressive developments of the ISM stimulated a variety of alternative approaches in soliton theory. Among those, we list the Hirota method [33–36, 44], the Painlevé analysis, Bäcklund transformations, multi-Hamiltonian theory [28, 59], classical r-matrix structure [22], symmetry approach [25, 64, 74] and algebraic study based on Lie algebras [26, 27] and Jordan algebras [32, 79–81]. Each approach has its own advantages in the study of soliton equations. For example, the Hirota method is a very powerful tool to obtain explicit expressions of soliton solutions and some other solutions. The Painlevé analysis is useful in testing integrability of novel systems. In comparison with alternative approaches, the ISM has some great advantages. It gives not only special solutions such as soliton solutions but also a procedure for solving the initial-value problem and obtaining conservation laws. The ISM can be interpreted as an extension of the Fourier transformation for nonlinear systems. In a certain sense the ISM linearizes soliton equations and enables us to superpose one solution on another. For some classes of soliton equations, it has been shown that the ISM gives a transformation from original dynamical variables to action-angle variables, which directly proves their complete integrability. Taking the above merits into account, we employ the ISM as the main approach which we use in this thesis.

Although more than thirty years has passed since the discovery of solitons and the ISM, it is still a matter of importance to obtain interesting generalizations of soliton equations with preserving the complete integrability. One direction is to consider multi-dimensional extensions of soliton equations. A typical example for this is the study of the Davey-Stewartson equation, which is a $(2+1)$-dimensional generalization of the $(1+1)$-dimensional NLS equation. Another direction, which we pursue in this thesis, is to study generalizations of soliton equations to systems which have multiple dependent variables (for short, multi-component soliton equations). Recently there have been a lot of researches focused on this field. One of the reasons is that multi-component soliton equations are significant in describing physical models which have some degree of internal freedom. The system of two-component NLS equations is a typical example which is used to describe diverse physical phenomena such as nonlinear propagation of two polarized electromagnetic waves [12, 37, 40, 46, 51, 63, 103]. Another reason may come from close interrelations between multi-component soliton equations and algebraic structure [79–81]. As is stressed in the above, the ISM provides a procedure to solve the initial-value problem of soliton equations. It is a prominent feature of the ISM, to be sure, but in practice it is often difficult to trace time evolution of initial values via the ISM. Thus, it is an important problem to find good schemes for numerical computation of soliton equations. One possible solution is to find discretizations of independent variables in soliton equations which preserves the complete integrability (for short, integrable discretizations). The discretized soliton equations are not only useful for numerical computation but also interesting as models of nonlinear population dynamics, nonlinear electric circuits and nonlinear lattice vibrations [19]. We can reproduce most of the properties of continuous soliton equations by taking an appropriate continuum limit for the corresponding integrable discretizations. Hence integrable discretizations may be regarded as a generalization of the soliton theory. In the 1990s integrable discretizations have been studied in connection with soliton cellular automata [67, 83]. In this theory, the concept of discretizations was extended to discretizations of dependent variables (referred to as ultra-discretizations).

The main theme of the thesis is to study multi-component soliton equations from a point of view of the ISM. Considering various generalizations of Lax pairs, we obtain novel multi-component extensions of soliton equations. For a part of the obtained multi-component soliton equations, we propose an integrable discretization within the framework of a discrete version of the ISM. The outline of the thesis is as follows.

In Chapter 2, we make a brief summary of general formulation of the ISM. As the first step in the ISM, we introduce an auxiliary linear problem and define the Lax pair and the zero-curvature condition. Considering some choices of 2×2 Lax matrix U, we introduce well-known soliton hierarchies by AKNS, Kaup-Newell, Takhtajan and WKI. For the AKNS hierarchy, we explain a concise outline of the ISM. We show that the formulation of the ISM can be extended for discrete space and time. The formulation includes an integrable discretization of the AKNS hierarchy proposed by Ablowitz and Ladik. Introducing an idea of gauge transformations in soliton theory, we construct the Chen-Lee-Liu hierarchy from the Kaup-Newell hierarchy.

In Chapter 3, we discuss multi-field generalizations of the AKNS formulation. We find a matrix NLS equation and a matrix mKdV equation. We show a method to construct an infinite number of integrals of motion, Hamiltonian structure and r-matrix representation for the matrix equations. Applying the ISM to the matrix AKNS hierarchy, we solve the initial-value problem and obtain multi-soliton solutions. We consider two reductions of the matrix AKNS formulation and obtain coupled NLS (cNLS) equations and coupled mKdV (cmKdV) equations. It is shown that the ISM is effective to the reduced systems by reflecting the reductions in the symmetry of scattering data. We prove that a superposition of the cNLS equations and the cmKdV equations gives a new type of coupled Hirota (cHirota) equations.

In Chapter 4, we propose a matrix generalization of the Chen-Lee-Liu equation on the basis of Lax formulation. Considering two reductions of the matrix Chen-Lee-Liu equation in terms of vectors, we obtain two simple types of coupled Chen-Lee-Liu equations. Recursion formulas for the conserved densities are explicitly given. Using gauge transformations, we clarify a connection between the obtained Lax formulations and multi-component generalizations of the AKNS formulation. By transformations of variables we derive a new system of coupled Kaup-Newell equations from one type of the coupled Chen-Lee-Liu equations.

In Chapter 5, we study matrix-valued DNLS-type equations, which were shown by Olver and Sokolov [73] to possess a higher symmetry. Introducing a transformation for the matrix Chen-Lee-Liu equation without a reduction, we prove that all systems but two in [73] have a Lax representation and that they each are connected with the others. For the remaining two systems, we explain a method to linearize them and to obtain the general solution.

In Chapter 6, we explain a multi-component generalization of the second flows in the HF hierarchy and the WKI hierarchy. For the multi-component system of the second HF flow, we find an integrable semi-discretization. We show that there is a correspondence via a gauge transformation between the multi-field HF flow and the multi-field WKI flow.

In Chapter 7, we discuss an integrable discretization of the multi-component soliton equations studied in Chapter 3. By a generalization of the Ablowitz-Ladik formulation, we obtain an integrable discretization of the matrix NLS equation and the matrix mKdV equation. We show a reduction of the discrete matrix mKdV equation to discrete cmKdV equations. By means of the discrete ISM, we solve the initial-value problem and derive the N-soliton solution for the discrete cmKdV equations. By a change of dependent variables for the discrete cmKdV equations, we obtain discrete cHirota equations, which include discrete cNLS equations with a special choice of parameters. We explain an essential difference between the continuous cNLS equations and the discrete cNLS equations from a point of view of the ISM.

The final chapter is devoted to summary and concluding remarks.

Chapter 2

Inverse Scattering Method

In this chapter, we briefly explain a general idea of the ISM. Before studying multi-field generalizations in later chapters, we summarize the class of soliton equations associated with Lax pairs of 2×2 matrix form. A matrix Lax pair was introduced by Zakharov and Shabat [108, 109] for the first time to solve the NLS equation via the ISM. Ablowitz, Kaup, Newell and Segur (AKNS) [4] refined the method by Zakharov and Shabat and established the ISM in a simple form. After the success of AKNS, some remarkable variations of the AKNS formulation were proposed. Typical ones are due to

(a) Kaup and Newell for a derivative NLS (DNLS) equation [48],

(b) Takhtajan for the Heisenberg ferromagnet (HF) equation [82],

(c) Wadati, Konno and Ichikawa (WKI) for a new hierarchy of integrable systems [96],

(d) Ablowitz and Ladik for a discrete NLS equation [6, 7].

We describe Lax formulations proposed by AKNS and (a)–(d). For the AKNS formulation we state the outline of the ISM concisely. Lastly, we introduce the idea of gauge transformations in soliton theory.

2.1 Lax Formulation

We consider a set of linear equations,

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \tag{2.1}$$

where the subscripts denote partial differentiations. Here Ψ is an L-component vector, and U, V are $L \times L$ matrices which contain an essential parameter, say ζ. The compatibility condition, $\partial_t \partial_x \Psi = \partial_x \partial_t \Psi$ is satisfied if

$$U_t - V_x + [U,\, V] = O. \tag{2.2}$$

Here $[\,\cdot\,,\,\cdot\,]$ is the commutator, $[U,\, V] \equiv UV - VU$. If we choose the form of matrices U and V appropriately, eq. (2.2) becomes equivalent to a set of nonlinear evolution equations independent of the parameter ζ. In such cases we call ζ the spectral parameter (or eigenvalue), U and V the Lax pair (or Lax matrices) and eq. (2.2) the zero-curvature condition (or simply, Lax equation). The expression (2.2) is an important step to perform the ISM. For the time being, we explain how to choose the form of the Lax pair by taking some examples.

2.2 AKNS Formulation

AKNS studied soliton equations associated with the 2×2 Lax matrix U:

$$U = \begin{bmatrix} -\mathrm{i}\zeta & q \\ r & \mathrm{i}\zeta \end{bmatrix}. \tag{2.3}$$

They showed that appropriate choices of the ζ-dependence of the Lax matrix V generate a class of soliton equations. If we expand V from ζ^2 to ζ^0,

$$V = \mathrm{i}\zeta^2 \begin{bmatrix} -2 & \\ & 2 \end{bmatrix} + \zeta \begin{bmatrix} & 2q \\ 2r & \end{bmatrix} + \mathrm{i} \begin{bmatrix} -qr & q_x \\ -r_x & qr \end{bmatrix},$$

we obtain a generalized form of the NLS equation,

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} - 2q^2 r = 0, \\ &\mathrm{i}r_t - r_{xx} + 2r^2 q = 0. \end{aligned} \tag{2.4}$$

The system includes the self-focusing case ($r = -q^*$; $*$ denotes the complex conjugation) and the defocusing case ($r = q^*$) of the NLS equation, which were studied in [108] and [109] respectively.

Further, expanding V from ζ^3 to ζ^0,

$$\begin{aligned} V = {} & \mathrm{i}\zeta^3 \begin{bmatrix} -4 & \\ & 4 \end{bmatrix} + \zeta^2 \begin{bmatrix} & 4q \\ 4r & \end{bmatrix} + \mathrm{i}\zeta \begin{bmatrix} -2qr & 2q_x \\ -2r_x & 2qr \end{bmatrix} \\ & + \begin{bmatrix} q_x r - q r_x & -q_{xx} + 2q^2 r \\ -r_{xx} + 2r^2 q & -q_x r + q r_x \end{bmatrix}, \end{aligned} \tag{2.5}$$

we obtain a pair of equations,

$$\begin{aligned} &q_t + q_{xxx} - 6q_x q r = 0, \\ &r_t + r_{xxx} - 6r_x r q = 0. \end{aligned} \tag{2.6}$$

This system reduces to the modified KdV (mKdV) equation for $r = \pm q$, the complex mKdV equation for $r = \pm q^*$ and the KdV equation for $r = -1$ respectively.

On the other hand, assuming that V has a term proportional to the inverse of ζ, i.e. $1/\zeta$, we also obtain physically important equations. The choice of V given by

$$V = \frac{\mathrm{i}}{\zeta} \begin{bmatrix} w & -\frac{1}{2}q_t \\ \frac{1}{2}r_t & -w \end{bmatrix},$$

yields a set of evolution equations:

$$\begin{aligned} w_x &= \frac{1}{2}(qr)_t, \\ q_{tx} &= 4wq, \\ r_{tx} &= 4wr. \end{aligned} \tag{2.7}$$

Setting

$$w = \frac{1}{4}\cos u, \qquad q = -r = -\frac{1}{2}u_x,$$

in eq. (2.7), we obtain the sine-Gordon equation:

$$u_{xt} = \sin u.$$

Meanwhile, choosing

$$w = \frac{1}{4}\cosh u, \qquad q = r = \frac{1}{2}u_x,$$

we obtain the sinh-Gordon equation:

$$u_{xt} = \sinh u.$$

It is surprising that only one choice of U given by eq. (2.3) yields a number of nonlinear systems of physical significance. In fact we can generate an infinite number of nonlinear evolution equations, which altogether we call the AKNS hierarchy, by expanding V from $O(\zeta^n)$ $(n = -1, 1, 2, \ldots)$ term. As will be explained later in this chapter, we can comprehensively deal with all of the members in the AKNS hierarchy in applying the ISM.

2.3 Kaup-Newell Formulation

Kaup and Newell introduced a new choice of the 2×2 Lax matrix U,

$$U = \begin{bmatrix} -\mathrm{i}\zeta^2 & \zeta q \\ \zeta r & \mathrm{i}\zeta^2 \end{bmatrix}, \tag{2.8}$$

in [47, 48]. They showed that a new hierarchy of soliton equations is derived in correspondence with various power expansions of the Lax matrix V with respect to ζ. For instance, putting eq. (2.8) and

$$\begin{aligned} V = {} & \mathrm{i}\zeta^4 \begin{bmatrix} -2 & \\ & 2 \end{bmatrix} + \zeta^3 \begin{bmatrix} & 2q \\ 2r & \end{bmatrix} + \mathrm{i}\zeta^2 \begin{bmatrix} -qr & \\ & qr \end{bmatrix} \\ & + \zeta \begin{bmatrix} & \mathrm{i}q_x + q^2 r \\ -\mathrm{i}r_x + r^2 q & \end{bmatrix}, \end{aligned} \tag{2.9}$$

into the zero-curvature condition (2.2), we obtain a generalized system of the DNLS equation,

$$\begin{aligned} \mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 r)_x &= 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}(r^2 q)_x &= 0. \end{aligned} \tag{2.10}$$

We call this system the Kaup-Newell equation for convenience. The Kaup-Newell equation has been solved via the ISM under the reduction $r = \pm q^*$ and appropriate boundary conditions [48, 50]. It is to be remarked that the massive Thirring model is also a member of the hierarchy generated by the Lax matrix (2.8), which we call the Kaup-Newell hierarchy. This fact will be shown in Section 2.8 in connection with gauge transformations.

2.4 Takhtajan Formulation

Takhtajan proposed a new eigenvalue problem in order to solve the HF equation in [82]. The Lax pair is given by

$$U = \mathrm{i}\zeta S, \qquad V = 2\mathrm{i}\zeta^2 S + \zeta S S_x. \tag{2.11}$$

Here S is a square matrix which satisfies

$$S^2 = I, \tag{2.12}$$

with I being the identity matrix. Substitution of eq. (2.11) into the Lax equation (2.2) gives the equation of motion for S,

$$\mathrm{i}S_t = \frac{1}{2}(SS_{xx} - S_{xx}S). \tag{2.13}$$

As a reduction of S given by

$$S = s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 = \boldsymbol{\sigma} \cdot \boldsymbol{S}, \tag{2.14}$$

where σ_1, σ_2, σ_3 are the Pauli matrices,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we obtain the HF spin chain:

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx}, \ \ |\boldsymbol{S}|^2 = 1, \ \ \boldsymbol{S} = (s_1, s_2, s_3). \tag{2.15}$$

Under the boundary condition

$$\lim_{x \to \pm\infty} \boldsymbol{S} = (0, 0, 1),$$

Takhtajan solved the HF equation by means of the ISM. Assuming the form of V to be a higher-order polynomial in ζ, we can construct higher flows of the HF hierarchy (see Chapter 6).

2.5 WKI Formulation

WKI showed that a new series of integrable nonlinear evolution equations is constructed via a new formulation of the ISM. In their formulation, the Lax matrix U is assumed to have the following form:

$$U = \begin{bmatrix} -\mathrm{i}\zeta & \zeta q \\ \zeta r & \mathrm{i}\zeta \end{bmatrix} = \zeta \begin{bmatrix} -\mathrm{i} & q \\ r & \mathrm{i} \end{bmatrix}.$$

Choosing

$$V = \frac{2\zeta^2}{\sqrt{1-qr}} \begin{bmatrix} -\mathrm{i} & q \\ r & \mathrm{i} \end{bmatrix} + \mathrm{i}\zeta \Big\{ \frac{1}{\sqrt{1-qr}} \begin{bmatrix} 0 & q \\ -r & 0 \end{bmatrix} \Big\}_x,$$

we obtain from eq. (2.2) the WKI equation with the linearized dispersion relation $\omega = k^2$;

$$\begin{aligned} \mathrm{i}q_t + \Big(\frac{q}{\sqrt{1-qr}} \Big)_{xx} &= 0, \\ \mathrm{i}r_t - \Big(\frac{r}{\sqrt{1-rq}} \Big)_{xx} &= 0. \end{aligned} \tag{2.16}$$

Choosing

$$\begin{aligned} V = {} & \frac{4\zeta^3}{\sqrt{1-qr}} \begin{bmatrix} -\mathrm{i} & q \\ r & \mathrm{i} \end{bmatrix} + \frac{\zeta^2}{(1-qr)^{\frac{3}{2}}} \begin{bmatrix} q_x r - q r_x & 2\mathrm{i}q_x \\ -2\mathrm{i}r_x & -q_x r + q r_x \end{bmatrix} \\ & + \zeta \Big\{ \frac{1}{(1-qr)^{\frac{3}{2}}} \begin{bmatrix} 0 & -q_x \\ -r_x & 0 \end{bmatrix} \Big\}_x, \end{aligned}$$

we obtain from eq. (2.2) the WKI equation with the linearized dispersion relation $\omega = -k^3$;

$$\begin{aligned} q_t + \left\{ \frac{q_x}{(1-qr)^{\frac{3}{2}}} \right\}_{xx} &= 0, \\ r_t + \left\{ \frac{r_x}{(1-rq)^{\frac{3}{2}}} \right\}_{xx} &= 0. \end{aligned} \tag{2.17}$$

Equations (2.16) and (2.17) are respectively looked upon as the first flow and the second flow in the WKI hierarchy. WKI gave a formula which generates an infinite series of conservation laws for the WKI flows. The ISM for the WKI equations was studied in [53, 78].

2.6 ISM for the AKNS Hierarchy

In previous sections, we have summarized Lax formulations and have given a few examples of the associated hierarchy. Now we shall concisely explain how to apply the ISM to the obtained soliton hierarchies. Here, for brevity, we only explain the outline of the ISM for the AKNS hierarchy. Most of the proofs of formulae in this section are omitted since they are either a special case of the proofs for the multi-component systems in Chapter 3 or very similar to the proofs in Chapter 3.

The spatial part of the Lax formulation by AKNS defines the following scattering problem:

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta & q \\ r & \mathrm{i}\zeta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{2.18}$$

Hereafter in this section we assume that q and r vanish rapidly as $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} q(x,t) = \lim_{x\to\pm\infty} r(x,t) = 0. \tag{2.19}$$

Let us define solutions ϕ, $\bar{\phi}$, ψ, $\bar{\psi}$ of eq. (2.18), which exhibit the following asymptotic behaviors:

$$\begin{aligned} \phi &\sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} \quad \text{as} \quad x \to -\infty, \\ \bar{\phi} &\sim \begin{bmatrix} 0 \\ -1 \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x} \quad \text{as} \quad x \to -\infty, \\ \psi &\sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x} \quad \text{as} \quad x \to +\infty, \\ \bar{\psi} &\sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} \quad \text{as} \quad x \to +\infty. \end{aligned}$$

These solutions are called Jost functions in the scattering theory. Because a pair of the Jost functions ϕ and $\bar{\phi}$, or ψ and $\bar{\psi}$, forms a fundamental system of the solutions of the scattering problem (2.18), one of the two pairs is expressed as a linear combination of the other,

$$\begin{aligned} \phi(x,\zeta) &= a(\zeta)\bar{\psi}(x,\zeta) + b(\zeta)\psi(x,\zeta), \\ \bar{\phi}(x,\zeta) &= \bar{b}(\zeta)\bar{\psi}(x,\zeta) - \bar{a}(\zeta)\psi(x,\zeta). \end{aligned}$$

Here the x-independent coefficients $a(\zeta)$, $\bar{a}(\zeta)$, $b(\zeta)$ and $\bar{b}(\zeta)$ are called scattering data. In accordance with the boundary conditions (2.19), we assume the following expression of the Jost functions ψ and $\bar{\psi}$:

$$\psi = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x} + \int_x^\infty \begin{bmatrix} K_1(x,s) \\ K_2(x,s) \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta s}\mathrm{d}s, \tag{2.20a}$$

$$\bar{\psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} + \int_x^\infty \begin{bmatrix} \bar{K}_1(x,s) \\ \bar{K}_2(x,s) \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta s}\mathrm{d}s. \tag{2.20b}$$

Substituting eqs. (2.20a) and (2.20b) into eq. (2.18), we obtain the following relations between the physical variables and the kernels:

$$q(x) = -2K_1(x,x), \quad r(x) = -2\bar{K}_2(x,x).$$

It can be shown that the kernels are related to the scattering data via the following integral equations (Gel'fand-Levitan-Marchenko equations):

$$K_1(x,y) = \bar{F}(x+y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 K_1(x,s_2)F(s_2+s_1)\bar{F}(s_1+y), \tag{2.21a}$$

$$\bar{K}_2(x,y) = -F(x+y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \bar{K}_2(x,s_2)\bar{F}(s_2+s_1)F(s_1+y), \tag{2.21b}$$

where $F(x)$ and $\bar{F}(x)$ are defined in terms of the scattering data by

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{b(\xi)}{a(\xi)}\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\xi - \mathrm{i}\sum_{j=1}^N c_j \mathrm{e}^{\mathrm{i}\zeta_j x},$$

$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\bar{b}(\xi)}{\bar{a}(\xi)}\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}\xi + \mathrm{i}\sum_{k=1}^{\bar{N}} \bar{c}_k \mathrm{e}^{-\mathrm{i}\bar{\zeta}_k x}.$$

Here we have assumed that $a(\zeta)$ ($\bar{a}(\zeta)$) has N ($\bar{N}$) isolated simple zeros $\{\zeta_1, \zeta_2, \cdots, \zeta_N\}$ ($\{\bar{\zeta}_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_{\bar{N}}\}$) in the upper (lower) half plane of ζ and is nonzero on the real axis. c_j ($\bar{c}_k$) is the residue of $b(\zeta)/a(\zeta)$ ($\bar{b}(\zeta)/\bar{a}(\zeta)$) at $\zeta = \zeta_j$ ($\zeta = \bar{\zeta}_k$).

Next we consider time dependence of the scattering data. We assume the following asymptotic form of the Lax matrix V:

$$V \to \begin{bmatrix} H(\zeta) & \\ & -H(\zeta) \end{bmatrix} \quad \text{as } x \to \pm\infty,$$

under the boundary conditions (2.19), where $H(\zeta)$ is a complex function of ζ. After some calculation, we find that time dependences of the scattering data are given by

$$a(\zeta,t) = a(\zeta,0), \tag{2.22a}$$
$$b(\zeta,t) = b(\zeta,0)\mathrm{e}^{-2H(\zeta)t}, \tag{2.22b}$$
$$c_j(t) = c_j(0)\mathrm{e}^{-2H(\zeta_j)t}, \tag{2.22c}$$

and

$$\bar{a}(\zeta,t) = \bar{a}(\zeta,0), \tag{2.23a}$$
$$\bar{b}(\zeta,t) = \bar{b}(\zeta,0)\mathrm{e}^{2H(\zeta)t}, \tag{2.23b}$$
$$\bar{c}_k(t) = \bar{c}_k(0)\mathrm{e}^{2H(\bar{\zeta}_k)t}. \tag{2.23c}$$

Due to these relations, we can easily prove that

$$\Big\{\partial_t - 2H\Big(\frac{\mathrm{i}}{2}\partial_x\Big)\Big\}\{-2\bar{F}(2x,t)\} = 0, \tag{2.24a}$$

$$\Big\{\partial_t + 2H\Big(-\frac{\mathrm{i}}{2}\partial_x\Big)\Big\}\{2F(2x,t)\} = 0. \tag{2.24b}$$

On the other hand, time-evolution equations for physical variables q, r have the following form (see [1]):

$$\Big\{\partial_t - 2H\Big(\frac{\mathrm{i}}{2}\partial_x\Big)\Big\}q + (\text{ nonlinear terms }) = 0, \tag{2.25a}$$

$$\Big\{\partial_t + 2H\Big(-\frac{\mathrm{i}}{2}\partial_x\Big)\Big\}r + (\text{ nonlinear terms }) = 0. \tag{2.25b}$$

Comparing eq. (2.25) with eq. (2.24), we find that we have a procedure to linearize soliton equations associated with eq. (2.18). This enables us to superpose one solution of eq. (2.25) on another by considering at the level of $\bar{F}$ and F. We can solve the initial-value problem in the following steps.

(a) Linearization: $q(x,0) \to -2\bar{F}(2x,0),\ \ r(x,0) \to 2F(2x,0)$.
For given potentials at $t = 0$, $q(x,0)$ and $r(x,0)$, we solve the direct problem of scattering and obtain the scattering data $\{b(\xi)/a(\xi), \bar{b}(\xi)/\bar{a}(\xi), \zeta_j, \bar{\zeta}_k, \mathrm{i}c_j, \mathrm{i}\bar{c}_k\}$.

(b) Linear time evolution: $\bar{F}(2x,0) \to \bar{F}(2x,t),\ \ F(2x,0) \to F(2x,t)$.
The time dependences of the scattering data are given by eqs. (2.22) and (2.23). Equations (2.22a) and (2.23a) show that the eigenvalues $\{\zeta_j\}$ and $\{\bar{\zeta}_k\}$ are time-independent.

(c) Nonlinearization: $-2\bar{F}(2x,t) \to q(x,t),\ \ 2F(2x,t) \to r(x,t)$.
We substitute the time-dependent scattering data into the Gel'fand-Levitan-Marchenko equations (2.21a) and (2.21b). Solving the equations, we reconstruct the time-dependent potentials,

$$q(x,t) = -2K_1(x,x;t), \qquad r(x,t) = -2\bar{K}_2(x,x;t).$$

This step corresponds to solving the inverse problem of scattering.

This solution directly proves the complete integrability of the AKNS hierarchy, that is, the initial-value problem is solvable.

Many physically significant equations appear in the case of either $r = \pm q^*$ or $r = \pm q$. For these two cases, it is important to reflect the symmetry of the Lax matrix U in the symmetry of the scattering data. When $r = \pm q^*$, we have

$$\bar{a}(\zeta) = \{a(\zeta^*)\}^*, \qquad \bar{b}(\zeta) = \mp\{b(\zeta^*)\}^*,$$

and consequently

$$\bar{N} = N, \qquad \bar{\zeta}_k = \zeta_k^*, \qquad \bar{c}_j = \mp c_j^*.$$

When $r = \pm q$, we have

$$\bar{a}(\zeta) = a(-\zeta), \qquad \bar{b}(\zeta) = \mp b(-\zeta),$$

and consequently

$$\bar{N} = N, \qquad \bar{\zeta}_k = -\zeta_k, \qquad \bar{c}_j = \mp c_j.$$

As has been mentioned above, the initial-value problem for the AKNS hierarchy can be solved in accordance with the steps (a)–(c). However, it is generally very difficult to solve the direct and inverse problems of scattering (steps (a) and (c)). An exception arises in the reflection-free case: $b(\xi) = \bar{b}(\xi) = 0$ for real ξ. In this case, since F and $\bar{F}$ are finite summations of exponential functions, we may rigorously solve the Gel'fand-Levitan-Marchenko equations. With reductions such as $r = -q^*$, we obtain soliton solutions. The ISM enables one not only to solve the initial-value problem but to construct an infinite set of conserved quantities. This is accomplished by expanding the logarithms of the time-independent scattering data, $\log a(\zeta)$ and $\log \bar{a}(\zeta)$, in negative powers of ζ. It is an interesting and important fact that the time-independent scattering data are generating functions of the integrals of motion. This approach, however, does not yield local conservation laws. We shall introduce an alternative approach in later chapters, which gives not only the densities but also the corresponding fluxes. Through this section, we have made a round-up of the general idea of the ISM for 2×2 Lax pairs by taking up the AKNS formulation as an example. We shall consider generalizations of the ISM formulation to study multi-component soliton systems later in this thesis.

2.7 Discrete Soliton Equations

The ISM is useful not only in the study of soliton equations in continuous space-time but also in investigating soliton equations in discretized space-time. Sometimes the ISM approach leads us into finding integrable discretizations of continuous soliton equations. To show this, we first consider Lax formulation for the semi-discrete case, where "semi-discrete" means that space is discrete and time is continuous. As a natural semi-discretization of eq. (2.1), we introduce an auxiliary linear problem,

$$\Psi_{n+1} = L_n \Psi_n, \quad \Psi_{n,t} = M_n \Psi_n. \tag{2.26}$$

Here , t denotes a differentiation with respect to time t. The compatibility condition $\partial_t E_n \Psi_n = E_n \partial_t \Psi_n$, where E_n is the shift operator in space, $E_n \Phi_n \equiv \Phi_{n+1}$, is satisfied if

$$L_{n,t} + L_n M_n - M_{n+1} L_n = O. \tag{2.27}$$

L_n and M_n, which contain an arbitrary parameter, constitute the Lax pair in the semi-discrete case. Equation (2.27) is a semi-discrete version of the zero-curvature condition (Lax equation). Comparing eq. (2.1) with eq. (2.26), we naively expect the following relations,

$$\lim_{\delta x \to 0} \frac{1}{\delta x}(L_n - I) = U, \tag{2.28a}$$
$$\lim_{\delta x \to 0} M_n = V, \tag{2.28b}$$

where δx is the lattice spacing for semi-discrete systems. It is easily proved that, with the relations (2.28), eq. (2.27) reduces to eq. (2.2) in the continuum limit of space: $\delta x \to 0$. If we have expressed a semi-discrete system in the form of eq. (2.27), we can usually expect that the system is completely integrable via the ISM. Hence, one effective strategy to find an integrable space discretization of continuous soliton equations is to search for a Lax pair which yields a set of evolution equations due to eq. (2.27). Here we have two remarks with regard to this approach:

(a) There are often plural schemes of integrable discretization for a continuous soliton equation. For some of those cases, there are plural L_n-matrices which reduce to the same U-matrix in the continuum limit of space (see also (b)).

(b) It is not always correct that a Lax pair for a continuous system and a Lax pair for its integrable discretization are related through eq. (2.28). One counter-example is a discrete version of the cNLS equations, which we study in Chapter 3 and Chapter 7.

Now, on the basis of the above-mentioned strategy, let us search an integrable semi-discretization of the AKNS hierarchy. Since eq. (2.28a) roughly means $L_n \sim I + \delta x \cdot U$, according to eq. (2.3), the Lax matrix L_n for a discrete AKNS hierarchy is expected to satisfy

$$L_n = \begin{bmatrix} 1 - \delta x \cdot \mathrm{i}\zeta & \delta x \cdot q_n \\ \delta x \cdot r_n & 1 + \delta x \cdot \mathrm{i}\zeta \end{bmatrix} + O(\delta x^2).$$

We can prove from eq. (2.27) that the quantity $\sum_n \log \det L_n$ is time-independent under appropriate boundary conditions. Thus $\log \det L_n$ should be the sum of a conserved density and a function of the spectral parameter. Combining the information, we find a candidate for the L_n-matrix of a discrete AKNS hierarchy:

$$L_n = \begin{bmatrix} z & q_n \\ r_n & 1/z \end{bmatrix}. \tag{2.29}$$

Here $z = \exp(-\delta x \cdot \mathrm{i}\zeta)$ is the spectral parameter and we have changed the scaling of q_n, r_n by $\delta x \cdot q_n \to q_n$, $\delta x \cdot r_n \to r_n$. After some trials, we find how to choose the form of M_n-matrix. For instance, choosing

$$\begin{aligned} M_n &= z^2 \begin{bmatrix} a & \\ & 0 \end{bmatrix} + z \begin{bmatrix} & aq_n \\ ar_{n-1} & \end{bmatrix} + \begin{bmatrix} -aq_n r_{n-1} + c_1 & \\ & br_n q_{n-1} + c_2 \end{bmatrix} \\ &\quad + \frac{1}{z} \begin{bmatrix} & -bq_{n-1} \\ -br_n & \end{bmatrix} + \frac{1}{z^2} \begin{bmatrix} 0 & \\ & -b \end{bmatrix} \\ &= \begin{bmatrix} z^2 a - aq_n r_{n-1} + c_1 & zaq_n - \frac{1}{z} bq_{n-1} \\ zar_{n-1} - \frac{1}{z} br_n & -\frac{1}{z^2} b + br_n q_{n-1} + c_2 \end{bmatrix}, \end{aligned} \tag{2.30}$$

we obtain

$$q_{n,t} - aq_{n+1} - bq_{n-1} + (c_2 - c_1)q_n + aq_{n+1}r_n q_n + bq_n r_n q_{n-1} = 0, \tag{2.31a}$$

$$r_{n,t} + br_{n+1} + ar_{n-1} + (c_1 - c_2)r_n - br_{n+1}q_n r_n - ar_n q_n r_{n-1} = 0. \tag{2.31b}$$

The Lax pair, (2.29) and (2.30), was first proposed by Ablowitz and Ladik [6]. Thus we call this formulation the Ablowitz-Ladik formulation.

Setting

$$a = -b = 1, \quad c_1 = c_2 (= 0),$$

in eq. (2.31), we obtain

$$\begin{aligned} q_{n,t} &= (1 - q_n r_n)(q_{n+1} - q_{n-1}), \\ r_{n,t} &= (1 - r_n q_n)(r_{n+1} - r_{n-1}). \end{aligned}$$

This system is interpreted as an integrable semi-discretization of the general form of the mKdV equation (2.6) [6]. Reductions, $r_n = \pm q_n$, $r_n = \pm q_n^*$ and $r_n = -1$, give a semi-discrete version of the mKdV equation, the complex mKdV equation and the KdV equation respectively.

Setting

$$a = b = \mathrm{i}, \qquad c_2 - c_1 = 2\mathrm{i},$$

in eq. (2.31), we obtain

$$\begin{aligned} &\mathrm{i}q_{n,t} + (q_{n+1} + q_{n-1} - 2q_n) - q_n r_n (q_{n+1} + q_{n-1}) = 0, \\ &\mathrm{i}r_{n,t} - (r_{n+1} + r_{n-1} - 2r_n) + r_n q_n (r_{n+1} + r_{n-1}) = 0. \end{aligned} \tag{2.32}$$

This system is interpreted as an integrable discretization of the general form of the NLS equation (2.4). The reduction $r_n = \pm q_n^*$ gives a discrete version of the NLS equation.

Once one has obtained an integrable semi-discretization of a continuous system, it is often easy to find an integrable discretization of both space and time (integrable full-discretization for short). An auxiliary linear problem in the full-discrete case is given by

$$\Psi_{n+1} = L_n \Psi_n, \quad \tilde{\Psi}_n = V_n \Psi_n,$$

where the tilde $\tilde{}$ denotes the time shift in discrete time $l \in \mathbb{Z}$: $\tilde{f}_n = f_n^{l+1} \equiv E_l f_n^l$. The compatibility condition, $E_l E_n \Psi_n = E_n E_l \Psi_n$, is satisfied if

$$\tilde{L}_n V_n = V_{n+1} L_n. \tag{2.33}$$

L_n and V_n, and eq. (2.33) are respectively the Lax pair and the zero-curvature condition (or Lax equation) in the full-discrete case.

Empirically we can obtain an integrable time discretization of a semi-discrete system by choosing the same L_n-matrix as the one for semi-discretization. This is true for a full-discretization of the Ablowitz-Ladik formulation (see [7]). The ISM for the full-discrete system can be performed only by changing the time dependence of the scattering data (see Section 7.4 for an example).

2.8 Gauge Transformations

It is now well-known that a variety of soliton equations are connected with others in many ways. In this section, we introduce one aspect of this fact; gauge transformations in the ISM formulation.

If we consider a change of gauge in the linear problem (2.1),

$$\Psi = g\Phi,$$

the linear problem and the Lax pair are respectively changed into

$$\Phi_x = U'\Phi, \quad \Phi_t = V'\Phi,$$

and

$$U' = g^{-1}Ug - g^{-1}g_x, \tag{2.34a}$$
$$V' = g^{-1}Vg - g^{-1}g_t. \tag{2.34b}$$

A semi-discrete variation and a full-discrete variation of the transformation formula (2.34) are respectively

$$\begin{aligned} L_n' &= g_{n+1}^{-1} L_n g_n, \\ M_n' &= g_n^{-1} M_n g_n - g_n^{-1} g_{n,t}, \end{aligned}$$

and

$$\begin{aligned} L_n' &= g_{n+1}^{-1} L_n g_n, \\ V_n' &= \tilde{g}_n^{-1} V_n g_n. \end{aligned}$$

Here we consider a gauge transformation for the Kaup-Newell equation (2.10) as an example. We introduce a new pair of variables q and r by

$$\begin{aligned} \mathsf{q} &= q \exp\Big\{-\frac{\mathrm{i}}{2}\int^x qr \mathrm{d}x'\Big\}, \\ \mathsf{r} &= r \exp\Big\{\frac{\mathrm{i}}{2}\int^x rq \mathrm{d}x'\Big\}. \end{aligned} \tag{2.35}$$

Using eq. (2.35) and the first conservation law for eq. (2.10), i.e.

$$\mathrm{i}(qr)_t + \Big(q_x r - q r_x - \mathrm{i}\frac{3}{2} q^2 r^2\Big)_x = 0,$$

we obtain from eq. (2.10)

$$\begin{aligned} &\mathrm{i}\mathsf{q}_t + \mathsf{q}_{xx} - \mathrm{i}\mathsf{q}\mathsf{r}\mathsf{q}_x = 0, \\ &\mathrm{i}\mathsf{r}_t - \mathsf{r}_{xx} - \mathrm{i}\mathsf{r}\mathsf{q}\mathsf{r}_x = 0. \end{aligned} \tag{2.36}$$

This is a type of DNLS equation proposed by Chen, Lee and Liu [20]. We call eq. (2.36) the Chen-Lee-Liu equation. Substitution of the inverse of the transformation (2.35), i.e.

$$\begin{aligned} q &= \mathsf{q} \exp\Big\{\frac{\mathrm{i}}{2}\int^x \mathsf{q}\mathsf{r}\, \mathrm{d}x'\Big\}, \\ r &= \mathsf{r} \exp\Big\{-\frac{\mathrm{i}}{2}\int^x \mathsf{r}\mathsf{q}\, \mathrm{d}x'\Big\}, \end{aligned}$$

into eqs. (2.8) and (2.9) gives a Lax pair for the Chen-Lee-Liu equation (2.36). We shall show that integral terms in the Lax pair can be canceled by an appropriate choice of the gauge. Choosing

$$g = \begin{bmatrix} \mathrm{e}^{\frac{\mathrm{i}}{4}\int^x \mathsf{q}\mathsf{r} dx'} & \\ & \mathrm{e}^{-\frac{\mathrm{i}}{4}\int^x \mathsf{q}\mathsf{r} dx'} \end{bmatrix},$$

we obtain from eq. (2.34) a local Lax pair without integral terms for the Chen-Lee-Liu equation:

$$U' = \mathrm{i}\zeta^2 \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} + \zeta \begin{bmatrix} & \mathsf{q} \\ \mathsf{r} & \end{bmatrix} + \mathrm{i} \begin{bmatrix} -\frac{1}{4}\mathsf{q}\mathsf{r} & \\ & \frac{1}{4}\mathsf{q}\mathsf{r} \end{bmatrix}, \tag{2.37}$$

$$\begin{aligned} V' = {} & \mathrm{i}\zeta^4 \begin{bmatrix} -2 & \\ & 2 \end{bmatrix} + \zeta^3 \begin{bmatrix} & 2\mathsf{q} \\ 2\mathsf{r} & \end{bmatrix} + \mathrm{i}\zeta^2 \begin{bmatrix} -\mathsf{q}\mathsf{r} & \\ & \mathsf{q}\mathsf{r} \end{bmatrix} + \zeta \begin{bmatrix} & \mathrm{i}\mathsf{q}_x + \frac{1}{2}\mathsf{q}^2\mathsf{r} \\ -\mathrm{i}\mathsf{r}_x + \frac{1}{2}\mathsf{r}^2\mathsf{q} & \end{bmatrix} \\ & + \mathrm{i} \begin{bmatrix} -\frac{\mathrm{i}}{4}(\mathsf{q}_x\mathsf{r} - \mathsf{q}\mathsf{r}_x) - \frac{1}{8}\mathsf{q}^2\mathsf{r}^2 & \\ & \frac{\mathrm{i}}{4}(\mathsf{q}_x\mathsf{r} - \mathsf{q}\mathsf{r}_x) + \frac{1}{8}\mathsf{q}^2\mathsf{r}^2 \end{bmatrix}. \end{aligned} \tag{2.38}$$

We call the series of soliton equations associated with the Lax matrix (2.37) the Chen-Lee-Liu hierarchy, which is equivalent to the Kaup-Newell hierarchy through the gauge transformation. A general form of the massive Thirring model,

$$\begin{aligned}
&\mathsf{q}_t - \mathrm{i}m\mathsf{p} + \mathrm{i}\frac{1}{2}\mathsf{p}\,\mathsf{s}\,\mathsf{q} = 0,\\
&\mathsf{r}_t + \mathrm{i}m\mathsf{s} - \mathrm{i}\frac{1}{2}\mathsf{p}\,\mathsf{s}\,\mathsf{r} = 0,\\
&\mathsf{p}_x - \mathrm{i}m\mathsf{q} + \mathrm{i}\frac{1}{2}\mathsf{q}\,\mathsf{r}\,\mathsf{p} = 0,\\
&\mathsf{s}_x + \mathrm{i}m\mathsf{r} - \mathrm{i}\frac{1}{2}\mathsf{q}\,\mathsf{r}\,\mathsf{s} = 0,
\end{aligned} \tag{2.39}$$

is a member of the Chen-Lee-Liu hierarchy since a Lax pair for the model is given by eq. (2.37) and

$$V' = \mathrm{i}\frac{m^2}{4\zeta^2}\begin{bmatrix} -1 & \\ & 1 \end{bmatrix} + \frac{m}{2\zeta}\begin{bmatrix} & \mathsf{p} \\ \mathsf{s} & \end{bmatrix} + \mathrm{i}\begin{bmatrix} -\frac{1}{4}\mathsf{ps} & \\ & \frac{1}{4}\mathsf{ps} \end{bmatrix}.$$

Here m is a nonzero constant. The system (2.39) reduces to the massive Thirring model:

$$\begin{aligned}
&\mathsf{q}_t = \mathrm{i}m\mathsf{p} \mp \mathrm{i}\frac{1}{2}|\mathsf{p}|^2\mathsf{q} = 0,\\
&\mathsf{p}_x = \mathrm{i}m\mathsf{q} \mp \mathrm{i}\frac{1}{2}|\mathsf{q}|^2\mathsf{p} = 0,
\end{aligned}$$

when $\mathsf{r} = \pm\mathsf{q}^*$ and $\mathsf{s} = \pm\mathsf{p}^*$ (with double signs of the same order).

We have explained only one example of gauge transformations: the transformation between the Kaup-Newell hierarchy and the Chen-Lee-Liu hierarchy. In fact, it has been proved that each of the AKNS formulation, the Kaup-Newell formulation, the Chen-Lee-Liu formulation, the Takhtajan formulation and the WKI formulation is connected with the others via gauge transformations (see [97] for the details). We shall study generalizations of the gauge transformations for multi-component systems later.

2.9 Summary

We have summarized the framework of the ISM briefly in this chapter. After introducing the Lax formulation as the compatibility condition of a linear problem, we have illustrated 2×2 Lax pairs by giving examples such as the AKNS formulation, the Kaup-Newell formulation, the Takhtajan formulation and the WKI formulation. It has been made clear that each choice of the Lax matrix U generates the associated soliton hierarchy.

Once we have a Lax pair, we expect that the system is solvable via the ISM. Starting from a Lax pair, we have explained the scenario of the ISM for the AKNS hierarchy. It has been shown that the ISM is essentially a transformation theory between the physical variables and the scattering data. Defining two functions $\bar{F}$, F in terms of the scattering data, we have obtained a linearization technique of the AKNS hierarchy. The linearization enables us to solve the initial-value problem of the hierarchy and to obtain conserved quantities and soliton solutions (see later chapters for the details).

We have studied the Lax formulation in the semi-discrete case and the full-discrete case. By inference we have rediscovered an integrable semi-discretization of the AKNS formulation, which is called the Ablowitz-Ladik formulation, and have obtained a semi-discrete mKdV equation, a semi-discrete NLS equation, etc. We did not apply the ISM to the discrete AKNS formulation since it turns out to be a special case of the ISM in Chapter 7 with a slight modification.

We have introduced the idea of gauge transformations in soliton theory. By applying a gauge transformation to a Lax pair for the Kaup-Newell equation, we have derived a local Lax pair for the Chen-Lee-Liu equation. Assuming an alternative choice of the temporal counterpart of the Lax pair, we have obtained a Lax pair for the massive Thirring model. In later chapters it is shown that the gauge transformations are of great assistance in the study of soliton equations.

Chapter 3

Generalization of AKNS Formulation

The ISM has been applied to a variety of soliton equations [1, 2, 68]. Among the soliton equations, the NLS equation and the mKdV equation have been studied extensively because of their simplicity and physical significance [33, 65, 98, 99, 108, 109]. As we have studied in Chapter 2, both of the equations are members of the AKNS hierarchy. Manakov [60] studied a system of coupled NLS (cNLS) equations and applied the ISM to the system. Generalization of the mKdV equation to a multi-component system has been studied by some authors [10, 29, 77, 104]. One example is a vector version of the mKdV equation proposed by Yajima and Oikawa [104]. Sasa and Satsuma [77] solved the initial-value problem of the system and constructed multi-soliton solutions. Another example is a matrix version of the mKdV equation studied by Athorne and Fordy [10].

In this chapter, we study a multi-field generalization of the NLS equation and the mKdV equation. First, we propose a matrix generalization of the AKNS-type Lax pair to obtain a matrix version of the NLS equation and the mKdV equation. By applying the ISM to the matrix NLS equation and the matrix mKdV equation, we solve the initial-value problem and obtain soliton solutions. Next, we consider a reduction of the matrix generalization of the ISM. As a reduction of the matrix NLS equation, we reproduce the ISM for the cNLS equations studied by Manakov. As a reduction of the matrix mKdV equation, we solve a system of coupled mKdV (cmKdV) equations by the ISM for the first time. Lastly, we propose a superposed system of the cNLS equations and the cmKdV equations.

3.1 General Formulation

3.1.1 Lax pair for the matrix mKdV equation

We introduce the following form of the Lax pair,

$$U = \mathrm{i}\zeta \begin{bmatrix} -I_1 & O \\ O & I_2 \end{bmatrix} + \begin{bmatrix} O & Q \\ R & O \end{bmatrix}, \tag{3.1}$$

$$\begin{aligned} V = {} & \mathrm{i}\zeta^3 \begin{bmatrix} -4I_1 & O \\ O & 4I_2 \end{bmatrix} + \zeta^2 \begin{bmatrix} O & 4Q \\ 4R & O \end{bmatrix} + \mathrm{i}\zeta \begin{bmatrix} -2QR & 2Q_x \\ -2R_x & 2RQ \end{bmatrix} \\ & + \begin{bmatrix} Q_xR - QR_x & -Q_{xx} + 2QRQ \\ -R_{xx} + 2RQR & R_xQ - RQ_x \end{bmatrix}, \end{aligned} \tag{3.2}$$

where ζ is the spectral parameter which does not depend on time, $\zeta_t = 0$. I_1 and I_2 are respectively the $p \times p$ and $q \times q$ unit matrices ; Q is a $p \times q$ matrix (made up of p rows and q columns); R is a $q \times p$ matrix. Obviously, this is a matrix generalization of the Lax pair (2.3) and (2.5) for the system (2.6).

Substituting eqs. (3.1) and (3.2) into eq. (2.2), we get a set of matrix equations,

$$Q_t + Q_{xxx} - 3Q_xRQ - 3QRQ_x = O, \tag{3.3a}$$

$$R_t + R_{xxx} - 3R_xQR - 3RQR_x = O. \tag{3.3b}$$

Suppose that R is connected with the Hermitian conjugate of Q by

$$R = \varepsilon Q^\dagger, \quad \varepsilon = \pm 1. \tag{3.4}$$

Then, eq. (3.3) is reduced to

$$Q_t + Q_{xxx} - 3\varepsilon(Q_xQ^\dagger Q + QQ^\dagger Q_x) = O, \quad \varepsilon = \pm 1. \tag{3.5}$$

If we restrict Q to be a real matrix, eq. (3.5) becomes equivalent to what Athorne and Fordy studied [10]. We call eq. (3.3) or eq. (3.5) the matrix mKdV equation. We shall consider the ISM for eq. (3.5) with $\varepsilon = -1$ in Section 3.2.

3.1.2 Lax pair for the matrix NLS equation

Let us employ another form of the Lax matrix V:

$$V = \mathrm{i}\zeta^2 \begin{bmatrix} -2I_1 & O \\ O & 2I_2 \end{bmatrix} + \zeta \begin{bmatrix} O & 2Q \\ 2R & O \end{bmatrix} + \mathrm{i} \begin{bmatrix} -QR & Q_x \\ -R_x & RQ \end{bmatrix}. \tag{3.6}$$

Substituting eqs. (3.1) and (3.6) into eq. (2.2), we get a set of matrix equations,

$$\mathrm{i}Q_t + Q_{xx} - 2QRQ = O, \tag{3.7a}$$

$$\mathrm{i}R_t - R_{xx} + 2RQR = O. \tag{3.7b}$$

Under the reduction (3.4), eq. (3.7) is cast into

$$\mathrm{i}Q_t + Q_{xx} - 2\varepsilon QQ^\dagger Q = O, \quad \varepsilon = \pm 1. \tag{3.8}$$

We call eq. (3.7) or eq. (3.8) the matrix NLS equation for convenience [110]. By changing the time dependence in the ISM in Section 3.2, we can solve the initial-value problem for the system (3.8) with $\varepsilon = -1$.

3.1.3 Conservation laws

In this subsection, we present a systematic method to construct local conservation laws for the matrix mKdV equation and the matrix NLS equation with $p = q = l$. It is to be remarked that we can assume $p = q$ without any loss of generality. In fact, if p is larger than q, we can change Q and R into $p \times p$ square matrices by appending the $p \times (p - q)$ zero matrix and the $(p - q) \times p$ zero matrix to Q and R respectively.

We start from a special class of eq. (2.1),

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{3.9}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{3.10}$$

where all the entries in eqs. (3.9) and (3.10) are assumed to be $l \times l$ square matrices. The following is an extension of the method for the $l = 1$ case [94]. If we define a square matrix Γ by

$$\Gamma \equiv \Psi_2 \Psi_1^{-1},$$

we can prove the following relations from eqs. (2.2), (3.9) and (3.10),

$$\{\mathrm{tr}(U_{12}\Gamma + U_{11})\}_t = \{\mathrm{tr}(V_{12}\Gamma + V_{11})\}_x, \tag{3.11}$$

$$\Gamma_x = U_{21} + U_{22}\Gamma - \Gamma U_{11} - \Gamma U_{12}\Gamma. \tag{3.12}$$

Equation (3.12) is interpreted as the matrix Riccati equation. Assuming that U is expressed as eq. (3.1), we have

$$U_{11} = -\mathrm{i}\zeta I, \quad U_{22} = \mathrm{i}\zeta I, \quad U_{12} = Q, \quad U_{21} = R, \tag{3.13}$$

where Q and R are square matrices in this case. Then eqs. (3.11) and (3.12) are cast into the following equations,

$$\{\mathrm{tr}\,(Q\Gamma)\}_t = \{\text{some function of } \Gamma,\ Q,\ R \text{ and } \zeta\}_x, \tag{3.14}$$

$$2\mathrm{i}\zeta Q\Gamma = -QR + Q(Q^{-1} \cdot Q\Gamma)_x + (Q\Gamma)^2. \tag{3.15}$$

Equation (3.14) is of the form of a local conservation law. It shows that $\mathrm{tr}\,(Q\Gamma)$ is a generating function of the conserved densities. We expand $Q\Gamma$ with respect to the spectral parameter ζ as follows,

$$Q\Gamma = \sum_{j=1}^{\infty} \frac{1}{(2\mathrm{i}\zeta)^j} F_j. \tag{3.16}$$

Substituting eq. (3.16) into eq. (3.15), we obtain a recursion formula,

$$F_{j+1} = -\delta_{j,0} QR + Q(Q^{-1}F_j)_x + \sum_{k=1}^{j-1} F_k F_{j-k}, \quad j = 0, 1, \ldots\ . \tag{3.17}$$

Each $\mathrm{tr}\,F_j$ is a conserved density for all positive integers j. Using the formula (3.17), we obtain the first four conserved densities,

$$F_1 = -QR, \quad F_1' = -RQ,$$

$$\mathrm{tr}F_2 = \mathrm{tr}\{-QR_x\},$$

$$\mathrm{tr}F_3 = \mathrm{tr}\{-QR_{xx} + QRQR\},$$

$$\mathrm{tr}F_4 = \mathrm{tr}\{-QR_{xxx} + 2QR_xQR + QRQ_xR + 2QRQR_x\}.$$

It should be noted that all elements of matrices F_1 and F_1' are conserved densities for the matrix mKdV equation (3.3) and the matrix NLS equation (3.7). This fact can be proved simply by a direct calculation.

3.1.4 Hamiltonian structure of the matrix mKdV equation and the matrix NLS equation

Let us consider Hamiltonian structure of the matrix mKdV equation and the matrix NLS equation with the condition that Q and R are $l \times l$ square matrices. A set of the Hamiltonian and the Poisson bracket for the matrix mKdV equation (3.3) is given by

$$H = \mathrm{tr} \int \{\mathrm{i}F_4\} \mathrm{d}x = \mathrm{tr} \int \Big\{ -\mathrm{i}QR_{xxx} + \mathrm{i}\frac{3}{2} QR(QR_x - Q_xR) \Big\} \mathrm{d}x, \tag{3.18}$$

and

$$\{Q(x) \underset{,}{\otimes} Q(y)\} = \{R(x) \underset{,}{\otimes} R(y)\} = O,$$
$$\{Q(x) \underset{,}{\otimes} R(y)\} = \mathrm{i}\delta(x-y)\Pi,$$

where $\{X \underset{,}{\otimes} Y\}^{ij}_{kl} = \{X_{ij}, Y_{kl}\}$ for matrices X, Y and Π denotes the $l^2 \times l^2$ permutation matrix. For the matrix NLS equation, the Hamiltonian is given by

$$H' = \mathrm{tr} \int \{-F_3\} \mathrm{d}x = \mathrm{tr} \int \{QR_{xx} - QRQR\} \mathrm{d}x,$$

instead of eq. (3.18), while the Poisson bracket is the same. We can rewrite the Poisson bracket between the elements of Q and R explicitly as follows,

$$\{Q_{ij}(x), Q_{kl}(y)\} = \{R_{ij}(x), R_{kl}(y)\} = 0,$$

$$\{Q_{ij}(x), R_{kl}(y)\} = \mathrm{i}\delta_{il}\delta_{jk}\delta(x-y) = \mathrm{i}\Pi^{ij}_{kl}\delta(x-y).$$

Indeed, we can check the following relations,

$$Q_t = \{Q, H\} = -Q_{xxx} + 3Q_xRQ + 3QRQ_x,$$

$$R_t = \{R, H\} = -R_{xxx} + 3R_xQR + 3RQR_x,$$

which are nothing but eq. (3.3). The same fact can be confirmed for the matrix NLS equation (3.7).

3.1.5 r-matrix representation of the matrix mKdV equation and the matrix NLS equation

In Section 3.1.3, we have shown that the matrix mKdV equation (3.3) and the matrix NLS equation (3.7) have an infinite number of conservation laws. Now we show that all the integrals of motion are in involution. In the following, we consider the systems on the infinite interval and assume the rapidly decreasing boundary conditions,

$$Q(x,t),\ R(x,t) \to O \quad \text{as } x \to \pm\infty. \tag{3.19}$$

If we define a classical r-matrix by

$$\mathrm{r}(\zeta_1, \zeta_2) \equiv \frac{1}{2(\zeta_1 - \zeta_2)} \begin{bmatrix} \Pi & & & \\ & O & \Pi & \\ & \Pi & O & \\ & & & \Pi \end{bmatrix},$$

the following relation,

$$\{U(x;\zeta_1) \underset{,}{\otimes} U(y;\zeta_2)\} = \delta(x-y)\Big[\mathrm{r}(\zeta_1,\zeta_2),\ U(x;\zeta_1) \otimes \begin{bmatrix} I & O \\ O & I \end{bmatrix} + \begin{bmatrix} I & O \\ O & I \end{bmatrix} \otimes U(x;\zeta_2)\Big], \tag{3.20}$$

is satisfied. Here $[A,\ B] \equiv AB - BA$ is the commutator and U is given by eq. (3.1). The transition matrix $T(x,y;\zeta)$ is defined by

$$T(x,y;\zeta) = \mathcal{E}\exp\Big\{\int_y^x U(z,\zeta)\mathrm{d}z\Big\},$$

with $\mathcal{E}$ being the path-ordering operator. From eq. (3.20), we obtain a relation for the transition matrix [54],

$$\{T(x,y;\zeta_1) \underset{,}{\otimes} T(x,y;\zeta_2)\} = [\mathrm{r}(\zeta_1,\zeta_2),\ T(x,y;\zeta_1) \otimes T(x,y;\zeta_2)]. \tag{3.21}$$

Taking the trace on both sides of eq. (3.21) , we get

$$\{\log\tau(\zeta_1), \log\tau(\zeta_2)\} = 0, \tag{3.22}$$

where $\tau(\zeta)$ is defined by

$$\tau(\zeta) = \mathrm{tr}\, T(\infty, -\infty; \zeta).$$

Expanding eq. (3.22) with respect to the spectral parameters ζ_1 and ζ_2, we have the involutiveness of the conserved quantities $\{J_n\}$,

$$\{J_n, J_m\} = 0, \quad n, m = 1, 2, \dots\ .$$

This fact indicates the complete integrability of the matrix mKdV equation and the matrix NLS equation.

3.2 Inverse Scattering Method for Matrix AKNS Hierarchy

In this section we consider the scattering problem associated with $2l \times 2l$ matrix (3.1) under the constraint (3.4) with $\varepsilon = -1$ and the boundary conditions (3.19), that is,

$$\Psi_x = U\Psi, \quad U = \begin{bmatrix} -\mathrm{i}\zeta I & Q \\ R & \mathrm{i}\zeta I \end{bmatrix}, \quad R = -Q^\dagger, \tag{3.23}$$

$$Q, R(=-Q^\dagger) \to O \quad \text{as} \quad x \to \pm\infty. \tag{3.24}$$

The results of this section are applicable to the matrix mKdV equation, the matrix NLS equation and other members of the hierarchy which altogether we call matrix AKNS hierarchy. Here we have assumed without any loss of generality that Q and R are $l \times l$ square matrices (see the explanation in Section 3.1.3). We consider a reduction to rectangular matrices Q and R in Section 3.3.1. The main idea in what follows is a modification of the analysis in [100, 102] for the matrix KdV equation.

3.2.1 Scattering problem

Let $\Psi(\zeta)$ and $\Phi(\zeta)$ be solutions of eq. (3.23) composed of $2l$ rows and l columns. We can show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\Psi^\dagger(\zeta^*)\Phi(\zeta)\} = O.$$

Hence we define an x-independent matrix function $W[\Psi, \Phi]$,

$$W[\Psi, \Phi] \equiv \Psi^\dagger(\zeta^*)\Phi(\zeta).$$

We introduce Jost functions ϕ, $\bar{\phi}$ and ψ, $\bar{\psi}$ which satisfy the boundary conditions,

$$\phi \sim \begin{bmatrix} I \\ O \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} \qquad \text{as} \quad x \to -\infty, \tag{3.25a}$$

$$\bar{\phi} \sim \begin{bmatrix} O \\ -I \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x} \qquad \text{as} \quad x \to -\infty, \tag{3.25b}$$

and

$$\psi \sim \begin{bmatrix} O \\ I \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x} \qquad \text{as} \quad x \to +\infty, \tag{3.25c}$$

$$\bar{\psi} \sim \begin{bmatrix} I \\ O \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} \qquad \text{as} \quad x \to +\infty. \tag{3.25d}$$

Here O and I are respectively the $l \times l$ zero matrix and the $l \times l$ unit matrix. It can be shown that $\phi\mathrm{e}^{\mathrm{i}\zeta x}$, $\psi\mathrm{e}^{-\mathrm{i}\zeta x}$ are analytic in the upper half plane of ζ, and $\bar{\phi}\mathrm{e}^{-\mathrm{i}\zeta x}$, $\bar{\psi}\mathrm{e}^{\mathrm{i}\zeta x}$ are analytic in the lower half plane of ζ when Q and R approach O rapidly at $x \to \pm\infty$. We assume the following integral representation of the Jost functions ψ and $\bar{\psi}$,

$$\psi = \begin{bmatrix} O \\ I \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x} + \int_x^\infty K(x,s)\mathrm{e}^{\mathrm{i}\zeta s}\mathrm{d}s, \tag{3.26}$$

$$\bar{\psi} = \begin{bmatrix} I \\ O \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} + \int_x^\infty \bar{K}(x,s)\mathrm{e}^{-\mathrm{i}\zeta s}\mathrm{d}s, \tag{3.27}$$

where $K(x,s)$ and $\bar{K}(x,s)$ are column vectors whose elements are $n \times n$ square matrices,

$$K(x,s) = \begin{bmatrix} K_1(x,s) \\ K_2(x,s) \end{bmatrix}, \qquad \bar{K}(x,s) = \begin{bmatrix} \bar{K}_1(x,s) \\ \bar{K}_2(x,s) \end{bmatrix}.$$

We substitute eq. (3.26) into eq. (3.23) and get the relations for K_1 and K_2,

$$\lim_{s\to+\infty} \begin{bmatrix} K_1(x,s) \\ K_2(x,s) \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix},$$

$$-2K_1(x,x) = Q(x),$$

$$(\partial_x - \partial_s)K_1(x,s) = Q(x)K_2(x,s) \qquad (s > x),$$

$$(\partial_x + \partial_s)K_2(x,s) = R(x)K_1(x,s) \qquad (s > x).$$

Similarly, substituting eq. (3.27) into eq. (3.23), we get for $\bar{K}_1$ and $\bar{K}_2$,

$$\lim_{s\to+\infty}\begin{bmatrix}\bar{K}_1(x,s)\\ \bar{K}_2(x,s)\end{bmatrix}=\begin{bmatrix}O\\ O\end{bmatrix},$$

$$-2\bar{K}_2(x,x)=R(x),$$

$$(\partial_x-\partial_s)\bar{K}_2(x,s)=R(x)\bar{K}_1(x,s)\qquad(s>x),$$

$$(\partial_x+\partial_s)\bar{K}_1(x,s)=Q(x)\bar{K}_2(x,s)\qquad(s>x).$$

Because a pair of the Jost functions ϕ and $\bar{\phi}$, or ψ and $\bar{\psi}$ forms a fundamental system of solutions of eq. (3.23), we can set

$$\phi(x,\zeta)=\bar{\psi}(x,\zeta)A(\zeta)+\psi(x,\zeta)B(\zeta), \tag{3.28a}$$

$$\bar{\phi}(x,\zeta)=\bar{\psi}(x,\zeta)\bar{B}(\zeta)-\psi(x,\zeta)\bar{A}(\zeta). \tag{3.28b}$$

Here the coefficients $A(\zeta)$, $\bar{A}(\zeta)$, $B(\zeta)$ and $\bar{B}(\zeta)$ are x-independent $l\times l$ matrices and called scattering data.

According to the asymptotic behaviors of the Jost functions (3.25a)–(3.25d), we get

$$\begin{aligned}
W[\phi,\phi]=W[\bar{\phi},\bar{\phi}]&=W[\psi,\psi]=W[\bar{\psi},\bar{\psi}]=I, && (3.29a)\\
W[\phi,\bar{\phi}]&=W[\psi,\bar{\psi}]=O, && (3.29b)\\
A(\zeta)&=W[\bar{\psi},\phi], && (3.29c)\\
\bar{A}(\zeta)&=-W[\psi,\bar{\phi}], && (3.29d)\\
B(\zeta)&=W[\psi,\phi], && (3.29e)\\
\bar{B}(\zeta)&=W[\bar{\psi},\bar{\phi}]. && (3.29f)
\end{aligned}$$

The expressions (3.29c) and (3.29d) show that $A(\zeta)$ and $\bar{A}(\zeta)$ are, respectively, analytic in the upper half plane and in the lower half plane. Using the above relations (3.29a)–(3.29f), we obtain the following relations among $A(\zeta)$, $\bar{A}(\zeta)$, $B(\zeta)$ and $\bar{B}(\zeta)$,

$$A^\dagger(\zeta^*)A(\zeta)+B^\dagger(\zeta^*)B(\zeta)=I, \tag{3.30a}$$

$$\bar{A}^\dagger(\zeta^*)\bar{A}(\zeta)+\bar{B}^\dagger(\zeta^*)\bar{B}(\zeta)=I, \tag{3.30b}$$

$$A^\dagger(\zeta^*)\bar{B}(\zeta)-B^\dagger(\zeta^*)\bar{A}(\zeta)=O. \tag{3.30c}$$

These relations are written as

$$\begin{bmatrix}A^\dagger(\zeta^*) & B^\dagger(\zeta^*)\\ \bar{B}^\dagger(\zeta^*) & -\bar{A}^\dagger(\zeta^*)\end{bmatrix}\begin{bmatrix}A(\zeta) & \bar{B}(\zeta)\\ B(\zeta) & -\bar{A}(\zeta)\end{bmatrix}=\begin{bmatrix}I & O\\ O & I\end{bmatrix},$$

which leads to the inversion of eq. (3.28),

$$\bar{\psi}(x,\zeta)=\phi(x,\zeta)A^\dagger(\zeta^*)+\bar{\phi}(x,\zeta)\bar{B}^\dagger(\zeta^*),$$

$$\psi(x,\zeta)=\phi(x,\zeta)B^\dagger(\zeta^*)-\bar{\phi}(x,\zeta)\bar{A}^\dagger(\zeta^*).$$

3.2.2 Gel'fand-Levitan-Marchenko equations

To derive the formula of the ISM concisely, we assume that $A(\zeta)$, $\bar{A}(\zeta)$, $B(\zeta)$ and $\bar{B}(\zeta)$ are entire functions. This assumption is true if the potentials Q and R decrease faster than any exponential function at $x \to \pm\infty$. The obtained result is, however, valid for larger classes of the potentials Q and R.

Multiplying $A(\zeta)^{-1}$ and $\bar{A}(\zeta)^{-1}$ from the right to eqs. (3.28a) and (3.28b) respectively, we get

$$\phi(x,\zeta)A(\zeta)^{-1} = \bar{\psi}(x,\zeta) + \psi(x,\zeta)B(\zeta)A(\zeta)^{-1}, \tag{3.31a}$$

$$\bar{\phi}(x,\zeta)\bar{A}(\zeta)^{-1} = -\psi(x,\zeta) + \bar{\psi}(x,\zeta)\bar{B}(\zeta)\bar{A}(\zeta)^{-1}. \tag{3.31b}$$

We operate

$$\frac{1}{2\pi}\int_C \mathrm{d}\zeta \mathrm{e}^{\mathrm{i}\zeta y} \quad (y > x)$$

on eq. (3.31a), where C is a semi-circle contour from $-\infty + \mathrm{i}0^+$ to $+\infty + \mathrm{i}0^+$ passing above all poles of $1/\mathrm{det}A(\zeta)$. After a standard calculation, we get the Gel'fand-Levitan-Marchenko equation,

$$\bar{K}(x,y) + \begin{bmatrix} O \\ I \end{bmatrix} F(x+y) + \int_x^\infty K(x,s)F(s+y)\mathrm{d}s = \begin{bmatrix} O \\ O \end{bmatrix} \qquad (y > x), \tag{3.32}$$

where $F(x)$ is defined by

$$F(x) = \frac{1}{2\pi}\int_C \mathrm{d}\zeta \mathrm{e}^{\mathrm{i}\zeta x} B(\zeta)A(\zeta)^{-1}.$$

We remark that $A(\zeta)^{-1}$ is given by

$$A(\zeta)^{-1} = \frac{1}{\mathrm{det}A(\zeta)}\breve{A}(\zeta),$$

where $\breve{A}$ is the cofactor matrix of A. We assume that $1/\mathrm{det}A(\zeta)$ has N isolated simple poles $\{\zeta_1, \zeta_2, \cdots, \zeta_N\}$ in the upper half plane and is regular on the real axis. Each of these poles determines one bound state. Then, by use of the residue theorem, we get an alternative expression of F,

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\xi \mathrm{e}^{\mathrm{i}\xi x} B(\xi)A(\xi)^{-1} - \mathrm{i}\sum_{j=1}^N C_j \mathrm{e}^{\mathrm{i}\zeta_j x}.$$

Here C_j is the residue matrix of $B(\zeta)A(\zeta)^{-1}$ at $\zeta = \zeta_j$.

Similarly, we operate

$$\frac{1}{2\pi}\int_{\bar{C}} \mathrm{d}\zeta \mathrm{e}^{-\mathrm{i}\zeta y} \quad (y > x)$$

on eq. (3.31b), where $\bar{C}$ is a semi-circle contour from $-\infty + \mathrm{i}0^-$ to $+\infty + \mathrm{i}0^-$ passing below all poles of $1/\mathrm{det}\bar{A}(\zeta)$. We get the counterpart of the Gel'fand-Levitan-Marchenko equation,

$$K(x,y) - \begin{bmatrix} I \\ O \end{bmatrix} \bar{F}(x+y) - \int_x^\infty \bar{K}(x,s)\bar{F}(s+y)\mathrm{d}s = \begin{bmatrix} O \\ O \end{bmatrix} \qquad (y > x), \tag{3.33}$$

where $\bar{F}(x)$ is defined by

$$\bar{F}(x) = \frac{1}{2\pi}\int_{\bar{C}} \mathrm{d}\zeta \mathrm{e}^{-\mathrm{i}\zeta x} \bar{B}(\zeta)\bar{A}(\zeta)^{-1}.$$

Assuming that $1/\det\bar{A}(\zeta)$ has $\bar{N}$ isolated simple poles $\{\bar{\zeta}_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_{\bar{N}}\}$ in the lower half plane and is regular on the real axis, we get an alternative expression of $\bar{F}$,

$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{e}^{-\mathrm{i}\xi x}\bar{B}(\xi)\bar{A}(\xi)^{-1} + \mathrm{i}\sum_{k=1}^{\bar{N}} \bar{C}_k \mathrm{e}^{-\mathrm{i}\bar{\zeta}_k x}.$$

Here $\bar{C}_k$ is the residue matrix of $\bar{B}(\zeta)\bar{A}(\zeta)^{-1}$ at $\zeta = \bar{\zeta}_k$.

3.2.3 Time dependence of the scattering data

Under the rapidly decreasing boundary conditions (3.24), the asymptotic form of the Lax matrix V for the matrix mKdV equation is given by

$$V \to \begin{bmatrix} -4\mathrm{i}\zeta^3 I & O \\ O & 4\mathrm{i}\zeta^3 I \end{bmatrix} \quad \text{as } x \to \pm\infty.$$

We define time-dependent Jost functions by

$$\phi^{(t)} \equiv \phi \mathrm{e}^{-4\mathrm{i}\zeta^3 t} \sim \begin{bmatrix} I \\ O \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x - 4\mathrm{i}\zeta^3 t} \qquad \text{as} \quad x \to -\infty,$$

$$\bar{\phi}^{(t)} \equiv \bar{\phi} \mathrm{e}^{4\mathrm{i}\zeta^3 t} \sim \begin{bmatrix} O \\ -I \end{bmatrix} \mathrm{e}^{\mathrm{i}\zeta x + 4\mathrm{i}\zeta^3 t} \qquad \text{as} \quad x \to -\infty.$$

From the relations

$$\frac{\partial \phi^{(t)}}{\partial t} = V\phi^{(t)}, \qquad \frac{\partial \bar{\phi}^{(t)}}{\partial t} = V\bar{\phi}^{(t)},$$

we get

$$\frac{\partial \phi}{\partial t} = (V + 4\mathrm{i}\zeta^3 I)\phi, \qquad \frac{\partial \bar{\phi}}{\partial t} = (V - 4\mathrm{i}\zeta^3 I)\bar{\phi}. \tag{3.34}$$

We substitute the definitions of the scattering data,

$$\phi(x,\zeta) = \bar{\psi}(x,\zeta)A(\zeta,t) + \psi(x,\zeta)B(\zeta,t),$$

$$\bar{\phi}(x,\zeta) = \bar{\psi}(x,\zeta)\bar{B}(\zeta,t) - \psi(x,\zeta)\bar{A}(\zeta,t),$$

into eq. (3.34). Then taking the limit $x \to +\infty$, we obtain

$$A_t(\zeta,t) = O,$$

$$B_t(\zeta,t) = 8\mathrm{i}\zeta^3 B(\zeta,t),$$

and

$$\bar{A}_t(\zeta,t) = O,$$

$$\bar{B}_t(\zeta,t) = -8\mathrm{i}\zeta^3 \bar{B}(\zeta,t).$$

The above relations lead to the following time dependences of the scattering data for the matrix mKdV equation:

$$A(\zeta,t) = A(\zeta,0), \tag{3.35a}$$

$$B(\xi,t)A(\xi,t)^{-1} = B(\xi,0)A(\xi,0)^{-1}\mathrm{e}^{8\mathrm{i}\xi^3 t}, \tag{3.35b}$$

$$C_j(t) = C_j(0)\mathrm{e}^{8\mathrm{i}\zeta_j^3 t}, \tag{3.35c}$$

and

$$\bar{A}(\zeta,t) = \bar{A}(\zeta,0),$$
$$\bar{B}(\xi,t)\bar{A}(\xi,t)^{-1} = \bar{B}(\xi,0)\bar{A}(\xi,0)^{-1}\mathrm{e}^{-8\mathrm{i}\xi^3 t}.$$
$$\bar{C}_k(t) = \bar{C}_k(0)\mathrm{e}^{-8\mathrm{i}\bar{\zeta}_k^3 t}.$$

Hence explicit time dependences of $F(x,t)$ and $\bar{F}(x,t)$ are given by

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{e}^{\mathrm{i}\xi x + 8\mathrm{i}\xi^3 t} B(\xi,0)A(\xi,0)^{-1} - \mathrm{i}\sum_{j=1}^{N} C_j(0)\mathrm{e}^{\mathrm{i}\zeta_j x + 8\mathrm{i}\zeta_j^3 t}, \tag{3.36}$$

$$\bar{F}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{e}^{-\mathrm{i}\xi x - 8\mathrm{i}\xi^3 t} \bar{B}(\xi,0)\bar{A}(\xi,0)^{-1} + \mathrm{i}\sum_{k=1}^{\bar{N}} \bar{C}_k(0)\mathrm{e}^{-\mathrm{i}\bar{\zeta}_k x - 8\mathrm{i}\bar{\zeta}_k^3 t}.$$

Similarly to the above discussion, the time dependences of the scattering data for the matrix NLS equation are given by

$$A(\zeta,t) = A(\zeta,0), \tag{3.37a}$$

$$B(\xi,t)A(\xi,t)^{-1} = B(\xi,0)A(\xi,0)^{-1}\mathrm{e}^{4\mathrm{i}\xi^2 t}, \tag{3.37b}$$

$$C_j(t) = C_j(0)\mathrm{e}^{4\mathrm{i}\zeta_j^2 t}, \tag{3.37c}$$

and

$$\bar{A}(\zeta,t) = \bar{A}(\zeta,0),$$
$$\bar{B}(\xi,t)\bar{A}(\xi,t)^{-1} = \bar{B}(\xi,0)\bar{A}(\xi,0)^{-1}\mathrm{e}^{-4\mathrm{i}\xi^2 t},$$
$$\bar{C}_k(t) = \bar{C}_k(0)\mathrm{e}^{-4\mathrm{i}\bar{\zeta}_k^2 t}.$$

It is remarkable that $-2\bar{F}(2x,t)$ and $2F(2x,t)$ satisfy the linearized dispersion relations, i.e.

$$(\partial_t + \partial_x^3)\{-2\bar{F}(2x,t)\} = 0, \quad (\partial_t + \partial_x^3)\{2F(2x,t)\} = 0,$$

for the matrix mKdV equation and

$$(\mathrm{i}\partial_t + \partial_x^2)\{-2\bar{F}(2x,t)\} = 0, \quad (\mathrm{i}\partial_t - \partial_x^2)\{2F(2x,t)\} = 0,$$

for the matrix NLS equation (see also Section 2.6).

3.2.4 Initial-value problem

Because of the constraint $R = -Q^\dagger$, we have some relations which make the further analysis simple.

First, we have

$$\det\bar{A}(\zeta) = \{\det A(\zeta^*)\}^*, \tag{3.38}$$

which is proved in Appendix A. This relation gives us a useful information about the total numbers and the positions of the poles of $A(\zeta)^{-1}$ and $\bar{A}(\zeta)^{-1}$,

$$\bar{N} = N, \quad \bar{\zeta}_k = \zeta_k^*. \tag{3.39}$$

Second, due to eq. (3.30c), we have

$$\bar{B}(\zeta)\bar{A}(\zeta)^{-1} = \{B(\zeta^*)A(\zeta^*)^{-1}\}^\dagger,$$

which leads to

$$\bar{B}(\xi)\bar{A}(\xi)^{-1} = \{B(\xi)A(\xi)^{-1}\}^\dagger \ \ (\xi : \text{real}), \tag{3.40a}$$

$$\bar{C}_k = C_k^\dagger. \tag{3.40b}$$

The relations (3.39) and (3.40) give a connection between $\bar{F}(x,t)$ and $F(x,t)$,

$$\bar{F}(x,t) = F(x,t)^\dagger. \tag{3.41}$$

Combining the above results, we arrive at

$$K_1(x,y;t) = F(x+y,t)^\dagger - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 K_1(x,s_2;t)F(s_2+s_1,t)F(s_1+y,t)^\dagger, \tag{3.42}$$

$$\bar{K}_2(x,y;t) = -F(x+y,t) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \bar{K}_2(x,s_2;t)F(s_2+s_1,t)^\dagger F(s_1+y,t), \tag{3.43}$$

where $F(x,t)$ is given by eq. (3.36).

We can solve the initial-value problem of the matrix mKdV equation, the matrix NLS equation and any other member of the matrix AKNS hierarchy by following the procedure (a)–(c) in Section 2.6.

As for the constraint $R = -Q^\dagger$, one comment is in order. Because $\bar{F}$ is connected with F by eq. (3.41), we can prove by the Neumann-Liouville expansion (see Appendix C) that the solution of eqs. (3.42) and (3.43) satisfies

$$\bar{K}_2(x,x;t) = -K_1(x,x;t)^\dagger.$$

This relation assures that the relation $R(x,t) = -Q(x,t)^\dagger$ holds at any time t.

3.2.5 Soliton solutions

Assuming the reflection-free condition,

$$B(\xi) = \bar{B}(\xi) = O \quad (\xi : \text{real}),$$

we can construct soliton solutions of the matrix AKNS hierarchy. In this case $F(x,t)$ is given by

$$F(x,t) = -\mathrm{i}\sum_{j=1}^N C_j(t)\mathrm{e}^{\mathrm{i}\zeta_j x}. \tag{3.44}$$

To solve eq. (3.42) with eq. (3.44), we set

$$K_1(x,y;t) = \mathrm{i}\sum_{k=1}^N P_k(x,t)C_k(t)^\dagger \mathrm{e}^{-\mathrm{i}\zeta_k^*(x+y)}. \tag{3.45}$$

Introducing eq. (3.45) into eq. (3.42), we have a set of algebraic equations,

$$P_k(x,t) - \sum_{l=1}^N\sum_{j=1}^N \frac{1}{(\zeta_j-\zeta_k^*)(\zeta_j-\zeta_l^*)} P_l(x,t)C_l(t)^\dagger C_j(t)\mathrm{e}^{2\mathrm{i}(\zeta_j-\zeta_l^*)x} = I. \tag{3.46}$$

We define a matrix S by

$$S_{lk} \equiv \delta_{lk} I - \sum_{j=1}^{N} \frac{\mathrm{e}^{2\mathrm{i}(\zeta_j - \zeta_l^*)x}}{(\zeta_j - \zeta_k^*)(\zeta_j - \zeta_l^*)} C_l(t)^\dagger C_j(t), \quad 1 \le l, k \le N.$$

Then eq. (3.46) is expressed as

$$(\, P_1 \; P_2 \; \cdots \; P_N \,) \begin{pmatrix} S_{11} & \cdots & S_{1N} \\ \vdots & \ddots & \vdots \\ S_{N1} & \cdots & S_{NN} \end{pmatrix} = (\underbrace{I \; I \; \cdots \; I}_{N}).$$

Thus the N-soliton solution of the matrix AKNS hierarchy is given by

$$\begin{aligned} Q(x,t) &= -2K_1(x,x;t) \\ &= -2\mathrm{i} \sum_{k=1}^{N} P_k(x,t) C_k(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_k^* x} \\ &= -2\mathrm{i} \, (\underbrace{I \; I \; \cdots \; I}_{N}) \, S^{-1} \begin{pmatrix} C_1(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_1^* x} \\ C_2(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_2^* x} \\ \vdots \\ C_N(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_N^* x} \end{pmatrix}. \end{aligned} \tag{3.47}$$

For instance, the one-soliton solution of the matrix mKdV equation (3.5) with $\varepsilon = -1$ is

$$\begin{aligned} Q(x,t) &= -2\mathrm{i} \left\{ I - \frac{\mathrm{e}^{8\mathrm{i}(\zeta_1^3 - \zeta_1^{*\,3})t}}{(\zeta_1 - \zeta_1^*)^2} C_1(0)^\dagger C_1(0) \mathrm{e}^{2\mathrm{i}(\zeta_1 - \zeta_1^*)x} \right\}^{-1} C_1(0)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_1^* x - 8\mathrm{i}\zeta_1^{*\,3} t} \\ &= -2\mathrm{i} \left\{ \mathrm{e}^{-\mathrm{i}(\zeta_1 - \zeta_1^*)x - 4\mathrm{i}(\zeta_1^3 - \zeta_1^{*\,3})t} I - \frac{1}{(\zeta_1 - \zeta_1^*)^2} C_1(0)^\dagger C_1(0) \mathrm{e}^{\mathrm{i}(\zeta_1 - \zeta_1^*)x + 4\mathrm{i}(\zeta_1^3 - \zeta_1^{*\,3})t} \right\}^{-1} \\ &\quad \cdot C_1(0)^\dagger \mathrm{e}^{-\mathrm{i}(\zeta_1 + \zeta_1^*)x - 4\mathrm{i}(\zeta_1^3 + \zeta_1^{*\,3})t}. \end{aligned}$$

3.3 Reductions

Now we study two important reductions of the ISM for the matrix AKNS hierarchy. In order to make the ISM applicable to the reduced systems, we have to reflect internal symmetry of Q and R in the scattering data.

3.3.1 Coupled NLS equations

One simple reduction of the matrix AKNS formulation is given by the following choice:

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad R = -Q^\dagger. \tag{3.48}$$

The first nontrivial flow of the reduced hierarchy is a system of coupled NLS (cNLS) equations,

$$\mathrm{i} q_{j,t} + q_{j,xx} + 2 \sum_{k=1}^{m} |q_k|^2 \cdot q_j = 0, \quad j = 1, 2, \ldots, m. \tag{3.49}$$

The $m=2$ case was solved by Manakov [60]. It is clear that the $2m \times 2m$ Lax matrices for the reduced hierarchy can be compressed into $(m+1)\times(m+1)$ matrices:

$$U = \begin{bmatrix} -\mathrm{i}\zeta & q_1 & \cdots & q_m \\ -q_1^* & \mathrm{i}\zeta & & \\ \vdots & & \ddots & \\ -q_m^* & & & \mathrm{i}\zeta \end{bmatrix}.$$

However, since we have applied the ISM to the matrix AKNS hierarchy for a square matrix Q, it is important to take account of the symmetry (3.48). The result is summarized as follows:

Proposition 1. (1) The reflection coefficient $B(\xi)A(\xi)^{-1}$ for real ξ is expressed as

$$B(\xi)A(\xi)^{-1} = \begin{bmatrix} g_1(\xi) & 0 & \cdots & 0 \\ g_2(\xi) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_m(\xi) & 0 & \cdots & 0 \end{bmatrix}.$$

(2) The residue matrices $\{C_1, \cdots, C_N\}$ are expressed as

$$\mathrm{i}C_j = \begin{bmatrix} \alpha_j^{(1)} & 0 & \cdots & 0 \\ \alpha_j^{(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_j^{(m)} & 0 & \cdots & 0 \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\alpha}_j & O \end{bmatrix}, \quad j = 1, 2, \ldots, N,$$

where $\{\alpha_j^{(i)}\}$ are complex functions of time t.

Hence F and $\bar{F}$ are expressed as

$$F(x,t) = \begin{bmatrix} f_1(x,t) & 0 & \cdots & 0 \\ f_2(x,t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_m(x,t) & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{F}(x,t) = F(x,t)^\dagger.$$

It should be noted that $-2\bar{F}(2x,t)$ and $2F(2x,t)$ have the same symmetry as Q and R respectively. Considering the above symmetry of the scattering data, we can solve the initial-value problem and obtain the N-soliton solution. Assuming the time dependence of the scattering data for the matrix NLS equation (cf. eq. (3.37)), we reproduce the result for the cNLS equations by Manakov [60]. From eq. (3.47), we obtain the N-soliton solution of the cNLS equations (3.49):

$$(q_1, q_2, \cdots, q_m) = 2\sum_{l=1}^{N}\sum_{k=1}^{N}(T^{-1})_{lk}\mathrm{e}^{-2\mathrm{i}\zeta_k^* x}\boldsymbol{\alpha}_k(t)^\dagger, \tag{3.50a}$$

where

$$T_{lk} = \delta_{lk} - \sum_{j=1}^{N}\frac{\mathrm{e}^{2\mathrm{i}(\zeta_j - \zeta_l^*)x}}{(\zeta_j - \zeta_k^*)(\zeta_j - \zeta_l^*)}\boldsymbol{\alpha}_l(t)^\dagger\boldsymbol{\alpha}_j(t), \quad 1 \le l, k \le N, \tag{3.50b}$$

$$\boldsymbol{\alpha}_j(t) = \boldsymbol{\alpha}_j(0)\mathrm{e}^{4\mathrm{i}\zeta_j^2 t}. \tag{3.50c}$$

The N-soliton solution exhibits remarkable behaviors characteristic of multi-component soliton equations. To show this, we plot the two-soliton solution ($N = 2$) in the two-component case for four choices of the parameter values. If we choose the parameter values appropriately, two solitons interact elastically with the other (Fig. 3.1). However, in general, the density distribution of each soliton among the two components can change due to the collision (Fig. 3.2) [76]. Thus the collision seems to be inelastic if we keep our eyes only on one component. For particular choices of the parameter values, we can observe annihilation and creation of soliton in one component (Fig. 3.3 and Fig. 3.4).

It is also remarkable that, by means of the reduction (3.48) for the result in Section 3.1.3 and Section 3.1.4, we obtain conservation laws and Hamiltonian structure for the reduced hierarchy starting from the cNLS equations. The appearance of Q^{-1} in eqs. (3.15) and (3.17) is just for a simplification of calculation and is not essential. The obtained conservation laws in Section 3.1.3 are valid even for choices of irregular matrix Q.

3.3.2 Coupled mKdV equations

As another simple reduction of the matrix AKNS formulation, we shall study a system of coupled mKdV (cmKdV) equations,

$$\frac{\partial u_i}{\partial t} + 6\Big(\sum_{j,k=0}^{M-1} C_{jk} u_j u_k\Big)\frac{\partial u_i}{\partial x} + \frac{\partial^3 u_i}{\partial x^3} = 0, \qquad i = 0, 1, \ldots, M-1, \tag{3.51}$$

where the constants C_{jk} are set to be symmetric with respect to the subscripts, $C_{jk} = C_{kj}$, without any loss of generality. Iwao and Hirota [43] obtained multi-soliton solutions of this system with the conditions $C_{jj} = 0$. The cmKdV equations for $M = 1, 2$ have been solved by the ISM. However, it has not been known whether the cmKdV equations for $M \geq 3$ and their hierarchy can be solved by the ISM or not.

We introduce an M-component vector field $\boldsymbol{u}$ and a constant $M \times M$ matrix G,

$$\boldsymbol{u} = (u_0, u_1, \cdots, u_{M-1})^T, \ \ G = (-C_{ij}),$$

where the symbol T means the transposition. Using this notation, we express the cmKdV equations (3.51) as

$$\boldsymbol{u}_t - 6(\,\boldsymbol{u}^T G \boldsymbol{u}\,)\,\boldsymbol{u}_x + \boldsymbol{u}_{xxx} = \boldsymbol{0}. \tag{3.52}$$

We assume that G is a real symmetric and regular matrix. Because a real symmetric matrix is diagonalized by a real orthogonal matrix, we set

$$G = P^T \Lambda P, \qquad P^T P = P P^T = I,$$

$$\Lambda = \mathrm{diag}(\lambda_0, \cdots, \lambda_{M-1}), \quad \lambda_j \neq 0 \ \text{ for all } j.$$

Thus, defining a new set of dependent variables $\boldsymbol{v} = (v_0, v_1, \cdots, v_{M-1})^T$ by $\boldsymbol{v} = P\boldsymbol{u}$, we transform eq. (3.52) into

$$\boldsymbol{v}_t - 6(\,\boldsymbol{v}^T \Lambda \boldsymbol{v}\,)\,\boldsymbol{v}_x + \boldsymbol{v}_{xxx} = \boldsymbol{0},$$

or more explicitly,

$$\frac{\partial v_i}{\partial t} - 6\Big(\sum_{j=0}^{M-1} \lambda_j v_j^2\Big)\frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad \lambda_j \neq 0, \qquad i = 0, 1, \ldots, M-1.$$

If we change a scale of v_i by $\sqrt{|\lambda_i|}\cdot v_i \to v_i$, we finally obtain "normalized" cmKdV equations,

$$\frac{\partial v_i}{\partial t} - 6\Big(\sum_{j=0}^{M-1} \varepsilon_j v_j^2\Big)\frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad \varepsilon_j = \mathrm{sgn}(\lambda_j) = \pm 1, \qquad i = 0, 1, \ldots, M-1. \tag{3.53}$$

Equation (3.53) is more convenient than eq. (3.51) to perform the ISM. Thus, we mainly deal with eq. (3.53) as the cmKdV equations in the following. In addition, we assume that the dependent variables $\{v_i\}$ are real and $\varepsilon_j = -1$ $(j = 0, 1, \ldots, M-1)$.

Suppose that Q and R are $2^{m-1} \times 2^{m-1}$ $(m \geq 2)$ matrices expressed as

$$Q^{(m)} = v_0 \mathbb{1} + \sum_{k=1}^{2m-1} v_k e_k, \qquad R^{(m)} = -v_0 \mathbb{1} + \sum_{k=1}^{2m-1} v_k e_k, \tag{3.54}$$

where $2^{m-1} \times 2^{m-1}$ matrices $\{e_1, \cdots, e_{2m-1}\}$ satisfy

$$\{e_i, e_j\}_+ = -2\delta_{ij}\mathbb{1}, \tag{3.55a}$$

$$e_k^\dagger = -e_k, \tag{3.55b}$$

$$\mathrm{tr}\, e_k = 0. \tag{3.55c}$$

Here $\mathbb{1}$ is the $2^{m-1} \times 2^{m-1}$ unit matrix. $\{\cdot,\cdot\}_+$ denotes the anti-commutator. Equation (3.55a) gives

$$\mathrm{tr}(e_i e_j) = -2^{m-1}\delta_{ij}. \tag{3.56}$$

An explicit representation of the matrices $\{e_j\}$ is given in Appendix B.

For $Q^{(m)}$ and $R^{(m)}$ defined by eq. (3.54), we can prove simple relations,

$$Q^{(m)}R^{(m)} = R^{(m)}Q^{(m)} = -\sum_{j=0}^{2m-1} v_j^2 \cdot \mathbb{1},$$

$$R^{(m)} = -Q^{(m)\dagger}.$$

Then substituting $Q^{(m)}$ and $R^{(m)}$ into Q and R in the matrix mKdV equation (3.3), we obtain the cmKdV equations,

$$\frac{\partial v_i}{\partial t} + 6\Big(\sum_{j=0}^{M-1} v_j^2\Big)\frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \qquad i = 0, 1, \ldots, M-1, \tag{3.57}$$

where we set $M = 2m$. Thus we have obtained a Lax representation for the cmKdV equations. Obviously we can obtain a Lax pair for a slightly general form of the cmKdV equations (3.53) by changing $v_j \to \sqrt{-\varepsilon_j}v_j$ in the above context. It should be noted that the Hamiltonian and the Poisson bracket for the matrix mKdV equation (3.3) become invalid for the cmKdV equations (3.57) with $M \geq 3$. This is because the degree of freedom of $Q^{(m)}$ and $R^{(m)}$ for the cmKdV equations is less than that for the matrix mKdV equation.

Next, we discuss the initial-value problem and soliton solutions of the cmKdV equations. Considering the scattering problem (3.23) with the potentials $Q^{(m)}$ and $R^{(m)}$ for $m \geq 2$, we can show the following restrictions on the scattering data.

Proposition 2. (1) The determinant of $A(\zeta)$ satisfies

$$\mathrm{det}A(\zeta) = \{\mathrm{det}A(-\zeta^*)\}^*,$$

as a complex function of ζ. Thus the poles of $1/\mathrm{det}A(\zeta)$ in the upper half plane should appear on the imaginary axis or as pairs which are situated symmetric with respect to the imaginary axis. Therefore, we can set the values of $2N$ poles as

$$\begin{aligned} \zeta_{2j-1} &= \xi_j + \mathrm{i}\eta_j, \\ \zeta_{2j} &= -\zeta_{2j-1}^* = -\xi_j + \mathrm{i}\eta_j, \end{aligned} \qquad j = 1, 2, \ldots, N, \tag{3.58}$$

where $\eta_j > 0$. The condition (3.58) should be interpreted as follows; if ζ_i is pure imaginary, it does not need its counterpart.

(2) The reflection coefficient $B(\xi)A(\xi)^{-1}$ for real ξ is expressed as

$$B(\xi)A(\xi)^{-1} = r^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} r^{(k)} e_k. \tag{3.59}$$

Here $r^{(0)}$ and $r^{(k)}$ are complex functions of ξ, t which satisfy

$$r^{(0)}(-\xi) = r^{(0)}(\xi)^*, \;\; r^{(k)}(-\xi) = r^{(k)}(\xi)^*. \tag{3.60}$$

(3) The residue matrices $\{C_1, C_2, \cdots, C_{2N-1}, C_{2N}\}$ are expressed as

$$\begin{aligned} \mathrm{i}C_{2j-1} &= c_j^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} c_j^{(k)} e_k, \\ \mathrm{i}C_{2j} &= c_j^{(0)\,*}\mathbb{I} + \sum_{k=1}^{2m-1} c_j^{(k)\,*} e_k, \end{aligned} \qquad j = 1, 2, \ldots, N, \tag{3.61}$$

where $c_j^{(0)}$, $c_j^{(k)}$ are complex functions of time t. For example, $\{C_1, C_2, \cdots, C_{2N}\}$ for the four-component cmKdV equations are given by

$$\mathrm{i}C_{2j-1} = \begin{bmatrix} \alpha_j & \beta_j \\ -\gamma_j & \delta_j \end{bmatrix}, \quad \mathrm{i}C_{2j} = \begin{bmatrix} \delta_j^* & \gamma_j^* \\ -\beta_j^* & \alpha_j^* \end{bmatrix},$$

in a different notation.

A proof of the statements is given in Appendix D.

Considering the above conditions, we have explicit expressions of F and $\bar{F}$ in terms of $\mathbb{I}$ and e_k,

$$\begin{aligned} F(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\xi,t)A(\xi,t)^{-1}\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\xi - \mathrm{i}\sum_{j=1}^{2N} C_j(t)\mathrm{e}^{\mathrm{i}\zeta_j x} \\ &= \frac{1}{2\pi}\int_0^{\infty}\Big\{(r^{(0)}\mathrm{e}^{\mathrm{i}\xi x} + r^{(0)\,*}\mathrm{e}^{-\mathrm{i}\xi x})\mathbb{I} + \sum_{k=1}^{2m-1}(r^{(k)}\mathrm{e}^{\mathrm{i}\xi x} + r^{(k)\,*}\mathrm{e}^{-\mathrm{i}\xi x})e_k\Big\}\,\mathrm{d}\xi \\ &\quad - \sum_{j=1}^{N}\Big\{(c_j^{(0)}\mathrm{e}^{\mathrm{i}\zeta_j x} + c_j^{(0)\,*}\mathrm{e}^{-\mathrm{i}\zeta_j^* x})\mathbb{I} + \sum_{k=1}^{2m-1}(c_j^{(k)}\mathrm{e}^{\mathrm{i}\zeta_j x} + c_j^{(k)\,*}\mathrm{e}^{-\mathrm{i}\zeta_j^* x})e_k\Big\}, \end{aligned}$$

$$
\begin{aligned}
\bar{F}(x,t) &= F(x,t)^{\dagger} \\
&= \frac{1}{2\pi}\int_0^{\infty}\Big\{(r^{(0)}\mathrm{e}^{\mathrm{i}\xi x} + r^{(0)\,*}\mathrm{e}^{-\mathrm{i}\xi x})\mathbb{1} - \sum_{k=1}^{2m-1}(r^{(k)}\mathrm{e}^{\mathrm{i}\xi x} + r^{(k)\,*}\mathrm{e}^{-\mathrm{i}\xi x})e_k\Big\}\,\mathrm{d}\xi \\
&\quad - \sum_{j=1}^{N}\Big\{(c_j^{(0)}\mathrm{e}^{\mathrm{i}\zeta_j x} + c_j^{(0)\,*}\mathrm{e}^{-\mathrm{i}\zeta_j^* x})\mathbb{1} - \sum_{k=1}^{2m-1}(c_j^{(k)}\mathrm{e}^{\mathrm{i}\zeta_j x} + c_j^{(k)\,*}\mathrm{e}^{-\mathrm{i}\zeta_j^* x})e_k\Big\}.
\end{aligned}
$$

Because $B(\xi,t)A(\xi,t)^{-1}$ and $C_j(t)$ depend on t as eqs. (3.35b) and (3.35c) for the matrix mKdV equation, the time dependences of $r^{(0)}$, $r^{(k)}$ and $c_j^{(0)}$, $c_j^{(k)}$ are given by

$$
r^{(0)}(\xi,t) = r^{(0)}(\xi,0)\mathrm{e}^{8\mathrm{i}\xi^3 t}, \quad r^{(k)}(\xi,t) = r^{(k)}(\xi,0)\mathrm{e}^{8\mathrm{i}\xi^3 t},
$$
$$
c_j^{(0)}(t) = c_j^{(0)}(0)\mathrm{e}^{8\mathrm{i}\zeta_j^3 t}, \quad c_j^{(k)}(t) = c_j^{(k)}(0)\mathrm{e}^{8\mathrm{i}\zeta_j^3 t}.
$$

It should be noted that $F(x,t)$ and $\bar{F}(x,t)$ are expressed as

$$
F(x,t) = f^{(0)}(x,t)\mathbb{1} + \sum_{k=1}^{2m-1} f^{(k)}(x,t)e_k,
$$

$$
\bar{F}(x,t) = f^{(0)}(x,t)\mathbb{1} - \sum_{k=1}^{2m-1} f^{(k)}(x,t)e_k,
$$

where the real functions $f^{(0)}(x,t)$ and $f^{(k)}(x,t)$ satisfy the linearized dispersion relation

$$
(\partial_t + \partial_{xxx})f^{(0)}(2x,t) = 0, \quad (\partial_t + \partial_{xxx})f^{(k)}(2x,t) = 0.
$$

Taking account of the conditions (3.58)–(3.61), we can solve the initial-value problem of the cmKdV equations by the ISM.

We replace N in Section 3.2 by $2N$ and obtain the N-soliton solution of the cmKdV equations (3.57) with $M(=2m)$ components,

$$
\begin{aligned}
Q^{(m)}(x,t) &= -2K_1(x,x;t) \\
&= -2\mathrm{i}\sum_{k=1}^{2N} P_k(x,t)C_k(t)^{\dagger}\mathrm{e}^{-2\mathrm{i}\zeta_k^* x} \\
&= -2\mathrm{i}\,(\underbrace{I\ I\ \cdots\ I}_{2N})\,S^{-1}\begin{pmatrix} C_1(t)^{\dagger}\mathrm{e}^{-2\mathrm{i}\zeta_1^* x} \\ C_2(t)^{\dagger}\mathrm{e}^{-2\mathrm{i}\zeta_2^* x} \\ \vdots \\ C_{2N}(t)^{\dagger}\mathrm{e}^{-2\mathrm{i}\zeta_{2N}^* x} \end{pmatrix},
\end{aligned}
\tag{3.62}
$$

where the matrix S is defined by

$$
S_{lk} \equiv \delta_{lk}I - \sum_{j=1}^{2N}\frac{\mathrm{e}^{2\mathrm{i}(\zeta_j-\zeta_l^*)x}}{(\zeta_j-\zeta_k^*)(\zeta_j-\zeta_l^*)}C_l(t)^{\dagger}C_j(t), \quad 1 \le l,k \le 2N.
$$

Strictly speaking, eq. (3.62) includes breathers besides solitons. In order to extract pure soliton solutions, we assume that each soliton seen in $\sum_j v_j(t)^2$ has a time-independent shape. By calculating an asymptotic behavior of the tails of solitons at $x \to +\infty$, we obtain the corresponding necessary conditions on the residue matrices,

$$
C_{2j-1}\bar{C}_{2j} = \bar{C}_{2j}C_{2j-1} = C_{2j}\bar{C}_{2j-1} = \bar{C}_{2j-1}C_{2j} = O, \qquad j = 1,2,\ldots,N. \tag{3.63}
$$

The conditions (3.63) are translated into

$$\sum_{i=0}^{2m-1} (c_j^{(i)})^2 = 0, \qquad j = 1, 2, \ldots, N.$$

As an example, we write down the pure one-soliton solution of the cmKdV equations (3.57). Choose $N = 1$ and set

$$\begin{aligned}
\zeta_1 &= \xi + \mathrm{i}\eta_1 = -\zeta_2^*, \\
-\mathrm{i}\bar{C}_1 &= \bar{c}_1^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} \bar{c}_1^{(k)} e_k, \\
-\mathrm{i}\bar{C}_2 &= \bar{c}_1^{(0)\,*} \mathbb{1} + \sum_{k=1}^{2m-1} \bar{c}_1^{(k)\,*} e_k.
\end{aligned}$$

Then, from eq. (3.62) we obtain

$$\begin{aligned}
Q^{(m)}(x,t) &= 2\eta_1 \operatorname{sech}\{2\eta_1 x - 8\eta_1(\eta_1^2 - 3\xi_1^2)t - x_0\} \left(2 \sum_{i=0}^{2m-1} |\bar{c}_1^{(i)}(0)|^2\right)^{-\frac{1}{2}} \\
&\quad \cdot \{-\mathrm{i}\bar{C}_1(0)\mathrm{e}^{-2\mathrm{i}\xi_1 x - 8\mathrm{i}\xi_1(\xi_1^2 - 3\eta_1^2)t} - \mathrm{i}\bar{C}_2(0)\mathrm{e}^{2\mathrm{i}\xi_1 x + 8\mathrm{i}\xi_1(\xi_1^2 - 3\eta_1^2)t}\},
\end{aligned}$$

where x_0 is defined by

$$\mathrm{e}^{-x_0} = 2\eta_1 \left(2 \sum_{i=0}^{2m-1} |\bar{c}_1^{(i)}(0)|^2\right)^{-\frac{1}{2}},$$

and $\sum_{i=0}^{2m-1} (\bar{c}_1^{(i)}(0))^2 = 0$. For each component, the one-soliton solution is rewritten as

$$\begin{aligned}
v_j(x,t) &= 2\eta_1 \operatorname{sech}\{2\eta_1 x - 8\eta_1(\eta_1^2 - 3\xi_1^2)t - x_0\} \left(2 \sum_{i=0}^{2m-1} |\bar{c}_1^{(i)}(0)|^2\right)^{-\frac{1}{2}} \\
&\quad \cdot \{\bar{c}_1^{(j)}(0)\mathrm{e}^{-2\mathrm{i}\xi_1 x - 8\mathrm{i}\xi_1(\xi_1^2 - 3\eta_1^2)t} + \bar{c}_1^{(j)\,*}(0)\mathrm{e}^{2\mathrm{i}\xi_1 x + 8\mathrm{i}\xi_1(\xi_1^2 - 3\eta_1^2)t}\}.
\end{aligned}$$

It is not evident whether eq. (3.62) can be expressed as

$$Q^{(m)}(x,t) = v_0(x,t)\mathbb{1} + \sum_{k=1}^{2m-1} v_k(x,t) e_k, \tag{3.64}$$

without using $e_i e_j$, $e_i e_j e_k$, etc. Noting the fact that summations and products of real quaternions are real quaternions, we can prove eq. (3.64) for $m = 2$ (four-component cmKdV equations) by using the Neumann-Liouville expansion (see Appendix C). It is an open problem to prove eq. (3.64) for general $M = 2m$.

The results in Section 3.1.3 assure that the cmKdV equations have an infinite number of conservation laws. We find that the first four conserved densities for the original cmKdV equations (3.51) are given by

$$\mathrm{I}_1 = \sum_{j,k} C_{jk} u_j u_k, \tag{3.65a}$$

$$\mathrm{I}_2 = u_j u_{k,x}, \quad \forall\, j,\, k \;\; (j \neq k), \tag{3.65b}$$

$$\mathrm{I}_3 = \left(\sum_{j,k} C_{jk} u_j u_k\right)^2 - \sum_{j,k} C_{jk} u_{j,x} u_{k,x}, \tag{3.65c}$$

$$\mathrm{I}_4 = \Big(\sum_{j,k} C_{jk}u_j u_k\Big)^3 - 3\sum_{j,k} C_{jk}u_j u_k \cdot \sum_{j,k} C_{jk}u_{j,x}u_{k,x} + \frac{1}{2}\sum_{j,k} C_{jk}u_{j,xx}u_{k,xx} - \frac{1}{2}\Big\{\Big(\sum_{j,k} C_{jk}u_j u_k\Big)_x\Big\}^2. \tag{3.65d}$$

We remark that the method in Section 3.1.3 does not give the quantity (3.65b).

3.3.3 Superposition

In the preceding two subsections, we have obtained an $(m+1)\times(m+1)$ Lax pair for the m-component cNLS equations and a $2^m \times 2^m$ Lax pair for the $2m$-component cmKdV equations. We shall show that the superposition of these two systems is also completely integrable via the ISM. In terms of complex variables,

$$q_j = v_{2j-2} + \mathrm{i}v_{2j-1}, \quad j = 1, 2, \ldots, m,$$

we obtain an alternative representation of the cmKdV equations (3.57),

$$q_{j,t} + 6\sum_{k=1}^{m} |q_k|^2 \cdot q_{j,x} + q_{j,xxx} = 0, \quad j = 1, 2, \ldots, m.$$

Thus, as the superposition of the cNLS equations and the cmKdV equations, we consider the following coupled system:

$$\mathrm{i}q_{j,t} + \mathrm{i}\gamma_1\Big\{q_{j,xxx} + 6\sum_{k=1}^{m} |q_k|^2 \cdot q_{j,x}\Big\} + \gamma_2\Big\{q_{j,xx} + 2\sum_{k=1}^{m}|q_k|^2 \cdot q_j\Big\} = 0, \quad j = 1, 2, \ldots, m. \tag{3.66}$$

This system is a multi-component generalization of the Hirota equation [34]. Thus we call eq. (3.66) the coupled Hirota (cHirota) equations in the following.

We introduce the following form of the Lax pair:

$$U = \mathrm{i}\zeta\begin{bmatrix} -I & \\ & I \end{bmatrix} + \begin{bmatrix} -\mathrm{i}\frac{\gamma_2}{6\gamma_1}H_1 & Q \\ R & -\mathrm{i}\frac{\gamma_2}{6\gamma_1}H_2 \end{bmatrix}, \tag{3.67}$$

$$\begin{aligned} V = {} & \gamma_1\Big\{\mathrm{i}\zeta^3\begin{bmatrix} -4I & \\ & 4I \end{bmatrix} + \zeta^2\begin{bmatrix} & 4Q \\ 4R & \end{bmatrix} + \mathrm{i}\zeta\begin{bmatrix} -2QR & 2Q_x \\ -2R_x & 2RQ \end{bmatrix} \\ & + \begin{bmatrix} Q_xR - QR_x & -Q_{xx} + 2QRQ \\ -R_{xx} + 2RQR & R_xQ - RQ_x \end{bmatrix}\Big\} + \gamma_2\Big\{\zeta\begin{bmatrix} & \frac{1}{3}(-H_1Q + QH_2) \\ \frac{1}{3}(H_2R - RH_1) & \end{bmatrix} \\ & + \frac{\mathrm{i}}{6}\begin{bmatrix} H_1QR - 2QH_2R + QRH_1 & 2(-H_1Q_x + Q_xH_2) \\ 2(-H_2R_x + R_xH_1) & H_2RQ - 2RH_1Q + RQH_2 \end{bmatrix}\Big\} \\ & + \gamma_2 \cdot \frac{\gamma_2}{\gamma_1}\Big\{\mathrm{i}\zeta\begin{bmatrix} \frac{1}{3}I & \\ & -\frac{1}{3}I \end{bmatrix} + \begin{bmatrix} & -\frac{2}{9}Q \\ -\frac{2}{9}R & \end{bmatrix}\Big\} + \gamma_2\Big(\frac{\gamma_2}{\gamma_1}\Big)^2\begin{bmatrix} \mathrm{i}\frac{1}{27}H_1 & \\ & \mathrm{i}\frac{1}{27}H_2 \end{bmatrix}. \end{aligned} \tag{3.68}$$

Here I is the $p \times p$ unit matrix. Q and R are $p \times p$ matrices. We assume the following relations for the constant matrices H_1 and H_2,

$$H_1Q - QH_2 = -2F_1QF_2, \tag{3.69a}$$

$$H_2R - RH_1 = 2F_2RF_1, \tag{3.69b}$$

where F_1 and F_2 satisfy

$$[H_1, F_1] = [H_2, F_2] = O, \quad (F_1)^2 = (F_2)^2 = I. \tag{3.70}$$

Substituting eqs. (3.67) and (3.68) into eq. (2.2), we get a set of matrix equations,

$$\begin{aligned} &\mathrm{i}Q_t + \mathrm{i}\gamma_1\{Q_{xxx} - 3Q_xRQ - 3QRQ_x\} \\ &+ \gamma_2\{F_1Q_{xx}F_2 - F_1QF_2RQ - QRF_1QF_2\} = O, &(3.71a)\\ &\mathrm{i}R_t + \mathrm{i}\gamma_1\{R_{xxx} - 3R_xQR - 3RQR_x\} \\ &- \gamma_2\{F_2R_{xx}F_1 - F_2RF_1QR - RQF_2RF_1\} = O. &(3.71b)\end{aligned}$$

We present a method to reduce the matrix equations (3.71) to the cHirota equations. For this purpose, we recursively define $2^{m-1} \times 2^{m-1}$ matrices $H_1^{(m)}$, $H_2^{(m)}$, $F_1^{(m)}$, $F_2^{(m)}$, $Q^{(m)}$ and $R^{(m)}$ by

$$H_1^{(1)} = -1, \quad H_2^{(1)} = 1, \quad F_1^{(1)} = 1, \quad F_2^{(1)} = 1, \tag{3.72}$$

$$H_1^{(m+1)} = \begin{bmatrix} H_1^{(m)} - I_{2^{m-1}} & \\ & H_2^{(m)} + I_{2^{m-1}} \end{bmatrix}, \quad H_2^{(m+1)} = \begin{bmatrix} H_2^{(m)} - I_{2^{m-1}} & \\ & H_1^{(m)} + I_{2^{m-1}} \end{bmatrix}, \tag{3.73}$$

$$F_1^{(m+1)} = \begin{bmatrix} F_1^{(m)} & \\ & -F_2^{(m)} \end{bmatrix}, \quad F_2^{(m+1)} = \begin{bmatrix} F_2^{(m)} & \\ & F_1^{(m)} \end{bmatrix}, \tag{3.74}$$

$$Q^{(1)} = q_1, \quad R^{(1)} = r_1, \tag{3.75}$$

$$Q^{(m+1)} = \begin{bmatrix} Q^{(m)} & q_{m+1}I_{2^{m-1}} \\ r_{m+1}I_{2^{m-1}} & -R^{(m)} \end{bmatrix}, \quad R^{(m+1)} = \begin{bmatrix} R^{(m)} & q_{m+1}I_{2^{m-1}} \\ r_{m+1}I_{2^{m-1}} & -Q^{(m)} \end{bmatrix}. \tag{3.76}$$

Here $I_{2^{m-1}}$ is the $2^{m-1} \times 2^{m-1}$ unit matrix. For the matrices defined by eqs. (3.72)–(3.76), we can prove the relations (3.69), (3.70) and a simple relation,

$$Q^{(m)}R^{(m)} = R^{(m)}Q^{(m)} = \sum_{k=1}^{m} q_kr_k \cdot I_{2^{m-1}},$$

by induction. Comparing $F_1^{(m)}Q^{(m)}F_2^{(m)}$ and $F_2^{(m)}R^{(m)}F_1^{(m)}$ with $Q^{(m)}$ and $R^{(m)}$ respectively, we observe that (-1) is multiplied to r_j in $F_1^{(m)}Q^{(m)}F_2^{(m)}$ and q_j in $F_2^{(m)}R^{(m)}F_1^{(m)}$. Then substituting $Q^{(m)}$, $R^{(m)}$, etc. into Q, R, etc. in the matrix equations (3.71), we obtain

$$\left.\begin{aligned} &\mathrm{i}q_{j,t} + \mathrm{i}\gamma_1\Big\{q_{j,xxx} - 6\sum_{k=1}^{m} q_kr_k \cdot q_{j,x}\Big\} + \gamma_2\Big\{q_{j,xx} - 2\sum_{k=1}^{m} q_kr_k \cdot q_j\Big\} = 0, \\ &\mathrm{i}r_{j,t} + \mathrm{i}\gamma_1\Big\{r_{j,xxx} - 6\sum_{k=1}^{m} r_kq_k \cdot r_{j,x}\Big\} - \gamma_2\Big\{r_{j,xx} - 2\sum_{k=1}^{m} r_kq_k \cdot r_j\Big\} = 0, \end{aligned}\right. \quad j = 1, 2, \ldots, m.$$

If we assume the reduction,

$$r_j = -\sigma_jq_j^*, \quad \sigma_j = \pm 1,$$

we obtain a slightly general version of the cHirota equations,

$$\mathrm{i}q_{j,t} + \mathrm{i}\gamma_1\Big\{q_{j,xxx} + 6\sum_{k=1}^{m} \sigma_k|q_k|^2 \cdot q_{j,x}\Big\} + \gamma_2\Big\{q_{j,xx} + 2\sum_{k=1}^{m} \sigma_k|q_k|^2 \cdot q_j\Big\} = 0, \quad j = 1, 2, \ldots, m. \tag{3.77}$$

It should be noted that $\gamma_1 = 0$ is the only singular value of parameters for the existence of the $2^m \times 2^m$ Lax pair for eq. (3.77). In the case $\gamma_1 = 0$, the Lax pair is given in terms of $(m+1) \times (m+1)$ matrices (see Section 3.3.1). In fact, the cHirota equations for $\gamma_1 \neq 0$ can be transformed to the cmKdV equations. If we change variables by

$$t = \frac{1}{\gamma_1}T, \quad x = X + \frac{\gamma_2^2}{3\gamma_1^2}T, \quad q_j = \mathrm{e}^{\mathrm{i}\left(\frac{\gamma_2}{3\gamma_1}x - \frac{2\gamma_2^3}{27\gamma_1^2}t\right)} u_j,$$

we find that the system of cHirota equations (3.77) is cast into the cmKdV equations,

$$u_{j,T} + \sum_{k=1}^{m} \sigma_k |u_k|^2 \cdot u_{j,X} + u_{j,XXX} = 0, \quad j = 1, 2, \ldots, m.$$

Hence, due to the discussion in Section 3.3.2, the system of cHirota equations (3.66) is completely integrable via the ISM. The Lax pair for the cHirota equations can also be transformed to that for the cmKdV equations through a gauge transformation.

3.4 Summary

In this chapter, we have considered a matrix generalization of the ISM for the AKNS hierarchy. As first two members of the matrix AKNS hierarchy, we obtain the matrix NLS equation and the matrix mKdV equation. We have shown the existence of an infinity of conservation laws, Hamiltonian structure and r-matrix representation for these two systems. The ISM has been applied to the matrix AKNS hierarchy. The initial-value problem is solved and the N-soliton solution is derived. As reductions of the matrix AKNS hierarchy, we have obtained the cNLS equations and the cmKdV equations. In accordance with the reductions, we obtain an infinite number of conserved quantities. Considering a reflection of the reductions in the scattering data, we have successfully applied the ISM to the reduced systems. We have introduced a superposed system of the cNLS equations and the cmKdV equations, which may be called cHirota equations. It has been shown that the cHirota equations can be solved via the ISM since the model is transformed into the cmKdV equations. The cHirota equations describe interactions among different modes in optical fibers with higher-order effects and seem to be physically significant.

Iwao and Hirota obtained a Pfaffian representation for an N-soliton solution of the cmKdV equations by means of the so-called Hirota method [43]. We stress that the initial-value problem of the cmKdV equations has been solved in the present chapter for the first time. In addition, it directly proves the complete integrability of the cmKdV equations. Our scheme enables one to construct more general solutions than the already known solutions.

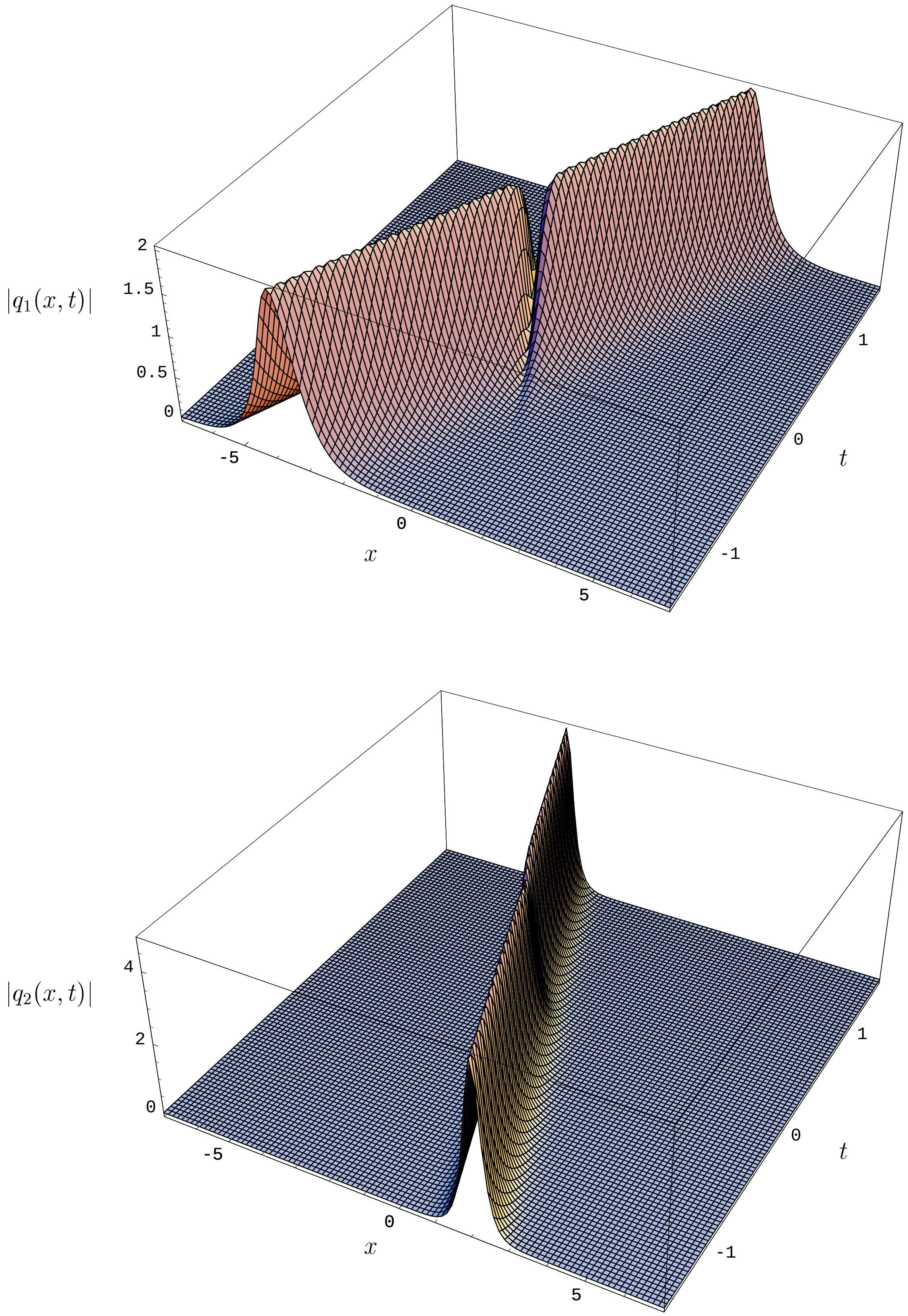


Figure 3.1: Two-soliton solution of the cNLS equations (3.49) with $m = 2$. The parameters in eq. (3.50) are chosen as $N = 2$, $\zeta_1 = -1/2 + \mathrm{i}$, $\zeta_2 = 2/5 + 12\mathrm{i}/5$, $\boldsymbol{\alpha}_1(0)^\dagger = (-\mathrm{i}, 0)$, $\boldsymbol{\alpha}_2(0)^\dagger = (0, -\mathrm{i})$.

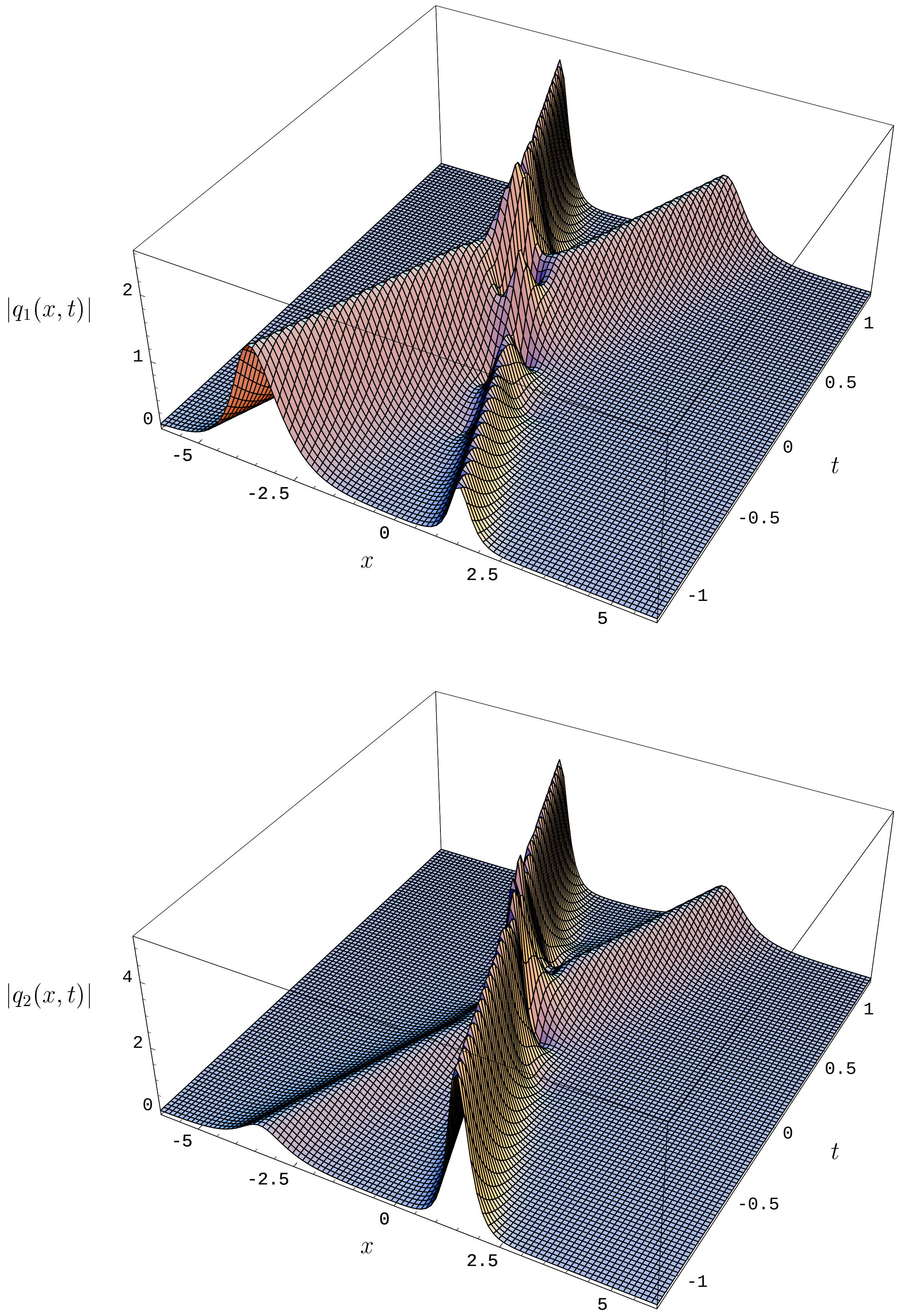


Figure 3.2: Two-soliton solution of the cNLS equations (3.49) with $m = 2$. The parameters in eq. (3.50) are chosen as $N = 2$, $\zeta_1 = -1/2 + \mathrm{i}$, $\zeta_2 = 2/5 + 12\mathrm{i}/5$, $\boldsymbol{\alpha}_1(0)^\dagger = (-3\mathrm{i}/5, 4/5)$, $\boldsymbol{\alpha}_2(0)^\dagger = (-11\mathrm{i}/61, -60\mathrm{i}/61)$.

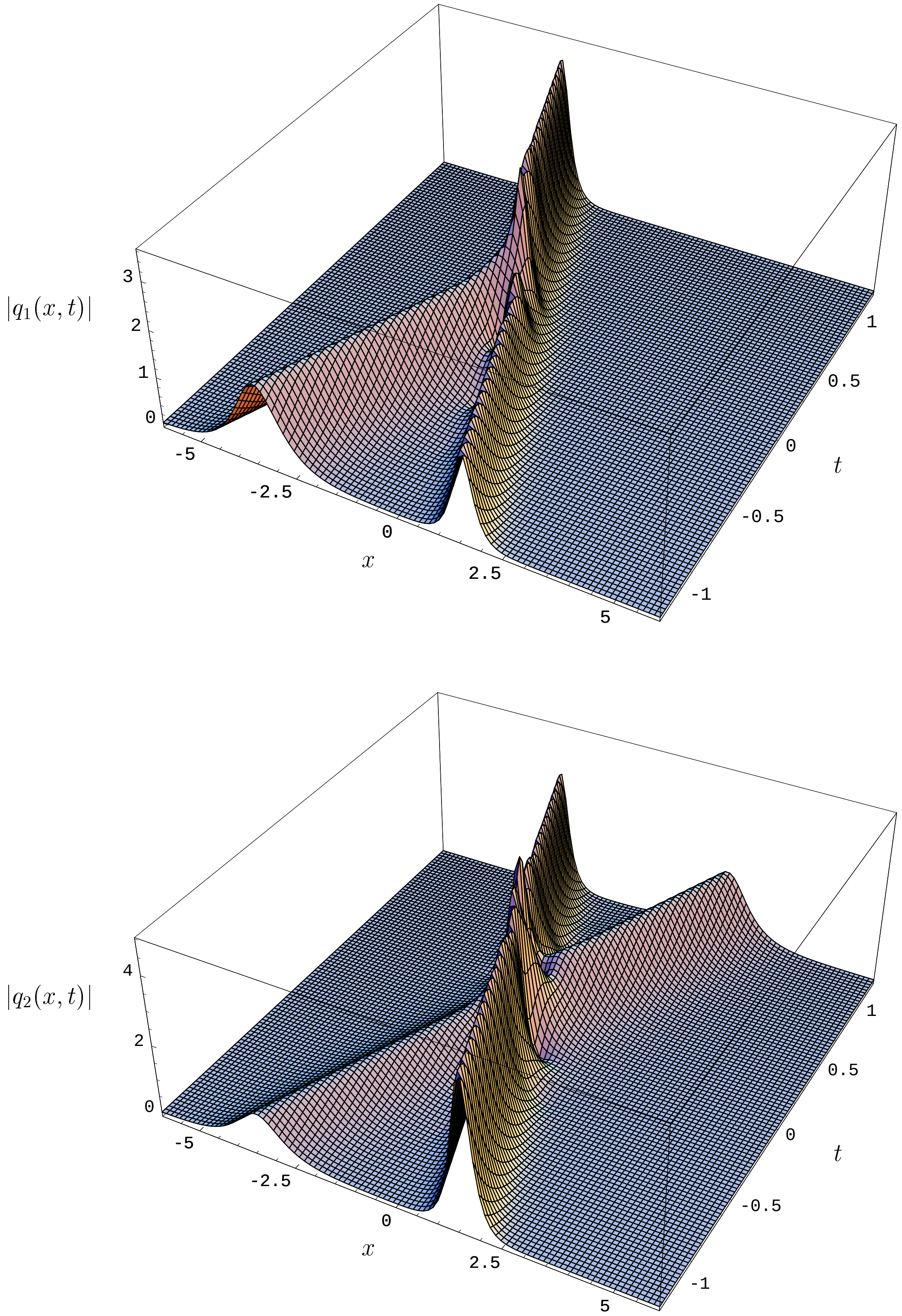
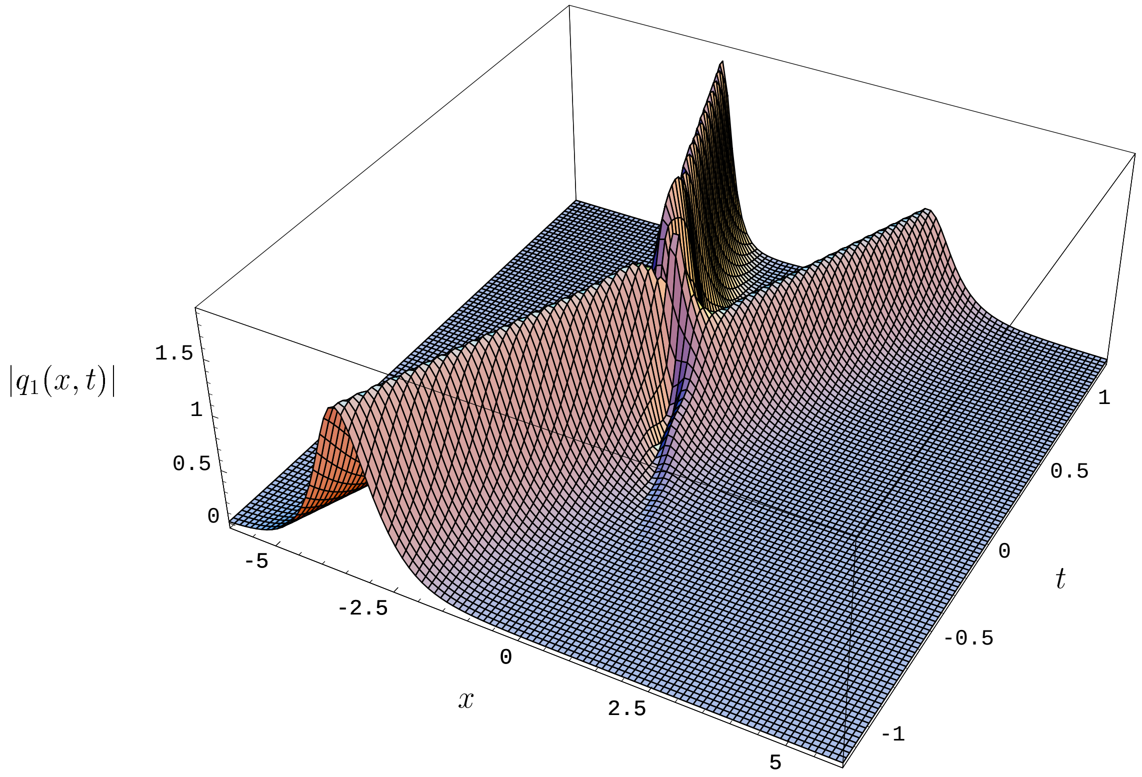


Figure 3.3: Two-soliton solution of the cNLS equations (3.49) with $m = 2$. The parameters in eq. (3.50) are chosen as $N = 2$, $\zeta_1 = -1/2 + \mathrm{i}$, $\zeta_2 = 2/5 + 12\mathrm{i}/5$, $\boldsymbol{\alpha}_1(0)^\dagger = (0, -\mathrm{i})$, $\boldsymbol{\alpha}_2(0)^\dagger = (5/13, -12\mathrm{i}/13)$.



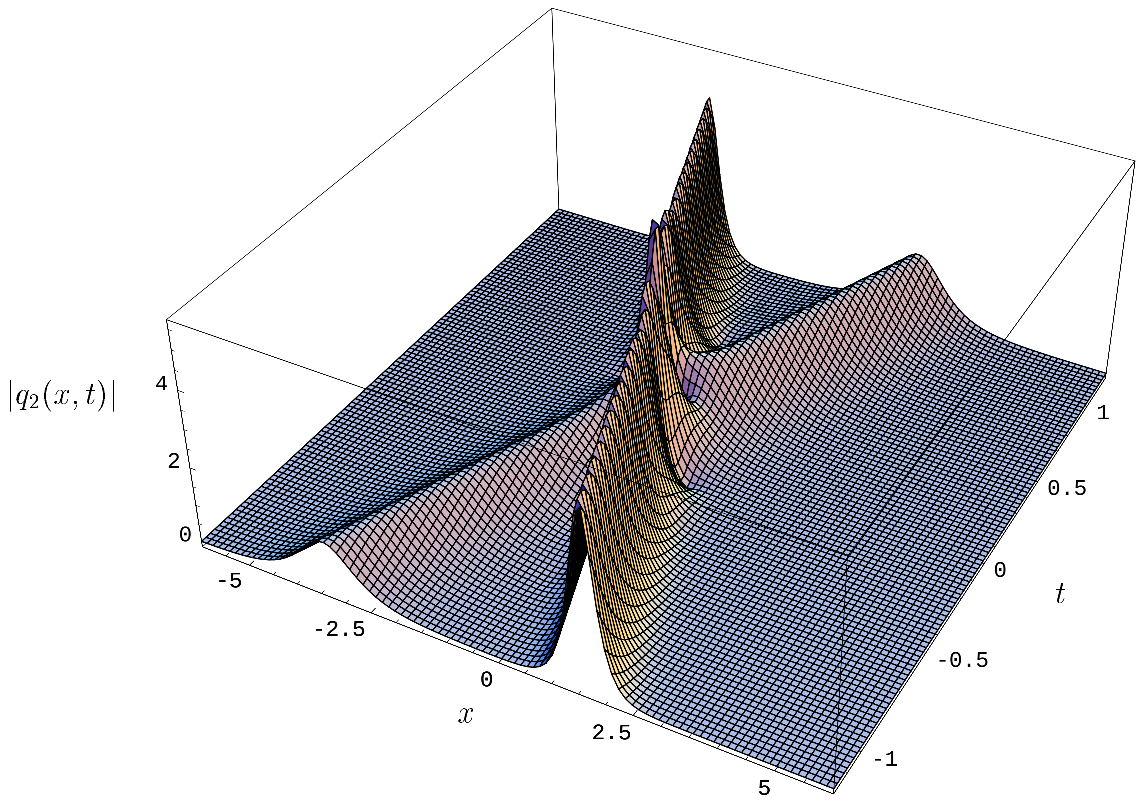


Figure 3.4: Two-soliton solution of the cNLS equations (3.49) with $m = 2$. The parameters in eq. (3.50) are chosen as $N = 2$, $\zeta_1 = -1/2 + \mathrm{i}$, $\zeta_2 = 2/5 + 12\mathrm{i}/5$, $\boldsymbol{\alpha}_1(0)^\dagger = (8/17, -15\mathrm{i}/17)$, $\boldsymbol{\alpha}_2(0)^\dagger = (0, -\mathrm{i}e^{\mathrm{i}})$.

Chapter 4

Coupled Chen-Lee-Liu Equations

As we studied in Chapter 2, there are two types of DNLS equation, i.e. the Kaup-Newell equation (2.10) and the Chen-Lee-Liu equation (2.36). There is a simple transformation of dependent variables which cast one into the other. In this sense, these two types of DNLS equation are gauge equivalent (see Section 2.8).

For the Kaup-Newell equation, it is known that there is a simple vector generalization [27, 66],

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} \mp \mathrm{i}(|\boldsymbol{q}|^2\boldsymbol{q})_x = \boldsymbol{0}, \tag{4.1}$$

which is completely integrable. Besides the system (4.1), Fordy [27] investigated various coupled versions of the Kaup-Newell equation by considering Hermitian symmetric spaces. Yajima studied a generalization of coupled DNLS equations by means of a gauge transformation in [105]. Meanwhile, multi-component extensions of the Chen-Lee-Liu equation have not been studied thoroughly from the ISM point of view.

In recent years, multi-field extensions or matrix generalizations of one-component classical integrable systems have been developed considerably by means of various approaches. Svinolupov, Sokolov, Habibullin and Yamilov clarified close connections between soliton equations and Jordan algebras or Jordan triple systems [32, 79–81]. A remarkable feature of their theory lies in the point that their approach exhausts all integrable cases in some classes of multi-field equations. Olver and Sokolov [72] surveyed integrable systems whose dependent variables take their values in an associative algebra, *e.g.* matrix-valued systems. They listed some classes of evolution equations on associative algebras which have higher-order symmetries.

In this chapter, we introduce a novel Lax formulation to get a matrix generalization of the Chen-Lee-Liu equation. As reductions of the matrix equation, we obtain two types of coupled Chen-Lee-Liu equations. Through a transformation of variables, one type is cast into the vector Kaup-Newell system (4.1). The other type is transformed into a new type of coupled Kaup-Newell equations. The latter type of the coupled Chen-Lee-Liu equations is shown to be connected with the cNLS equations. We restrict ourselves to a detailed study of two reduced systems of the matrix Chen-Lee-Liu equation. We treat the matrix Chen-Lee-Liu equation more generally in connection with the work of Olver and Sokolov [73] in the succeeding chapter.

4.1 Coupled DNLS equations

In this section, we consider a matrix generalization of the Chen-Lee-Liu equation by use of the ISM formulation. As reductions, we derive two new integrable systems of coupled Chen-Lee-Liu equations.

4.1.1 Lax formulation

We choose the following form of the Lax pair as an extension of eqs. (2.37) and (2.38),

$$U = \mathrm{i}\zeta^2 \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \zeta \begin{bmatrix} & \mathsf{Q} \\ \mathsf{R} & \end{bmatrix} + \mathrm{i} \begin{bmatrix} O & \\ & \frac{1}{2}\mathsf{RQ} \end{bmatrix}, \tag{4.2}$$

$$\begin{aligned} V = {} & \mathrm{i}\zeta^4 \begin{bmatrix} -2I_1 & \\ & 2I_2 \end{bmatrix} + \zeta^3 \begin{bmatrix} & 2\mathsf{Q} \\ 2\mathsf{R} & \end{bmatrix} + \mathrm{i}\zeta^2 \begin{bmatrix} -\mathsf{QR} & \\ & \mathsf{RQ} \end{bmatrix} \\ & + \zeta \begin{bmatrix} & \mathrm{i}\mathsf{Q}_x + \frac{1}{2}\mathsf{QRQ} \\ -\mathrm{i}\mathsf{R}_x + \frac{1}{2}\mathsf{RQR} & \end{bmatrix} + \mathrm{i} \begin{bmatrix} O & \\ & \frac{\mathrm{i}}{2}(\mathsf{RQ}_x - \mathsf{R}_x\mathsf{Q}) + \frac{1}{4}\mathsf{RQRQ} \end{bmatrix}. \end{aligned} \tag{4.3}$$

I_1 and I_2 are respectively the $p \times p$ and the $q \times q$ identity matrices. Q is a $p \times q$ matrix and R is a $q \times p$ matrix. Substituting eqs. (4.2) and (4.3) into eq. (2.2) and equating the terms with the same powers of ζ, we get a set of nonlinear evolution equations,

$$\mathrm{i}\mathsf{Q}_t + \mathsf{Q}_{xx} - \mathrm{i}\mathsf{QRQ}_x = O, \tag{4.4a}$$

$$\mathrm{i}\mathsf{R}_t - \mathsf{R}_{xx} - \mathrm{i}\mathsf{R}_x\mathsf{QR} = O. \tag{4.4b}$$

Comments are in order. First, the equation obtained in the order $O(\zeta^0 = 1)$ of eq. (2.2) is automatically satisfied because of eq. (4.4). Thus, we have no restrictions on the sizes of Q and R, that is, on p and q. This fact enables us to consider various multi-field extensions of the Chen-Lee-Liu equation by choosing the forms of Q and R appropriately. Second, the trace of U, $\mathrm{tr}\, U$, depends on dynamical variables in this formulation. Third, Olver and Sokolov showed that the matrix equation (4.4) possesses at least one higher symmetry [72], which leads to a conjecture for the integrability of eq. (4.4). The existence of the Lax pair gives a definite support to the complete integrability of the model.

4.1.2 Coupled Chen-Lee-Liu equations (type I)

As a reduction of eq. (4.4), we choose Q and R to be a row vector and a column vector respectively,

$$\mathsf{Q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_m), \quad \mathsf{R} = (\mathsf{r}_1, \mathsf{r}_2, \cdots, \mathsf{r}_m)^T.$$

Here the superscript T stands for the transposition. Then, we obtain a coupled version of the Chen-Lee-Liu equation,

$$\left.\begin{aligned} & \mathrm{i}\mathsf{q}_{j,t} + \mathsf{q}_{j,xx} - \mathrm{i}\sum_{k=1}^{m} \mathsf{q}_k\mathsf{r}_k \cdot \mathsf{q}_{j,x} = 0, \\ & \mathrm{i}\mathsf{r}_{j,t} - \mathsf{r}_{j,xx} - \mathrm{i}\sum_{k=1}^{m} \mathsf{r}_k\mathsf{q}_k \cdot \mathsf{r}_{j,x} = 0, \end{aligned}\right. \quad j = 1, 2, \ldots, m. \tag{4.5}$$

In the following, we call this system the coupled Chen-Lee-Liu I equations or, simply, the type I equations. The explicit form of the Lax matrix U is given by

$$U=\begin{bmatrix} -\mathrm{i}\zeta^2 & \zeta \mathsf{q}_1 & \zeta \mathsf{q}_2 & \cdots & \zeta \mathsf{q}_m \\ \zeta \mathsf{r}_1 & \mathrm{i}\zeta^2+\mathrm{i}\frac{1}{2}\mathsf{r}_1\mathsf{q}_1 & \mathrm{i}\frac{1}{2}\mathsf{r}_1\mathsf{q}_2 & \cdots & \mathrm{i}\frac{1}{2}\mathsf{r}_1\mathsf{q}_m \\ \zeta \mathsf{r}_2 & \mathrm{i}\frac{1}{2}\mathsf{r}_2\mathsf{q}_1 & \mathrm{i}\zeta^2+\mathrm{i}\frac{1}{2}\mathsf{r}_2\mathsf{q}_2 & \cdots & \mathrm{i}\frac{1}{2}\mathsf{r}_2\mathsf{q}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta \mathsf{r}_m & \mathrm{i}\frac{1}{2}\mathsf{r}_m\mathsf{q}_1 & \mathrm{i}\frac{1}{2}\mathsf{r}_m\mathsf{q}_2 & \cdots & \mathrm{i}\zeta^2+\mathrm{i}\frac{1}{2}\mathsf{r}_m\mathsf{q}_m \end{bmatrix}. \tag{4.6}$$

Under the reduction,

$$\mathsf{r}_j=\pm\mathsf{q}_j^*, \quad j=1,2,\ldots,m,$$

the system (4.5) is expressed in a compact form,

$$\mathrm{i}\boldsymbol{q}_t+\boldsymbol{q}_{xx}\mp\mathrm{i}|\boldsymbol{q}|^2\boldsymbol{q}_x=\boldsymbol{0},$$

with $\boldsymbol{q}$ being the vector, $\boldsymbol{q}=(\mathsf{q}_1,\mathsf{q}_2,\cdots,\mathsf{q}_m)$.

We can construct an infinite number of conservation laws from the zero-curvature condition for the Lax pair. If we set

$$\Psi=(\Psi_1,\Psi_2,\cdots,\Psi_{m+1})^T,$$

in eq. (2.1), we have

$$\Big(\sum_{j=1}^{m+1}U_{ij}\Psi_j\Psi_i^{-1}\Big)_t=\Big(\sum_{j=1}^{m+1}V_{ij}\Psi_j\Psi_i^{-1}\Big)_x, \tag{4.7}$$

and

$$(U_{jj}-U_{ii})\Psi_j\Psi_i^{-1}=-\sum_{k(\neq j)}U_{jk}\Psi_k\Psi_i^{-1}+(\Psi_j\Psi_i^{-1})_x+\Psi_j\Psi_i^{-1}\sum_{k(\neq i)}U_{ik}\Psi_k\Psi_i^{-1}, \tag{4.8}$$

by virtue of eqs. (2.1) and (2.2) [45]. Introducing a new set of variables $\{\Gamma_j\}$ by

$$\Gamma_j\equiv\Psi_{j+1}\Psi_1^{-1}, \quad j=1,2,\ldots,m,$$

we get from eqs. (4.7) and (4.8) with eq. (4.6),

$$\Big(\sum_{j=1}^{m}\mathsf{q}_j\Gamma_j\Big)_t=\Big(\zeta^{-1}V_{11}+\zeta^{-1}\sum_{j=1}^{m}V_{1j+1}\Gamma_j\Big)_x, \tag{4.9}$$

and

$$\mathsf{q}_j\Gamma_j=-\frac{1}{2\mathrm{i}\zeta}\mathsf{q}_j\mathsf{r}_j+\frac{2\mathrm{i}}{(2\mathrm{i}\zeta)^2}\mathsf{q}_j\Gamma_{j,x}+\frac{\mathsf{q}_j\mathsf{r}_j}{(2\mathrm{i}\zeta)^2}\sum_{k=1}^{m}\mathsf{q}_k\Gamma_k+\frac{1}{2\mathrm{i}\zeta}\mathsf{q}_j\Gamma_j\sum_{k=1}^{m}\mathsf{q}_k\Gamma_k.$$

Equation (4.9) shows that $\sum_j \mathsf{q}_j\Gamma_j$ is a generating function of conserved densities. We expand $\mathsf{q}_j\Gamma_j$ as

$$\mathsf{q}_j\Gamma_j=\sum_{l=1}^{\infty}\frac{1}{(2\mathrm{i}\zeta)^{2l-1}}f_j^{(l)},$$

to get a recursion formula for the conserved densities,

$$f_j^{(l)}=-\mathsf{q}_j\mathsf{r}_j\delta_{l,1}+2\mathrm{i}\mathsf{q}_j(\mathsf{q}_j^{-1}f_j^{(l-1)})_x+\sum_{n=2}^{l-1}f_j^{(n)}\sum_{k=1}^{m}f_k^{(l-n)}.$$

For example, the followings are the first four conserved densities for the coupled Chen-Lee-Liu I equations (4.5),

$$\mathrm{I}_1 = \sum_j \mathsf{q}_j \mathsf{r}_j, \tag{4.10a}$$

$$\mathrm{I}_2 = \mathsf{q}_j \mathsf{r}_{l,x}, \quad \forall\, j, l, \tag{4.10b}$$

$$\mathrm{I}_3 = -4\sum_j \mathsf{q}_{j,x}\mathsf{r}_{j,x} + \mathrm{i}\sum_j (\mathsf{q}_j\mathsf{r}_{j,x} - \mathsf{q}_{j,x}\mathsf{r}_j)\cdot\sum_k \mathsf{q}_k\mathsf{r}_k, \tag{4.10c}$$

$$\begin{aligned}\mathrm{I}_4 = &-4\mathrm{i}\sum_j(\mathsf{q}_{j,x}\mathsf{r}_{j,xx} - \mathsf{q}_{j,xx}\mathsf{r}_{j,x}) + 8\sum_j \mathsf{q}_{j,x}\mathsf{r}_{j,x}\sum_k \mathsf{q}_k\mathsf{r}_k - \Big\{\sum_j(\mathsf{q}_j\mathsf{r}_{j,x} - \mathsf{q}_{j,x}\mathsf{r}_j)\Big\}^2 \\ &+ \Big\{\Big(\sum_j \mathsf{q}_j\mathsf{r}_j\Big)_x\Big\}^2 - \mathrm{i}\sum_j(\mathsf{q}_j\mathsf{r}_{j,x} - \mathsf{q}_{j,x}\mathsf{r}_j)\cdot\Big(\sum_k \mathsf{q}_k\mathsf{r}_k\Big)^2.\end{aligned} \tag{4.10d}$$

4.1.3 Coupled Chen-Lee-Liu equations (type II)

As another reduction of eq. (4.4), we choose Q and R to be a column vector and a row vector respectively,

$$\mathsf{Q} = (\hat{\mathsf{q}}_1, \hat{\mathsf{q}}_2, \cdots, \hat{\mathsf{q}}_m)^T, \qquad \mathsf{R} = (\hat{\mathsf{r}}_1, \hat{\mathsf{r}}_2, \cdots, \hat{\mathsf{r}}_m).$$

Here we use the hat to distinguish dependent variables from those for type I. In this case, eq. (4.4) reduces to

$$\left.\begin{aligned}&\mathrm{i}\hat{\mathsf{q}}_{j,t} + \hat{\mathsf{q}}_{j,xx} - \mathrm{i}\sum_{k=1}^{m}\hat{\mathsf{q}}_{k,x}\hat{\mathsf{r}}_k\cdot\hat{\mathsf{q}}_j = 0,\\ &\mathrm{i}\hat{\mathsf{r}}_{j,t} - \hat{\mathsf{r}}_{j,xx} - \mathrm{i}\sum_{k=1}^{m}\hat{\mathsf{r}}_{k,x}\hat{\mathsf{q}}_k\cdot\hat{\mathsf{r}}_j = 0,\end{aligned}\right. \quad j = 1, 2, \ldots, m. \tag{4.11}$$

We call this system the coupled Chen-Lee-Liu II equations or, simply, the type II equations in what follows. Hisakado proposed the coupled Chen-Lee-Liu II equations independently in [39]. The Lax matrix U for eq. (4.11) is given by

$$U = \begin{bmatrix} -\mathrm{i}\zeta^2 & & & \zeta\hat{\mathsf{q}}_1 \\ & \ddots & & \vdots \\ & & -\mathrm{i}\zeta^2 & \zeta\hat{\mathsf{q}}_m \\ \zeta\hat{\mathsf{r}}_1 & \cdots & \zeta\hat{\mathsf{r}}_m & \mathrm{i}\zeta^2 + \mathrm{i}\frac{1}{2}\sum_{k=1}^{m}\hat{\mathsf{r}}_k\hat{\mathsf{q}}_k \end{bmatrix}. \tag{4.12}$$

The difference in the structure of the Lax matrices (4.6) and (4.12) should be noteworthy.

Under the reduction,

$$\hat{\mathsf{r}}_j = \pm\hat{\mathsf{q}}_j^*, \qquad j = 1, 2, \ldots, m,$$

the system (4.11) is expressed in a vector form,

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} \mp \mathrm{i}\langle\boldsymbol{q}_x, \boldsymbol{q}^*\rangle\boldsymbol{q} = \boldsymbol{0},$$

with $\boldsymbol{q}$ being the vector, $\boldsymbol{q} = (\hat{\mathsf{q}}_1, \hat{\mathsf{q}}_2, \cdots, \hat{\mathsf{q}}_m)$. Here $\langle\ ,\ \rangle$ denotes the inner product between vectors.

We can calculate an infinite set of conservation laws for eq. (4.11) in the same manner as in Section 4.1.2. The recursion relations for the conserved densities are

$$g_j^{(l)} = -\hat{\mathsf{q}}_j\hat{\mathsf{r}}_j\delta_{l,1} + 2\mathrm{i}\hat{\mathsf{q}}_j(\hat{\mathsf{q}}_j^{-1}g_j^{(l-1)})_x + \sum_{n=1}^{l-2} g_j^{(n)}\sum_{k=1}^{m} g_k^{(l-n)},$$

which yield an infinite series of conserved densities $\sum_j g_j^{(l)}$. The first four conserved densities are, for instance,

$$\hat{\mathrm{I}}_1 = \sum_j \hat{\mathsf{q}}_j \hat{\mathsf{r}}_j, \tag{4.13a}$$

$$\hat{\mathrm{I}}_2 = \mathrm{i} \sum_j (\hat{\mathsf{q}}_j \hat{\mathsf{r}}_{j,x} - \hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_j), \tag{4.13b}$$

$$\hat{\mathrm{I}}_3 = -4 \sum_j \hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_{j,x} + \mathrm{i} \sum_j (\hat{\mathsf{q}}_j \hat{\mathsf{r}}_{j,x} - \hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_j) \cdot \sum_k \hat{\mathsf{q}}_k \hat{\mathsf{r}}_k, \tag{4.13c}$$

$$\begin{aligned} \hat{\mathrm{I}}_4 = & -4\mathrm{i} \sum_j (\hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_{j,xx} - \hat{\mathsf{q}}_{j,xx} \hat{\mathsf{r}}_{j,x}) + 4 \sum_j \hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_{j,x} \sum_k \hat{\mathsf{q}}_k \hat{\mathsf{r}}_k - 2 \Big\{ \sum_j (\hat{\mathsf{q}}_j \hat{\mathsf{r}}_{j,x} - \hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_j) \Big\}^2 \\ & + 2 \Big\{ \Big(\sum_j \hat{\mathsf{q}}_j \hat{\mathsf{r}}_j \Big)_x \Big\}^2 - \mathrm{i} \sum_j (\hat{\mathsf{q}}_j \hat{\mathsf{r}}_{j,x} - \hat{\mathsf{q}}_{j,x} \hat{\mathsf{r}}_j) \cdot \Big(\sum_k \hat{\mathsf{q}}_k \hat{\mathsf{r}}_k \Big)^2. \end{aligned} \tag{4.13d}$$

It is interesting to compare eq. (4.10b) and eq. (4.13b): $\hat{\mathrm{I}}_2$ is a conserved density only after taking the summation with respect to the subscript j.

4.2 Gauge Transformations

Let us investigate the structure of the Lax pairs given in the previous section in connection with the AKNS formulation. In the following, dependent variables are assumed to approach 0 as $|x| \to \infty$ for convenience.

4.2.1 type I

In terms of $\{\mathsf{q}_j\}$, $\{\mathsf{r}_j\}$ which satisfy eq. (4.5), we introduce a new set of variables $\{u_j\}$, $\{v_j\}$ by

$$\left. \begin{aligned} u_j &= a\mathsf{q}_j \exp\Big\{ -\frac{\mathrm{i}}{2} \int_{-\infty}^{x} \sum_{k=1}^{m} \mathsf{q}_k \mathsf{r}_k \mathrm{d}x' \Big\}, \\ v_j &= b\mathsf{r}_{j,x} \exp\Big\{ \frac{\mathrm{i}}{2} \int_{-\infty}^{x} \sum_{k=1}^{m} \mathsf{q}_k \mathsf{r}_k \mathrm{d}x' \Big\}, \end{aligned} \right. \quad j = 1, 2, \ldots, m. \tag{4.14}$$

Here the constants a and b satisfy $ab = -\mathrm{i}/2$. Using eq. (4.14) and the first conservation law for eq. (4.5), we obtain

$$\mathrm{i}u_{j,t} + u_{j,xx} - 2 \sum_{k=1}^{m} u_k v_k \cdot u_j = \Big[\mathrm{i}\mathsf{q}_{j,t} + \mathsf{q}_{j,xx} - \mathrm{i} \sum_{k=1}^{m} \mathsf{q}_k \mathsf{r}_k \cdot \mathsf{q}_{j,x} \Big] a \exp\Big\{ -\frac{\mathrm{i}}{2} \int_{-\infty}^{x} \sum_{k=1}^{m} \mathsf{q}_k \mathsf{r}_k \mathrm{d}x' \Big\},$$

$$\mathrm{i}v_{j,t} - v_{j,xx} + 2 \sum_{k=1}^{m} v_k u_k \cdot v_j = \Big[\mathrm{i}\mathsf{r}_{j,t} - \mathsf{r}_{j,xx} - \mathrm{i} \sum_{k=1}^{m} \mathsf{r}_k \mathsf{q}_k \cdot \mathsf{r}_{j,x} \Big]_x b \exp\Big\{ \frac{\mathrm{i}}{2} \int_{-\infty}^{x} \sum_{k=1}^{m} \mathsf{q}_k \mathsf{r}_k \mathrm{d}x' \Big\}.$$

Hence, we conclude that if $\{\mathsf{q}_j\}$ and $\{\mathsf{r}_j\}$ satisfy the coupled Chen-Lee-Liu I equations, $\{u_j\}$ and $\{v_j\}$ satisfy the nonreduced cNLS equations,

$$\left. \begin{aligned} &\mathrm{i}u_{j,t} + u_{j,xx} - 2 \sum_{k=1}^{m} u_k v_k \cdot u_j = 0, \\ &\mathrm{i}v_{j,t} - v_{j,xx} + 2 \sum_{k=1}^{m} v_k u_k \cdot v_j = 0, \end{aligned} \right. \quad j = 1, 2, \ldots, m. \tag{4.15}$$

By means of the transformation (4.14), the second conserved densities (4.10b) are changed into the first conserved densities for eq. (4.15),

$$\mathrm{I}_2' = u_j v_l, \quad \forall\, j, l.$$

The Lax matrix for the cNLS equations (4.15) is obtained from eq. (4.6) through a gauge transformation,

$$g = \begin{bmatrix} 2\mathrm{i}b\zeta \mathrm{e}^K & 0 & \cdots & 0 \\ -b\mathsf{r}_1 \mathrm{e}^K & 1 & & \\ \vdots & & \ddots & \\ -b\mathsf{r}_m \mathrm{e}^K & & & 1 \end{bmatrix}, \qquad K = \frac{\mathrm{i}}{2}\int_{-\infty}^{x} \sum_{k=1}^{m} \mathsf{q}_k \mathsf{r}_k \mathrm{d}x',$$

as

$$U' = g^{-1}Ug - g^{-1}g_x = \begin{bmatrix} -\mathrm{i}\zeta^2 & u_1 & \cdots & u_m \\ v_1 & \mathrm{i}\zeta^2 & & \\ \vdots & & \ddots & \\ v_m & & & \mathrm{i}\zeta^2 \end{bmatrix}.$$

We thus have shown that the scattering problem for the coupled Chen-Lee-Liu I equations associated with eq. (4.6) is gauge equivalent to the reduction of the matrix AKNS formulation studied in Section 3.3.1.

4.2.2 type II

Next, let us consider a gauge transformation of the Lax matrix for the coupled Chen-Lee-Liu II equations. By virtue of a gauge transformation,

$$g = \begin{bmatrix} 2\mathrm{i}b\zeta & & & \\ & \ddots & & \\ & & 2\mathrm{i}b\zeta & \\ -b\hat{\mathsf{r}}_1 & \cdots & -b\hat{\mathsf{r}}_m & 1 \end{bmatrix},$$

the Lax matrix (4.12) is cast into

$$\begin{aligned} U' &= g^{-1}Ug - g^{-1}g_x \\ &= \begin{bmatrix} -\mathrm{i}\zeta^2 & & & \\ & \ddots & & \\ & & -\mathrm{i}\zeta^2 & \\ & & & \mathrm{i}\zeta^2 \end{bmatrix} + \begin{bmatrix} \mathrm{i}\frac{1}{2}\hat{\mathsf{q}}_1\hat{\mathsf{r}}_1 & \cdots & \mathrm{i}\frac{1}{2}\hat{\mathsf{q}}_1\hat{\mathsf{r}}_m & a\hat{\mathsf{q}}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{i}\frac{1}{2}\hat{\mathsf{q}}_m\hat{\mathsf{r}}_1 & \cdots & \mathrm{i}\frac{1}{2}\hat{\mathsf{q}}_m\hat{\mathsf{r}}_m & a\hat{\mathsf{q}}_m \\ b\hat{\mathsf{r}}_{1,x} & \cdots & b\hat{\mathsf{r}}_{m,x} & 0 \end{bmatrix}, \end{aligned}$$

with $ab = -\mathrm{i}/2$. It is noticed that there is no longer the spectral parameter ζ in the potential part of U'. We can easily eliminate dependent variables in the diagonal elements of U' by a further transformation. Thus, the gauge-transformed Lax formulation is embedded in the $(m+1) \times (m+1)$ matrix generalization of the AKNS formulation [3, 110].

4.3 Generalizations

Let us consider generalizations of the coupled Chen-Lee-Liu equations (type I & II) via transformations of variables.

4.3.1 type I

We first transform the coupled Chen-Lee-Liu I equations into a system with cubic terms without differentiation. Let us change independent and dependent variables by

$$T = t, \qquad X = x + \frac{2\beta}{\alpha}t,$$

$$\begin{aligned} \mathsf{q}_j &= \sqrt{\alpha}Q_j \exp\Big\{-\mathrm{i}\frac{\beta}{\alpha}X + \mathrm{i}\Big(\frac{\beta}{\alpha}\Big)^2 T\Big\}, \\ \mathsf{r}_j &= \sqrt{\alpha}R_j \exp\Big\{\mathrm{i}\frac{\beta}{\alpha}X - \mathrm{i}\Big(\frac{\beta}{\alpha}\Big)^2 T\Big\}, \end{aligned} \qquad j = 1, 2, \ldots, m.$$

From eq. (4.5), we get a system of equations,

$$\begin{aligned} &\mathrm{i}Q_{j,T} + Q_{j,XX} - \mathrm{i}\alpha \sum_{k=1}^{m} Q_k R_k \cdot Q_{j,X} - \beta \sum_{k=1}^{m} Q_k R_k \cdot Q_j = 0, \\ &\mathrm{i}R_{j,T} - R_{j,XX} - \mathrm{i}\alpha \sum_{k=1}^{m} R_k Q_k \cdot R_{j,X} + \beta \sum_{k=1}^{m} R_k Q_k \cdot R_j = 0, \end{aligned} \qquad j = 1, 2, \ldots, m.$$

We further utilize a kind of gauge transformation,

$$\begin{aligned} q_j &= Q_j \exp\Big\{-2\mathrm{i}\delta \int_{-\infty}^{X} \sum_{k=1}^{m} Q_k R_k \mathrm{d}X'\Big\}, \\ r_j &= R_j \exp\Big\{2\mathrm{i}\delta \int_{-\infty}^{X} \sum_{k=1}^{m} Q_k R_k \mathrm{d}X'\Big\}, \end{aligned} \qquad j = 1, 2, \ldots, m,$$

and get an extension of eq. (4.5),

$$\begin{aligned} &\mathrm{i}q_{j,T} + q_{j,XX} - \beta \sum_{k=1}^{m} q_k r_k \cdot q_j + 4\mathrm{i}\delta \sum_{k=1}^{m} q_k r_{k,X} \cdot q_j \\ &\quad + \mathrm{i}(4\delta - \alpha) \sum_{k=1}^{m} q_k r_k \cdot q_{j,X} + \delta(4\delta + \alpha)\Big(\sum_{k=1}^{m} q_k r_k\Big)^2 q_j = 0, \\ &\mathrm{i}r_{j,T} - r_{j,XX} + \beta \sum_{k=1}^{m} r_k q_k \cdot r_j + 4\mathrm{i}\delta \sum_{k=1}^{m} r_k q_{k,X} \cdot r_j \\ &\quad + \mathrm{i}(4\delta - \alpha) \sum_{k=1}^{m} r_k q_k \cdot r_{j,X} - \delta(4\delta + \alpha)\Big(\sum_{k=1}^{m} r_k q_k\Big)^2 r_j = 0, \end{aligned} \qquad j = 1, 2, \ldots, m.$$

For a choice $4\delta + \alpha = 0$, the system reads as

$$\begin{aligned} &\mathrm{i}q_{j,T} + q_{j,XX} - \mathrm{i}\alpha \sum_{k=1}^{m} q_k r_{k,X} \cdot q_j - 2\mathrm{i}\alpha \sum_{k=1}^{m} q_k r_k \cdot q_{j,X} - \beta \sum_{k=1}^{m} q_k r_k \cdot q_j = 0, \\ &\mathrm{i}r_{j,T} - r_{j,XX} - \mathrm{i}\alpha \sum_{k=1}^{m} r_k q_{k,X} \cdot r_j - 2\mathrm{i}\alpha \sum_{k=1}^{m} r_k q_k \cdot r_{j,X} + \beta \sum_{k=1}^{m} r_k q_k \cdot r_j = 0, \end{aligned} \qquad j = 1, 2, \ldots, m. \tag{4.16}$$

For $m = 1$, this is equivalent to the Kaup-Newell equation with a cubic term [95]. Thus, the system (4.16) is interpreted as a new multi-field extension of the Kaup-Newell equation. We can write an explicit expression of the corresponding Lax pair. For instance, the Lax matrix for eq. (4.16) with $\alpha = 1$, $\beta = 0$ is given by

$$U = \begin{bmatrix} -\mathrm{i}\zeta^2 + \mathrm{i}\frac{1}{2}\sum_{k=1}^m q_k r_k & \zeta q_1 & \zeta q_2 & \cdots & \zeta q_m \\ \zeta r_1 & \mathrm{i}\zeta^2 + \mathrm{i}\frac{1}{2} r_1 q_1 & \mathrm{i}\frac{1}{2} r_1 q_2 & \cdots & \mathrm{i}\frac{1}{2} r_1 q_m \\ \zeta r_2 & \mathrm{i}\frac{1}{2} r_2 q_1 & \mathrm{i}\zeta^2 + \mathrm{i}\frac{1}{2} r_2 q_2 & \cdots & \mathrm{i}\frac{1}{2} r_2 q_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta r_m & \mathrm{i}\frac{1}{2} r_m q_1 & \mathrm{i}\frac{1}{2} r_m q_2 & \cdots & \mathrm{i}\zeta^2 + \mathrm{i}\frac{1}{2} r_m q_m \end{bmatrix}.$$

4.3.2 type II

Following the same procedure as type I, we obtain a generalization of the coupled Chen-Lee-Liu II equations (4.11). By a change of variables,

$$T = t, \qquad X = x + \frac{2\beta}{\alpha} t,$$

$$\begin{aligned} \hat{\mathsf{q}}_j &= \sqrt{\alpha}\hat{Q}_j \exp\Big\{-\mathrm{i}\frac{\beta}{\alpha} X + \mathrm{i}\Big(\frac{\beta}{\alpha}\Big)^2 T\Big\}, \\ \hat{\mathsf{r}}_j &= \sqrt{\alpha}\hat{R}_j \exp\Big\{\mathrm{i}\frac{\beta}{\alpha} X - \mathrm{i}\Big(\frac{\beta}{\alpha}\Big)^2 T\Big\}, \end{aligned} \qquad j = 1, 2, \ldots, m,$$

we get a system of equations,

$$\begin{aligned} &\mathrm{i}\hat{Q}_{j,T} + \hat{Q}_{j,XX} - \mathrm{i}\alpha \sum_{k=1}^m \hat{Q}_{k,X}\hat{R}_k \cdot \hat{Q}_j - \beta \sum_{k=1}^m \hat{Q}_k \hat{R}_k \cdot \hat{Q}_j = 0, \\ &\mathrm{i}\hat{R}_{j,T} - \hat{R}_{j,XX} - \mathrm{i}\alpha \sum_{k=1}^m \hat{R}_{k,X}\hat{Q}_k \cdot \hat{R}_j + \beta \sum_{k=1}^m \hat{R}_k \hat{Q}_k \cdot \hat{R}_j = 0, \end{aligned} \qquad j = 1, 2, \ldots, m.$$

By virtue of a gauge transformation,

$$\begin{aligned} \hat{q}_j &= \hat{Q}_j \exp\Big\{-2\mathrm{i}\delta \int_{-\infty}^X \sum_{k=1}^m \hat{Q}_k \hat{R}_k \mathrm{d}X'\Big\}, \\ \hat{r}_j &= \hat{R}_j \exp\Big\{2\mathrm{i}\delta \int_{-\infty}^X \sum_{k=1}^m \hat{Q}_k \hat{R}_k \mathrm{d}X'\Big\}, \end{aligned} \qquad j = 1, 2, \ldots, m,$$

we obtain

$$\begin{aligned} &\mathrm{i}\hat{q}_{j,T} + \hat{q}_{j,XX} - \beta \sum_{k=1}^m \hat{q}_k \hat{r}_k \cdot \hat{q}_j - \mathrm{i}\alpha \sum_{k=1}^m \hat{q}_{k,X}\hat{r}_k \cdot \hat{q}_j + 4\mathrm{i}\delta \sum_{k=1}^m \hat{q}_k \hat{r}_{k,X} \cdot \hat{q}_j \\ &\quad + 4\mathrm{i}\delta \sum_{k=1}^m \hat{q}_k \hat{r}_k \cdot \hat{q}_{j,X} + \delta(4\delta + \alpha)\Big(\sum_{k=1}^m \hat{q}_k \hat{r}_k\Big)^2 \hat{q}_j = 0, \\ &\mathrm{i}\hat{r}_{j,T} - \hat{r}_{j,XX} + \beta \sum_{k=1}^m \hat{r}_k \hat{q}_k \cdot \hat{r}_j - \mathrm{i}\alpha \sum_{k=1}^m \hat{r}_{k,X}\hat{q}_k \cdot \hat{r}_j + 4\mathrm{i}\delta \sum_{k=1}^m \hat{r}_k \hat{q}_{k,X} \cdot \hat{r}_j \\ &\quad + 4\mathrm{i}\delta \sum_{k=1}^m \hat{r}_k \hat{q}_k \cdot \hat{r}_{j,X} - \delta(4\delta + \alpha)\Big(\sum_{k=1}^m \hat{r}_k \hat{q}_k\Big)^2 \hat{r}_j = 0, \end{aligned} \qquad j = 1, 2, \ldots, m.$$

In the case $4\delta+\alpha=0$, the system coincides with the vector Kaup-Newell system (4.1) in a nonreduced form,

$$\begin{aligned} &\mathrm{i}\hat{q}_{j,T}+\hat{q}_{j,XX}-\mathrm{i}\alpha\Big(\sum_{k=1}^{m}\hat{q}_k\hat{r}_k\cdot\hat{q}_j\Big)_X-\beta\sum_{k=1}^{m}\hat{q}_k\hat{r}_k\cdot\hat{q}_j=0,\\ &\mathrm{i}\hat{r}_{j,T}-\hat{r}_{j,XX}-\mathrm{i}\alpha\Big(\sum_{k=1}^{m}\hat{r}_k\hat{q}_k\cdot\hat{r}_j\Big)_X+\beta\sum_{k=1}^{m}\hat{r}_k\hat{q}_k\cdot\hat{r}_j=0,\end{aligned}\qquad j=1,2,\ldots,m. \tag{4.17}$$

This shows that the coupled Chen-Lee-Liu II equations are gauge equivalent to the vector Kaup-Newell system. The Lax matrix for eq. (4.17) with $\alpha=1$, $\beta=0$ is given by [27, 66]

$$U=\begin{bmatrix} -\mathrm{i}\zeta^2 & & & \zeta\hat{q}_1\\ & \ddots & & \vdots\\ & & -\mathrm{i}\zeta^2 & \zeta\hat{q}_m\\ \zeta\hat{r}_1 & \cdots & \zeta\hat{r}_m & \mathrm{i}\zeta^2 \end{bmatrix}.$$

This can also be derived from eq. (4.12) by means of a gauge transformation.

4.4 Summary

In this chapter, we have found a matrix generalization of a Lax pair for the Chen-Lee-Liu equation. As vector reductions of the matrix Chen-Lee-Liu equation, we have obtained two types of coupled Chen-Lee-Liu equations, (4.5) and (4.11). As is often the case with one-component soliton systems [26, 27, 32, 72, 79–81], the Chen-Lee-Liu equation also has plural multi-field generalizations. Using the Lax pairs, the conservation laws and the gauge transformations, we have studied the properties of the two types of coupled Chen-Lee-Liu equations in detail. An important step to obtain the coupled DNLS equations in our theory is the introduction of U by eq. (4.2). The form of U seems unusual, because we more or less assume $\mathrm{tr}\,U$ to be a constant (often equal to 0) for soliton systems. The Lax pair is, however, transformed into a multi-component generalization of the AKNS formulation, which is solvable by the ISM [1, 3, 11, 18]. Instead of applying the ISM, we give formulas for obtaining conserved quantities.

In the two-component case, the coupled Chen-Lee-Liu equations (type I & II) and the related models may have physical significances. They may describe wave propagations in birefringent optical fibers with nonlinear effects such as the Raman scattering and the Kerr effect.

We can construct other flows of the hierarchies by employing the corresponding time dependences of the scattering problems (ζ-dependences of the Lax matrix V). These flows have in common the conserved densities for the original flows. Expanding the matrix V from $O(\zeta^6)$ to $O(1)$, we get

$$\begin{aligned} &\mathsf{q}_{j,t}+\frac{1}{2}\mathsf{q}_{j,xxx}-\mathrm{i}\frac{3}{4}\Big(\sum_k\mathsf{q}_{k,x}\mathsf{r}_k\mathsf{q}_{j,x}+\sum_k\mathsf{q}_k\mathsf{r}_k\mathsf{q}_{j,xx}\Big)-\frac{3}{8}\Big(\sum_k\mathsf{q}_k\mathsf{r}_k\Big)^2\mathsf{q}_{j,x}=0,\\ &\mathsf{r}_{j,t}+\frac{1}{2}\mathsf{r}_{j,xxx}+\mathrm{i}\frac{3}{4}\Big(\sum_k\mathsf{r}_{k,x}\mathsf{q}_k\mathsf{r}_{j,x}+\sum_k\mathsf{r}_k\mathsf{q}_k\mathsf{r}_{j,xx}\Big)-\frac{3}{8}\Big(\sum_k\mathsf{r}_k\mathsf{q}_k\Big)^2\mathsf{r}_{j,x}=0,\end{aligned}$$

for the coupled Chen-Lee-Liu I hierarchy (cf. eq. (4.6)), and

$$\hat{\mathsf{q}}_{j,t} + \frac{1}{2}\hat{\mathsf{q}}_{j,xxx} - \mathrm{i}\frac{3}{4}\Big(\sum_k \hat{\mathsf{q}}_{k,x}\hat{\mathsf{r}}_k\hat{\mathsf{q}}_{j,x} + \sum_k \hat{\mathsf{q}}_{k,xx}\hat{\mathsf{r}}_k\hat{\mathsf{q}}_j\Big) - \frac{3}{8}\sum_k \hat{\mathsf{q}}_k\hat{\mathsf{r}}_k \sum_l \hat{\mathsf{q}}_{l,x}\hat{\mathsf{r}}_l\hat{\mathsf{q}}_j = 0,$$
$$\hat{\mathsf{r}}_{j,t} + \frac{1}{2}\hat{\mathsf{r}}_{j,xxx} + \mathrm{i}\frac{3}{4}\Big(\sum_k \hat{\mathsf{r}}_{k,x}\hat{\mathsf{q}}_k\hat{\mathsf{r}}_{j,x} + \sum_k \hat{\mathsf{r}}_{k,xx}\hat{\mathsf{q}}_k\hat{\mathsf{r}}_j\Big) - \frac{3}{8}\sum_k \hat{\mathsf{r}}_k\hat{\mathsf{q}}_k \sum_l \hat{\mathsf{r}}_{l,x}\hat{\mathsf{q}}_l\hat{\mathsf{r}}_j = 0,$$

for the coupled Chen-Lee-Liu II hierarchy (cf. eq. (4.12)). On the other hand, by expanding V from $O(\zeta^{-2})$ to $O(1)$, we obtain coupled versions of the massive Thirring model in a light-cone frame [87] (see Appendix E).

Chapter 5

DNLS-Type Equations

In the previous chapter, we have studied two types of coupled Chen-Lee-Liu equations as vector reductions for the matrix Chen-Lee-Liu equation. In this chapter, we shall consider a transformation of dependent variables for the general form of matrix Chen-Lee-Liu equation without any reduction. We obtain several matrix-valued systems of the DNLS type within the framework of ISM. The motivation of the study originates in the recent work of Olver and Sokolov. In [72, 73] Olver and Sokolov made a detailed investigation on DNLS-type systems of the form

$$\begin{aligned} P_t &= P_{xx} + f(P, S, P_x, S_x), \\ S_t &= -S_{xx} + g(P, S, P_x, S_x). \end{aligned} \tag{5.1}$$

Here P and S take values in an associative algebra. For simplicity, in the following we regard P and S as matrix-valued. f and g are non-commutative polynomials of weight 5, where the weights of ∂_t, ∂_x, P and S are respectively assigned to be 4, 2, 1 and 1. They made a complete list of the DNLS-type systems (5.1) which have one higher symmetry of the following form with weight 9,

$$\begin{aligned} P_\tau &= P_{xxxx} + \tilde{f}(P, S, P_x, S_x, P_{xx}, S_{xx}, P_{xxx}, S_{xxx}), \\ S_\tau &= -S_{xxxx} + \tilde{g}(P, S, P_x, S_x, P_{xx}, S_{xx}, P_{xxx}, S_{xxx}). \end{aligned}$$

Here the commutativity of the two flows, i.e. $\partial_t \partial_\tau P = \partial_\tau \partial_t P$, $\partial_t \partial_\tau S = \partial_\tau \partial_t S$, works as a strong constraint on the form of f and g in order for the non-commutative polynomials $\tilde{f}$ and $\tilde{g}$ to exist. See [25, 64, 74] for a more detailed explanation of the symmetry approach.

If a system had one higher symmetry, it was believed that the system had an infinite series of symmetries and was thus completely integrable. However, a system proposed by Bakirov was recently proved to be a counter-example to this empirical law [13, 14]. Thus, there is no guarantee that the systems in the list by Olver and Sokolov are really integrable, although the counter-example seems very exceptional. The aim of this chapter is to establish the complete integrability of all the matrix-valued systems given in [73]. In a previous chapter, we introduced a Lax pair for the matrix generalization of the Chen-Lee-Liu equation (4.4) which is a member of the list by Olver and Sokolov. In the present chapter, we generalize the Lax pair for eq. (4.4) to be applicable for several matrix systems of the DNLS type in [73]. For the rest of them, instead of giving a Lax pair, we shall derive the general solution to prove their complete integrability. Equation numbers without a 'section number'. refer to equations in [73].

5.1 'C-integrability' and 'S-integrability'

The entries of the list by Olver and Sokolov are divided into two kinds. Two systems in the list are interpreted as non-Abelian analogues of the following integrable system:

$$\begin{aligned} p_t &= p_{xx} + 2\alpha p^2 s_x + 2\alpha p p_x s - \alpha\beta p^3 s^2, \\ s_t &= -s_{xx} + 2\beta s^2 p_x + 2\beta s s_x p + \alpha\beta s^3 p^2, \end{aligned} \tag{5.2}$$

for a particular choice of the constants α and β. The system (5.2) is linearizable by a change of the dependent variables. Thus, we can construct the general solution of the system. We often refer to such linearizable systems as 'C-integrable' in Calogero's terminology [15]. On the other hand, excepting the two entries corresponding to eq. (5.2), the scalar-valued counterparts of the systems in the list by Olver and Sokolov are given by [88]

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + 4\mathrm{i}\delta q^2 r_x + \mathrm{i}(4\delta - \alpha) q q_x r + \delta(4\delta + \alpha) q^3 r^2 = 0, \\ &\mathrm{i}r_t - r_{xx} + 4\mathrm{i}\delta r^2 q_x + \mathrm{i}(4\delta - \alpha) r r_x q - \delta(4\delta + \alpha) r^3 q^2 = 0, \end{aligned} \tag{5.3}$$

for special choices of the constants α and δ. This system was generated via a gauge transformation for the DNLS equation by Kundu [55] (see Section 4.3). We can write down a Lax pair for the system (5.3) with the help of the gauge transformation. According to Calogero's terminology, these kinds of systems, which are linearizable by the inverse scattering formulation, are called 'S-integrable' systems [15].

5.2 'S-integrable' Systems

Let us begin with a brief summary of the matrix generalization of the Chen-Lee-Liu equation in Chapter 4. We introduced a Lax pair (4.2) and (4.3) and obtained a matrix Chen-Lee-Liu equation (4.4),

$$\begin{aligned} &\mathrm{i}\mathsf{Q}_t + \mathsf{Q}_{xx} - \mathrm{i}\mathsf{Q}\mathsf{R}\mathsf{Q}_x = O, \\ &\mathrm{i}\mathsf{R}_t - \mathsf{R}_{xx} - \mathrm{i}\mathsf{R}_x\mathsf{Q}\mathsf{R} = O. \end{aligned} \tag{5.4}$$

The system (5.4) was shown to possess at least one higher symmetry [72, 73]. The system is now proved to be completely integrable in the sense that it has a Lax pair and, as a result, an infinite number of conservation laws.

Next, we shall prove the complete integrability of other systems in [73] in the same sense. For this purpose, we introduce a transformation of dependent variables:

$$\mathsf{Q} = F^{-1} Q G^{-1}, \quad \mathsf{R} = GRF, \tag{5.5}$$

or equivalently

$$Q = F\mathsf{Q}G, \quad R = G^{-1}\mathsf{R}F^{-1}.$$

Here F and G are invertible matrices, which in general depend on Q and R (or Q and R). Then, time-evolution equations for Q and R, (5.4), are cast into those for Q and R:

$$\begin{aligned} &\mathrm{i}Q_t + Q_{xx} - \mathrm{i}QRQ_x - \mathrm{i}F_tF^{-1}Q + \mathrm{i}Q(G^{-1})_tG - (F_xF^{-1})_xQ + Q\{(G^{-1})_xG\}_x \\ &- 2F_xF^{-1}Q_x + 2Q_x(G^{-1})_xG - 2F_xF^{-1}Q(G^{-1})_xG + (F_xF^{-1})^2Q \\ &+ Q\{(G^{-1})_xG\}^2 + \mathrm{i}QRF_xF^{-1}Q - \mathrm{i}QRQ(G^{-1})_xG = O, \end{aligned} \tag{5.6a}$$

$$\begin{aligned}&\mathrm{i}R_t - R_{xx} - \mathrm{i}R_xQR - \mathrm{i}(G^{-1})_tGR + \mathrm{i}RF_tF^{-1} + \{(G^{-1})_xG\}_xR - R(F_xF^{-1})_x \\ &+ 2(G^{-1})_xGR_x - 2R_xF_xF^{-1} + 2(G^{-1})_xGRF_xF^{-1} - \{(G^{-1})_xG\}^2R \\ &- R(F_xF^{-1})^2 + \mathrm{i}(G^{-1})_xGRQR - \mathrm{i}RF_xF^{-1}QR = O. \end{aligned} \tag{5.6b}$$

A sufficient condition for eq. (5.6) to be local and closed equations is that F_xF^{-1}, F_tF^{-1}, $(G^{-1})_xG$ and $(G^{-1})_tG$ are expressed locally in closed forms in terms of Q and R, i.e. they do not include terms with integrals, infinite sums, etc. We impose this condition on F and G in what follows. A closed expression of the Lax pair for the transformed system is given by performing the gauge transformation,

$$\Psi = g\Phi, \qquad g = \begin{bmatrix} F^{-1} & \\ & G \end{bmatrix}.$$

Due to this transformation, the linear problem and the Lax pair for eq. (5.4) are changed into those for eq. (5.6):

$$\Phi_x = U'\Phi, \qquad \Phi_t = V'\Phi,$$

$$\begin{aligned} U' &= g^{-1}Ug - g^{-1}g_x \\ &= \mathrm{i}\zeta^2 \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \zeta \begin{bmatrix} & Q \\ R & \end{bmatrix} + \mathrm{i} \begin{bmatrix} -\mathrm{i}F_xF^{-1} & \\ & \frac{1}{2}RQ - \mathrm{i}(G^{-1})_xG \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} V' &= g^{-1}Vg - g^{-1}g_t \\ &= \mathrm{i}\zeta^4 \begin{bmatrix} -2I_1 & \\ & 2I_2 \end{bmatrix} + \zeta^3 \begin{bmatrix} & 2Q \\ 2R & \end{bmatrix} + \mathrm{i}\zeta^2 \begin{bmatrix} -QR & \\ & RQ \end{bmatrix} + \zeta \begin{bmatrix} & V_{12} \\ V_{21} & \end{bmatrix} \\ &\quad + \mathrm{i} \begin{bmatrix} -\mathrm{i}F_tF^{-1} & \\ & V_{22} \end{bmatrix}. \end{aligned}$$

Here

$$\begin{aligned} V_{12} &= \mathrm{i}Q_x + \frac{1}{2}QRQ - \mathrm{i}F_xF^{-1}Q + \mathrm{i}Q(G^{-1})_xG, \\ V_{21} &= -\mathrm{i}R_x + \frac{1}{2}RQR + \mathrm{i}(G^{-1})_xGR - \mathrm{i}RF_xF^{-1}, \\ V_{22} &= \mathrm{i}\frac{1}{2}(RQ_x - R_xQ) + \frac{1}{4}RQRQ - \mathrm{i}RF_xF^{-1}Q + \mathrm{i}\frac{1}{2}RQ(G^{-1})_xG + \mathrm{i}\frac{1}{2}(G^{-1})_xGRQ \\ &\quad - \mathrm{i}(G^{-1})_tG. \end{aligned}$$

The above transformation offers a powerful tool; it yields new integrable systems of the DNLS type by appropriate choices of F and G. To confirm this, we list six illustrative examples (a)–(f) with the definition of F and G, the evolution equations for Q and R and the transformed Lax matrix U':

(a)

$$F = I_1,$$

$$(G^{-1})_x = -\mathrm{i}\frac{1}{2}G^{-1}\mathsf{RQ} = -\mathrm{i}\frac{1}{2}RQG^{-1},$$

$$\begin{aligned}(G^{-1})_t &= G^{-1}\Big\{\frac{1}{2}(\mathsf{RQ}_x-\mathsf{R}_x\mathsf{Q})-\mathrm{i}\frac{1}{4}\mathsf{RQRQ}\Big\}\\ &=\Big\{\frac{1}{2}(RQ_x-R_xQ)-\mathrm{i}\frac{3}{4}RQRQ\Big\}G^{-1},\end{aligned}$$

$$\begin{aligned}\mathrm{i}Q_t+Q_{xx}-\mathrm{i}(QRQ)_x=O,\\ \mathrm{i}R_t-R_{xx}-\mathrm{i}(RQR)_x=O,\end{aligned} \tag{5.7}$$

$$U'=\mathrm{i}\zeta^2\begin{bmatrix}-I_1 & \\ & I_2\end{bmatrix}+\zeta\begin{bmatrix} & Q\\ R & \end{bmatrix}.$$

(b)

$$G^{-1}=I_2,$$

$$F_x=-\mathrm{i}\frac{1}{2}F\mathsf{QR}=-\mathrm{i}\frac{1}{2}QRF,$$

$$\begin{aligned}F_t &= F\Big\{\frac{1}{2}(\mathsf{Q}_x\mathsf{R}-\mathsf{QR}_x)-\mathrm{i}\frac{1}{4}\mathsf{QRQR}\Big\}\\ &=\Big\{\frac{1}{2}(Q_xR-QR_x)+\mathrm{i}\frac{1}{4}QRQR\Big\}F,\end{aligned}$$

$$\begin{aligned}\mathrm{i}Q_t+Q_{xx}+\mathrm{i}QR_xQ+\frac{1}{2}QRQRQ=O,\\ \mathrm{i}R_t-R_{xx}+\mathrm{i}RQ_xR-\frac{1}{2}RQRQR=O,\end{aligned} \tag{5.8}$$

$$U'=\mathrm{i}\zeta^2\begin{bmatrix}-I_1 & \\ & I_2\end{bmatrix}+\zeta\begin{bmatrix} & Q\\ R & \end{bmatrix}+\mathrm{i}\begin{bmatrix}-\frac{1}{2}QR & \\ & \frac{1}{2}RQ\end{bmatrix}.$$

(c)

$$F=I,$$

$$(G^{-1})_x=\mathrm{i}\frac{1}{2}\mathsf{QR}G^{-1}=\mathrm{i}\frac{1}{2}QRG^{-1},$$

$$\begin{aligned}(G^{-1})_t &= \Big\{\frac{1}{2}(\mathsf{QR}_x-\mathsf{Q}_x\mathsf{R})+\mathrm{i}\frac{1}{4}\mathsf{QRQR}\Big\}G^{-1}\\ &=\Big\{\frac{1}{2}(QR_x-Q_xR)+\mathrm{i}\frac{1}{4}QRQR-\mathrm{i}\frac{1}{2}Q^2R^2\Big\}G^{-1},\end{aligned}$$

$$\begin{aligned}\mathrm{i}Q_t+Q_{xx}-\mathrm{i}QRQ_x+\mathrm{i}Q^2R_x+\mathrm{i}Q_xQR-\frac{1}{2}Q^2RQR+\frac{1}{2}Q^3R^2+\frac{1}{2}QRQ^2R=O,\\ \mathrm{i}R_t-R_{xx}-\mathrm{i}R_xQR+\mathrm{i}Q_xR^2+\mathrm{i}QRR_x+\frac{1}{2}QRQR^2-\frac{1}{2}Q^2R^3-\frac{1}{2}QR^2QR=O,\end{aligned} \tag{5.9}$$

$$U'=\mathrm{i}\zeta^2\begin{bmatrix}-I & \\ & I\end{bmatrix}+\zeta\begin{bmatrix} & Q\\ R & \end{bmatrix}+\mathrm{i}\begin{bmatrix}O & \\ & \frac{1}{2}(RQ+QR)\end{bmatrix}.$$

(d)

$$G^{-1}=I,$$

$$F_x=\mathrm{i}\frac{1}{2}\mathsf{RQ}F=\mathrm{i}\frac{1}{2}RQF,$$

$$
\begin{aligned}
F_t &= \Big\{\frac{1}{2}(\mathsf{R}_x\mathsf{Q} - \mathsf{R}\mathsf{Q}_x) + \mathrm{i}\frac{1}{4}\mathsf{RQRQ}\Big\}F \\
&= \Big\{\frac{1}{2}(R_xQ - RQ_x) + \mathrm{i}\frac{1}{4}RQRQ + \mathrm{i}\frac{1}{2}R^2Q^2\Big\}F,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{i}Q_t + Q_{xx} - \mathrm{i}QRQ_x - \mathrm{i}R_xQ^2 - \mathrm{i}RQQ_x + \frac{1}{2}R^2Q^3 - \frac{1}{2}QR^2Q^2 = O, \\
&\mathrm{i}R_t - R_{xx} - \mathrm{i}R_xQR - \mathrm{i}R^2Q_x - \mathrm{i}R_xRQ - \frac{1}{2}R^3Q^2 + \frac{1}{2}R^2Q^2R = O,
\end{aligned}
\tag{5.10}
$$

$$
U' = \mathrm{i}\zeta^2\begin{bmatrix} -I & \\ & I \end{bmatrix} + \zeta\begin{bmatrix} & Q \\ R & \end{bmatrix} + \mathrm{i}\begin{bmatrix} \frac{1}{2}RQ & \\ & \frac{1}{2}RQ \end{bmatrix}.
$$

(e)

$$
G^{-1} = F,
$$

$$
F_x = -\mathrm{i}\frac{1}{2}F\mathsf{RQ} = -\mathrm{i}\frac{1}{2}RQF,
$$

$$
\begin{aligned}
F_t &= F\Big\{\frac{1}{2}(\mathsf{R}\mathsf{Q}_x - \mathsf{R}_x\mathsf{Q}) - \mathrm{i}\frac{1}{4}\mathsf{RQRQ}\Big\} \\
&= \Big\{\frac{1}{2}(RQ_x - R_xQ) - \mathrm{i}\frac{3}{4}RQRQ + \mathrm{i}\frac{1}{2}R^2Q^2\Big\}F,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{i}Q_t + Q_{xx} - \mathrm{i}QRQ_x - \mathrm{i}QR_xQ - \mathrm{i}Q_xRQ + \mathrm{i}R_xQ^2 + \mathrm{i}RQQ_x - RQRQ^2 \\
&\qquad + \frac{1}{2}R^2Q^3 + \frac{1}{2}RQ^2RQ = O, \\
&\mathrm{i}R_t - R_{xx} - \mathrm{i}R_xQR - \mathrm{i}RQ_xR - \mathrm{i}RQR_x + \mathrm{i}R^2Q_x + \mathrm{i}R_xRQ + R^2QRQ \\
&\qquad - \frac{1}{2}R^3Q^2 - \frac{1}{2}RQR^2Q = O,
\end{aligned}
\tag{5.11}
$$

$$
U' = \mathrm{i}\zeta^2\begin{bmatrix} -I & \\ & I \end{bmatrix} + \zeta\begin{bmatrix} & Q \\ R & \end{bmatrix} + \mathrm{i}\begin{bmatrix} -\frac{1}{2}RQ & \\ & O \end{bmatrix}.
$$

(f)

$$
G^{-1} = F,
$$

$$
F_x = -\mathrm{i}\frac{1}{2}F\mathsf{QR} = -\mathrm{i}\frac{1}{2}QRF,
$$

$$
\begin{aligned}
F_t &= F\Big\{\frac{1}{2}(\mathsf{Q}_x\mathsf{R} - \mathsf{Q}\mathsf{R}_x) - \mathrm{i}\frac{1}{4}\mathsf{QRQR}\Big\} \\
&= \Big\{\frac{1}{2}(Q_xR - QR_x) + \mathrm{i}\frac{1}{4}QRQR - \mathrm{i}\frac{1}{2}Q^2R^2\Big\}F,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{i}Q_t + Q_{xx} + \mathrm{i}QR_xQ - \mathrm{i}Q^2R_x - \mathrm{i}Q_xQR - \frac{1}{2}Q^2R^2Q - \frac{1}{2}Q^2RQR \\
&\qquad + \frac{1}{2}Q^3R^2 + \frac{1}{2}QRQRQ = O, \\
&\mathrm{i}R_t - R_{xx} + \mathrm{i}RQ_xR - \mathrm{i}Q_xR^2 - \mathrm{i}QRR_x + \frac{1}{2}RQ^2R^2 + \frac{1}{2}QRQR^2 \\
&\qquad - \frac{1}{2}Q^2R^3 - \frac{1}{2}RQRQR = O,
\end{aligned}
\tag{5.12}
$$

$$U' = \mathrm{i}\zeta^2 \begin{bmatrix} -I & \\ & I \end{bmatrix} + \zeta \begin{bmatrix} & Q \\ R & \end{bmatrix} + \mathrm{i} \begin{bmatrix} -\frac{1}{2}QR & \\ & \frac{1}{2}(RQ - QR) \end{bmatrix}.$$

For all of the six examples (a)–(f), the compatibility conditions for F and G^{-1}, i.e.

$$(F_x)_t = (F_t)_x, \quad \{(G^{-1})_x\}_t = \{(G^{-1})_t\}_x,$$

can be checked by a straightforward calculation with the help of eq. (5.4). As is clear from the construction of the gauge transformations, Q and R can be rectangular matrices for (a) and (b), while Q and R must be square matrices for (c)–(f).

Comparing the above results with those by Olver and Sokolov, we find that eq. (5.4) and eq. (10) in [73], eq. (5.7) and eq. (7) in [73], eq. (5.8) and eq. (12) in [73], eq. (5.9) and eq. (14) in [73], eq. (5.10) and eq. (16) in [73], eq. (5.11) and eq. (15) in [73], eq. (5.12) and eq. (17) in [73] are respectively identical up to scalings of variables. The system (5.7) is a well-known matrix generalization of the Kaup-Newell equation [27]. For systems (5.8), (5.9), (5.10), (5.11) and (5.12), or (12), (14)–(17) in [73], we have obtained the Lax representations by virtue of a non-commutative version of gauge transformations for the first time.

The systems (5.4), (5.11) and (5.12) are interpreted as matrix generalizations of eq. (5.3) with $\alpha = 1$, $\delta = 0$. The systems (5.7) and (5.10) reduce to eq. (5.3) with $\alpha = 1$, $\delta = -1/4$ in the commutative case. The systems (5.8) and (5.9) correspond to eq. (5.3) with $\alpha = 1$, $\delta = 1/4$ when Q and R are scalar-valued. The symmetry approach shows that matrix generalizations of eq. (5.3) are essentially exhausted by the above statement up to scalings and the transposition [73]. It is remarkable that the system (5.3) has matrix generalizations only for some choices of α and δ. This feature of matrix generalizations may be explained from our approach in the following way. In the case of scalar variables, eq. (5.3) is generated by the gauge transformation

$$\begin{aligned} q &= \frac{1}{\sqrt{\alpha}} \mathsf{q} \exp\Big\{-2\mathrm{i}\frac{\delta}{\alpha} \int^x \mathsf{rq} \mathrm{d}x'\Big\}, \\ r &= \frac{1}{\sqrt{\alpha}} \mathsf{r} \exp\Big\{2\mathrm{i}\frac{\delta}{\alpha} \int^x \mathsf{qr} \mathrm{d}x'\Big\}, \end{aligned}$$

for the Chen, Lee and Liu equation (2.36). Why can we not generalize this transformation to the matrix case for the continuous parameters α and δ? As an illustrative example, we choose F and G^{-1} to satisfy

$$F_x = \mathrm{i}\gamma \mathsf{RQ} F, \quad \lim_{x \to x_0} F = I, \quad G^{-1} = I,$$

where γ is a scalar parameter. The explicit form of F is given by

$$F = \mathcal{E} \exp\Big\{\mathrm{i}\gamma \int_{x_0}^x \mathsf{RQ} \mathrm{d}x'\Big\} = I + \sum_{n=1}^{\infty} (\mathrm{i}\gamma)^n A_n.$$

Here $\mathcal{E}$ is the path-ordering operator and

$$A_n = \int_{x_0}^x \mathrm{d}x_1 \int_{x_0}^{x_1} \mathrm{d}x_2 \cdots \int_{x_0}^{x_{n-1}} \mathrm{d}x_n \mathsf{R}(x_1)\mathsf{Q}(x_1) \cdots \mathsf{R}(x_n)\mathsf{Q}(x_n).$$

We can calculate the time derivative of F, F_t, with the help of eq. (5.4). However, it is observed that $F_t F^{-1}$ cannot be expressed in a closed form in general, i.e. it includes infinitely multiple integrals. An exception is the case of $\gamma = 1/2$, which gives

$$F_t = \Big\{\frac{1}{2}(\mathsf{R}_x\mathsf{Q} - \mathsf{R}\mathsf{Q}_x) + \mathrm{i}\frac{1}{4}\mathsf{RQRQ}\Big\}F,$$

and, as a result, leads to example (d). The example explains why the transformation (5.5) is effective for finite choices of F and G in the matrix case.

As we have studied in Section 2.8, the spatial part of the Lax formulation for the Chen-Lee-Liu equation is common with that for the massive Thirring model, i.e. they belong to the same hierarchy. Thus, employing an appropriate time part in the Lax formulations in correspondence with the Lax matrices U (or U') in this section, we can obtain new matrix generalizations of the massive Thirring model (see Appendix E).

5.3 'C-integrable' Systems

In the previous section, we have verified that all but two of the systems proposed by Olver and Sokolov are 'S-integrable', i.e. they have Lax representations and can be linearized by the ISM. In this section, we show that two systems left for further analysis are 'C-integrable', i.e. they can be linearized by a certain type of transformation of dependent variables which resembles that in the previous section.

The two systems, (11) and (13) in [73], are matrix generalizations of eq. (5.2) with $\alpha = 0$, $\beta = 1$. We briefly summarize a solution of eq. (5.2) before investigating eqs. (11) and (13). The pair of equations (5.2) is rewritten in a linearized form as

$$\begin{aligned}
&\big(p\mathrm{e}^{\alpha\int_{x_0}^{x} sp\,\mathrm{d}x'}\big)_t - \big(p\mathrm{e}^{\alpha\int_{x_0}^{x} sp\,\mathrm{d}x'}\big)_{xx} = 0,\\
&\big(s\mathrm{e}^{-\beta\int_{x_0}^{x} ps\,\mathrm{d}x'}\big)_t + \big(s\mathrm{e}^{-\beta\int_{x_0}^{x} ps\,\mathrm{d}x'}\big)_{xx} = 0,
\end{aligned}$$

under the boundary conditions: $\lim_{x\to x_0} p = \lim_{x\to x_0} s = 0$. In terms of functions $y(x,t)$ and $z(x,t)$, which satisfy a pair of heat equations

$$y_t - y_{xx} = 0, \qquad z_t + z_{xx} = 0,$$

and the boundary conditions, $\lim_{x\to x_0} y = \lim_{x\to x_0} z = 0$, the general solution of eq. (5.2) is given by

$$\begin{aligned}
p &= y\mathrm{e}^{-\alpha\int_{x_0}^{x} sp\,\mathrm{d}x'} = y\Big\{1 + (\alpha-\beta)\int_{x_0}^{x} zy\,\mathrm{d}x'\Big\}^{-\frac{\alpha}{\alpha-\beta}},\\
s &= z\mathrm{e}^{\beta\int_{x_0}^{x} ps\,\mathrm{d}x'} = z\Big\{1 + (\alpha-\beta)\int_{x_0}^{x} yz\,\mathrm{d}x'\Big\}^{\frac{\beta}{\alpha-\beta}},
\end{aligned}$$

for $\alpha \neq \beta$ (cf. [16] for $\alpha = -\beta$) and

$$\begin{aligned}
p &= y\mathrm{e}^{-\alpha\int_{x_0}^{x} sp\,\mathrm{d}x'} = y\mathrm{e}^{-\alpha\int_{x_0}^{x} zy\,\mathrm{d}x'},\\
s &= z\mathrm{e}^{\alpha\int_{x_0}^{x} ps\,\mathrm{d}x'} = z\mathrm{e}^{\alpha\int_{x_0}^{x} yz\,\mathrm{d}x'},
\end{aligned}$$

for $\alpha = \beta$.

We proceed to solve eqs. (11) and (13) in [73] by generalizing the above method to the matrix case. We write eq. (11) in [73]:

$$\begin{aligned}
&\mathsf{P}_t = \mathsf{P}_{xx},\\
&\mathsf{S}_t = -\mathsf{S}_{xx} + 2\mathsf{S}\mathsf{P}_x\mathsf{S} + 2\mathsf{SPS}_x.
\end{aligned} \tag{5.13}$$

Here P is an $n_1 \times n_2$ matrix and S is an $n_2 \times n_1$ matrix. The boundary conditions

$$\lim_{x \to x_0} \mathsf{P} = O, \quad \lim_{x \to x_0} \mathsf{S} = O,$$

are assumed for convenience. In terms of a matrix function A defined by

$$A_x = A(-\mathsf{SP}), \quad A_t = A(\mathsf{S}_x\mathsf{P} - \mathsf{SP}_x - \mathsf{SPSP}), \tag{5.14}$$

the time-evolution equation for S in eq. (5.13) is rewritten in a linearized form as

$$(A\mathsf{S})_t + (A\mathsf{S})_{xx} = O.$$

Here the consistency condition, $A_{xt} = A_{tx}$, is checked by a direct calculation using eq. (5.13). We introduce an $n_1 \times n_2$ matrix $Y(x,t)$ and an $n_2 \times n_1$ matrix $Z(x,t)$ which satisfy a pair of matrix heat equations

$$Y_t - Y_{xx} = O, \quad Z_t + Z_{xx} = O, \tag{5.15}$$

and the boundary conditions

$$\lim_{x \to x_0} Y = O, \quad \lim_{x \to x_0} Z = O.$$

The general solution of eq. (5.15) is obtained by means of the Fourier transformation. Thus, if we set

$$\mathsf{P} = Y, \quad \mathsf{S} = A^{-1}Z,$$

this gives the general solution of eq. (5.13). Due to the relation $A_x = -ZY$, we obtain

$$A = I_2 - \int_{x_0}^{x} Z(x',t)Y(x',t)\mathrm{d}x'.$$

Here we have assumed the boundary condition, $\lim_{x \to x_0} A = I_2$, with I_2 being the $n_2 \times n_2$ identity matrix. In conclusion, an explicit form of the general solution of eq. (5.13) is given by

$$\mathsf{P}(x,t) = Y(x,t), \quad \mathsf{S}(x,t) = \Big\{ I_2 - \int_{x_0}^{x} Z(x',t)Y(x',t)\mathrm{d}x' \Big\}^{-1} Z(x,t). \tag{5.16}$$

Finally, we shall derive the general solution of the only system left to solve, eq. (13) in [73]. For this purpose, we set $n_1 = n_2$ and perform a change of the dependent variables:

$$\mathsf{P} = AP, \quad \mathsf{S} = SA^{-1}.$$

Noting the fact that eq. (5.14) is rewritten in terms of the new variables P and S as

$$A_x = A(-SP), \quad A_t = A(S_xP - SP_x - SPSP + 2S^2P^2),$$

we obtain the evolution equations for P and S:

$$\begin{aligned} P_t &= P_{xx} - 2S_xP^2 - 2SPP_x + 2SPSP^2 - 2S^2P^3, \\ S_t &= -S_{xx} - 2S^2P_x - 2S_xSP + 2SPS_x + 2SP_xS \\ &\quad + 2SPS^2P + 2S^3P^2 - 2S^2PSP - 2S^2P^2S. \end{aligned} \tag{5.17}$$

This is nothing but the system (13) in [73] up to scalings of variables. Thus, by virtue of the derivation in the above, the general solution of eq. (5.17), which is an alternative expression of eq. (13), is obtained straightforwardly:

$$\begin{aligned} &P(x,t) = A^{-1}\mathsf{P} = \Big\{ I - \int_{x_0}^{x} Z(x',t)Y(x',t)\mathrm{d}x' \Big\}^{-1} Y(x,t), \\ &S(x,t) = \mathsf{S}A = \Big\{ I - \int_{x_0}^{x} Z(x',t)Y(x',t)\mathrm{d}x' \Big\}^{-1} Z(x,t) \Big\{ I - \int_{x_0}^{x} Z(x',t)Y(x',t)\mathrm{d}x' \Big\}. \end{aligned} \tag{5.18}$$

It should be noted that all of the matrices in the above expression are square matrices.

5.4 Summary

In this chapter, we have studied matrix-valued systems of the DNLS type. Applying a kind of gauge transformation to a matrix version of the Chen-Lee-Liu equation with a Lax pair proposed in Chapter 4, we have derived Lax representations for all but two of the systems proposed in [73]. Hence, these systems can be linearized through the inverse scattering formulation and proved to be 'S-integrable' in the terminology of Calogero. As has been clarified in Section 5.2, these systems are connected with the others through transformations of the dependent variables. It should be remarked, however, that the transformations cannot be written in a closed form in terms of the matrix-valued dependent variables. More explicitly, if F (or G^{-1}) is not the identity, we may not express F (or G^{-1}) for the examples (a)–(f) in Section 4 without using the path-ordering operator, infinitely multiple integrals, etc.

For the two systems in [73] for which it remains to prove their complete integrability, we have shown that both of them are linearizable and connected with each other by a change of dependent variables. The transformations which linearize the two systems can be explicitly written in a closed form in terms of the auxiliary variables Y and Z. This enables us to obtain the general solutions of the two systems, which directly proves their 'C-integrability'.

To summarize, we have proved for the first time that all the matrix-valued systems proposed in [73] can be integrated by the ISM or the transformations of the dependent variables. The dependent variables of the systems take values in either square matrices or, more generally, rectangular matrices of arbitrary size. However, it is noteworthy that not all of integrable multi-component systems can be expressed in terms of matrix variables of arbitrary size.

Chapter 6

HF Hierarchy and WKI Hierarchy

In recent years there have been a lot of progresses in the study of integrable systems with multiple components [21, 26, 27, 32, 43, 60, 72, 79–81, 84–88]. Among various approaches, the Lax formulation often helps us to obtain natural and simple multi-field extensions of single-component integrable systems [21, 26, 27, 60, 84–88]. From this viewpoint, in this chapter, we consider integrable systems derived from the eigenvalue problem [82, 96],

$$\begin{cases} \Psi_x = U\Psi, \quad U = \mathrm{i}\zeta U_1, \\ \Psi_t = V\Psi, \end{cases} \tag{6.1}$$

where U_1 is independent of the spectral parameter ζ. We call this problem the Takhtajan-Wadati-Konno-Ichikawa (T-WKI) type for brevity. As appropriate reductions of the corresponding compatibility condition (2.2), we obtain a multi-field generalization of the second flows of the Heisenberg ferromagnet (HF) equation and the Wadati-Konno-Ichikawa (WKI) equation for the first time. As is well-known [41, 97, 111], there is a gauge transformation between the HF hierarchy and the WKI hierarchy. We show that this correspondence can be generalized for the multi-component case. Considering a semi-discrete version of the eigenvalue problem (6.1), we find a semi-discretization of the coupled system of the second HF flow.

6.1 Heisenberg Ferromagnet System

6.1.1 Generalization of the second flow

Let us derive a multi-component extension of a higher flow in the HF hierarchy. In order to investigate a generalization of the higher HF equation, we consider a Lax pair:

$$U = \mathrm{i}\zeta S, \quad V = 4\mathrm{i}\zeta^3 S + 2\zeta^2 S S_x - \mathrm{i}\zeta\Big(S_{xx} + \frac{3}{2}S_x^2 S\Big), \tag{6.2}$$

with

$$S^2 = I. \tag{6.3}$$

Putting eq. (6.2) into the zero-curvature condition (2.2), we obtain the equation of motion for S [41],

$$S_t + S_{xxx} + \frac{3}{2}(S_x^2 S)_x = O. \tag{6.4}$$

It should be noted that this system is consistent with the condition (6.3), i.e. we can prove $(S^2)_t = S_t S + S S_t = O$ by use of eqs. (6.4) and (6.3). Of course, we can prove the same fact for the original system (2.15).

If we consider the reduction (2.14), the matrix equation (6.4) is cast into the second flow in the HF hierarchy. It is a vectorial equation for $\boldsymbol{S}$ with the constraint $|\boldsymbol{S}|^2 = 1$. To find a generalization with arbitrarily multiple components of the second flow, we assume that S is expressed as

$$S = \sum_{k=0}^{2m} \mathrm{i} s_k e_k \equiv S^{(m)},$$

in terms of anti-commutative matrices $\{e_i\}$;

$$\{e_i,\, e_j\} \equiv e_i e_j + e_j e_i = -2\delta_{ij} I, \quad 0 \le i, j \le 2m.$$

Then eq. (6.4) reduces to a simple multi-component system,

$$s_{j,t} + s_{j,xxx} + \frac{3}{2}\Big(\sum_{k=0}^{2m} s_{k,x}^2 \cdot s_j\Big)_x = 0, \quad j = 0, 1, \ldots, 2m. \tag{6.5a}$$

The constraint (6.3) is interpreted into

$$\sum_{j=0}^{2m} s_j^2 = 1. \tag{6.5b}$$

Because $\{e_i\}$ are elements of the Clifford algebra, we can construct their matrix representation (see Appendix B). The matrix representation shows that the system (6.5) has a $2^m \times 2^m$ Lax pair.

6.1.2 Semi-discretization

An integrable semi-discretization of the system (6.5) is given in an analogous way to [42]. As a semi-discrete version of the eigenvalue problem (2.1) with eq. (6.2), we consider the eigenvalue problem,

$$\Psi_{n+1} = L_n \Psi_n, \quad \Psi_{n,t} = M_n \Psi_n,$$

where

$$L_n = I + \lambda S_n, \tag{6.6a}$$

$$\begin{aligned} M_n &= \frac{4\lambda^2}{1-\lambda^2} \cdot S_{n-1}(S_n + S_{n-1})^{-1} + \frac{4\lambda}{1-\lambda^2} \cdot (S_n + S_{n-1})^{-1} \\ &= 4\sum_{j=1}^{\infty} \lambda^{2j} \cdot S_{n-1}(S_n + S_{n-1})^{-1} + 4\sum_{j=1}^{\infty} \lambda^{2j-1} \cdot (S_n + S_{n-1})^{-1}. \end{aligned} \tag{6.6b}$$

Here λ is a time-independent parameter and S_n is a matrix which satisfies

$$S_n^2 = I. \tag{6.7}$$

Substituting eq. (6.6) into eq. (2.27), we obtain

$$S_{n,t} + 4(S_n + S_{n-1})^{-1} - 4(S_{n+1} + S_n)^{-1} = O. \tag{6.8}$$

It is easy to check that $(S_n^2)_t = S_{n,t}S_n + S_nS_{n,t} = O$ due to eqs. (6.8) and (6.7). Thus, eq. (6.8) is consistent with eq. (6.7). In parallel with the reduction in the continuous case, we set

$$S_n = \sum_{k=0}^{2m} \mathrm{i}s_n^{(k)}e_k,$$

and obtain

$$s_{n,t}^{(j)} + \frac{2(s_n^{(j)} + s_{n-1}^{(j)})}{1 + \sum_{k=0}^{2m} s_n^{(k)}s_{n-1}^{(k)}} - \frac{2(s_{n+1}^{(j)} + s_n^{(j)})}{1 + \sum_{k=0}^{2m} s_{n+1}^{(k)}s_n^{(k)}} = 0, \quad j = 0, 1, \ldots, 2m, \tag{6.9}$$

with $\sum_{k=0}^{2m} s_n^{(k)\,2} = 1$. Equation (6.9) with $m = 1$ was derived in [75]. This system is interpreted as a semi-discretization of the coupled system (6.5). In fact, if we expand $s_{n\pm1}^{(j)}$ in powers of the lattice constant δx,

$$s_{n\pm1}^{(j)} = s^{(j)} \pm (\delta x)s_x^{(j)} + \frac{1}{2}(\delta x)^2 s_{xx}^{(j)} \pm \frac{1}{6}(\delta x)^3 s_{xxx}^{(j)} + \cdots,$$

the system (6.9) is rewritten as

$$s_t^{(j)} - 2(\delta x)s_x^{(j)} - \frac{1}{3}(\delta x)^3 s_{xxx}^{(j)} - \frac{1}{2}(\delta x)^3 \Big(\sum_{k=0}^{2m} s_x^{(k)\,2} \cdot s^{(j)}\Big)_x + O(\delta x^5) = 0.$$

Thus, up to a scaling of t and a Galilei transformation, the semi-discrete system (6.9) coincides with the system (6.5) in the continuum limit.

6.2 WKI System

Let us consider a multi-field generalization of the WKI equation with the linearized dispersion relation $\omega = -k^3$ (see eq. (2.17)). The generalization is interpreted as the one for the second flow of the WKI hierarchy, because the WKI hierarchy starts from an equation with the linearized dispersion relation $\omega = k^2$ (see eq. (2.16)). For this purpose, we choose the Lax matrices U and V as

$$U = \zeta \begin{bmatrix} -\mathrm{i}I & Q^{(m)} \\ R^{(m)} & \mathrm{i}I \end{bmatrix}, \tag{6.10}$$

$$\begin{aligned} V = {} & 4\zeta^3 f \begin{bmatrix} -\mathrm{i}I & Q^{(m)} \\ R^{(m)} & \mathrm{i}I \end{bmatrix} + \zeta^2 f^3 \begin{bmatrix} Q_x^{(m)}R^{(m)} - Q^{(m)}R_x^{(m)} & 2\mathrm{i}Q_x^{(m)} \\ -2\mathrm{i}R_x^{(m)} & R_x^{(m)}Q^{(m)} - R^{(m)}Q_x^{(m)} \end{bmatrix} \\ & + \zeta \left\{ f^3 \begin{bmatrix} O & -Q_x^{(m)} \\ -R_x^{(m)} & O \end{bmatrix} \right\}_x . \end{aligned} \tag{6.11}$$

Here $Q^{(m)}$ and $R^{(m)}$ are $2^{m-1} \times 2^{m-1}$ matrices which satisfy the constraint,

$$Q^{(m)}R^{(m)} = R^{(m)}Q^{(m)} = \sum_{k=1}^{m} q_k r_k \cdot I. \tag{6.12}$$

The scalar function f in eq. (6.11) is given by

$$f = \frac{1}{\sqrt{1 - \sum_{k=1}^{m} q_k r_k}}. \tag{6.13}$$

An explicit representation of $Q^{(m)}$ and $R^{(m)}$ which satisfy eq. (6.12) is given recursively by

$$Q^{(1)} = q_1, \quad R^{(1)} = r_1, \tag{6.14a}$$

$$Q^{(m+1)} = \begin{bmatrix} Q^{(m)} & q_{m+1} I_{2^{m-1}} \\ r_{m+1} I_{2^{m-1}} & -R^{(m)} \end{bmatrix}, \quad R^{(m+1)} = \begin{bmatrix} R^{(m)} & q_{m+1} I_{2^{m-1}} \\ r_{m+1} I_{2^{m-1}} & -Q^{(m)} \end{bmatrix}. \tag{6.14b}$$

It is proved by induction that eq. (6.12) is satisfied for integers $m \geq 1$. Substituting eqs. (6.10) and (6.11) with eqs. (6.12)–(6.14) into the Lax equation (2.2), we obtain a coupled version of the second WKI flow,

$$\begin{aligned} & q_{j,t} + \left\{ \left(1 - \sum_{k=1}^{m} q_k r_k\right)^{-\frac{3}{2}} q_{j,x} \right\}_{xx} = 0, \\ & r_{j,t} + \left\{ \left(1 - \sum_{k=1}^{m} r_k q_k\right)^{-\frac{3}{2}} r_{j,x} \right\}_{xx} = 0, \end{aligned} \qquad j = 1, 2, \ldots, m. \tag{6.15}$$

As is clear from the above discussion, the Lax formulation for the system (6.15) is given in terms of $2^m \times 2^m$ matrices.

6.3 Gauge Transformation

In previous sections, we have found new integrable multi-field systems (6.5) and (6.15), which are related to the T-WKI-type eigenvalue problem (6.1). It was shown that there is a gauge transformation between the system (6.5) with $m = 1$ and the system (6.15) with $m = 1$ [41, 97, 111]. In what follows, we shall prove that the gauge transformation is applicable to the multi-component systems. For the system (6.5), we perform a transformation of the independent variables,

$$\xi = \int_{x_0}^{x} s_0(x', t) \mathrm{d}x', \quad \tau = t.$$

Here we assume the boundary conditions, $s_{j,x} \to 0$ as $x \to x_0$ for $j = 0, 1, \ldots, 2m$. From the transformation, we obtain

$$\partial_x = s_0 \partial_\xi, \quad \partial_t = \partial_\tau - (s_0 s_{0,\xi}^2 + s_0^2 s_{0,\xi\xi} + \frac{3}{2} s_0^3 \sum_{k=0}^{2m} s_{k,\xi}^2) \partial_\xi.$$

Then the Lax formulation for the system (6.5) in Section 6.1.1 is transformed into

$$\Psi_\xi = U' \Psi, \quad \Psi_\tau = V' \Psi, \tag{6.16}$$

where

$$U' = \mathrm{i}\zeta X, \quad X \equiv \frac{1}{s_0} S, \tag{6.17a}$$

$$V' = 4\mathrm{i}\zeta^3 s_0 X + \zeta^2 s_0^3 (X X_\xi - X_\xi X) - \mathrm{i}\zeta (s_0^3 X_\xi)_\xi. \tag{6.17b}$$

Due to eq. (6.3), the matrix X satisfies the constraint,

$$X^2 = \frac{1}{s_0^2} I.$$

The compatibility condition of the transformed eigenvalue problem, $U'_\tau - V'_\xi + [U',\, V'] = O$, yields

$$X_\tau + (s_0^3 X_\xi)_{\xi\xi} = O. \tag{6.18}$$

If we take an appropriate representation of the anti-commutative matrices $\{e_i\}$, the matrix X,

$$X = \mathrm{i}e_0 + \sum_{k=1}^{2m} \mathrm{i}\Big(\frac{s_k}{s_0}\Big) e_k \equiv X^{(m)},$$

is expressed as

$$X^{(m)} = \begin{bmatrix} -I_{2^{m-1}} & -\mathrm{i}Q^{(m)} \\ -\mathrm{i}R^{(m)} & I_{2^{m-1}} \end{bmatrix}. \tag{6.19}$$

Here $Q^{(m)}$ and $R^{(m)}$ are given recursively by

$$Q^{(1)} = \frac{s_1}{s_0} + \mathrm{i}\frac{s_2}{s_0}, \qquad R^{(1)} = -\frac{s_1}{s_0} + \mathrm{i}\frac{s_2}{s_0}, \tag{6.20a}$$

$$Q^{(m+1)} = \begin{bmatrix} Q^{(m)} & \Big(\frac{s_{2m+1}}{s_0} + \mathrm{i}\frac{s_{2m+2}}{s_0}\Big) I_{2^{m-1}} \\ \Big(-\frac{s_{2m+1}}{s_0} + \mathrm{i}\frac{s_{2m+2}}{s_0}\Big) I_{2^{m-1}} & -R^{(m)} \end{bmatrix}, \tag{6.20b}$$

$$R^{(m+1)} = \begin{bmatrix} R^{(m)} & \Big(\frac{s_{2m+1}}{s_0} + \mathrm{i}\frac{s_{2m+2}}{s_0}\Big) I_{2^{m-1}} \\ \Big(-\frac{s_{2m+1}}{s_0} + \mathrm{i}\frac{s_{2m+2}}{s_0}\Big) I_{2^{m-1}} & -Q^{(m)} \end{bmatrix}. \tag{6.20c}$$

If we introduce a new set of variables $\{q_k\}$ and $\{r_k\}$ by $q_k = (s_{2k-1} + \mathrm{i}s_{2k})/s_0$, $r_k = (-s_{2k-1} + \mathrm{i}s_{2k})/s_0$, eq. (6.18) is reduced to

$$\begin{aligned} q_{j,\tau} + (s_0^3 q_{j,\xi})_{\xi\xi} = 0, \\ r_{j,\tau} + (s_0^3 r_{j,\xi})_{\xi\xi} = 0, \end{aligned} \qquad j = 1, 2, \ldots, m, \tag{6.21a}$$

where

$$s_0 = \pm\Big\{1 + \sum_{k=1}^{2m}\Big(\frac{s_k}{s_0}\Big)^2\Big\}^{-\frac{1}{2}} = \pm\frac{1}{\sqrt{1 - \sum_{k=1}^{m} q_k r_k}}. \tag{6.21b}$$

The system (6.21) essentially coincides with the coupled WKI system (6.15). Further, it is easily checked that the transformed Lax representation (6.16) with eqs. (6.17), (6.19) and (6.20) agrees with that for the coupled WKI system in Section 6.2. This shows that eq. (6.5) and eq. (6.15), or their Lax representations are connected by a gauge transformation.

6.4 Summary

In this chapter, we have obtained new coupled systems in the HF hierarchy and the WKI hierarchy. These two systems are proved to be connected with each other. It is noteworthy that Lax representations by use of the Clifford algebra yield coupled systems (6.5) and (6.15) in a simple manner. The technique is shown to be effective for some soliton equations in other chapters. Meanwhile just a replacement of scalar variables in 2×2 Lax matrices by vectors does not give a consistent equation of motion for the T-WKI-type eigenvalue problems as far as we have examined. We have found an integrable semi-discretization of the coupled system (6.5), i.e. eq. (6.9), by considering a semi-discrete version of the eigenvalue problem. For systems (6.5), (6.9) and (6.15), an infinite set of conservation laws can be obtained recursively on the basis of the Lax pairs [85, 86]. We do not give an explicit derivation of the conservation laws in this chapter.

For the original HF equation (2.15) and the first nontrivial flow in the WKI hierarchy, eq. (2.16), we have not found any simple generalization with multiple components so far. It remains an open problem to find a simple multi-field generalization of eq. (2.15) or eq. (2.16). The system (2.13) with the constraint (2.12) has a Lax representation (2.11) regardless of the size of S [111]. However, it is difficult to find an interesting reduction, except for the 2×2 matrix case. If there is no simple generalization of eq. (2.15) or eq. (2.16), it should be extremely interesting to ask why the higher flows do have a generalization and the original flows do not.

Chapter 7

Integrable Discretizations

Today it is widely accepted that the ISM is a very effective way to study a variety of soliton hierarchies (see, *e.g.* Chapter 2). As we have studied in previous chapters, we can obtain multi-component extensions of continuous soliton equations by generalizing Lax pairs of 2×2 matrices to the ones of larger matrices (see also [2, 11, 18, 49, 60, 77, 104]).

On the other hand, some discrete integrable models have been investigated by means of discrete versions of the ISM formulation [5–7, 17, 24, 58, 61, 62]). However, the family of discrete models which have been solved via the ISM is not large enough compared with that of continuous models. In fact, the study of multi-field discrete integrable systems has not been well-developed.

In this chapter, we propose a matrix generalization of the Ablowitz-Ladik formulation. The Ablowitz-Ladik formulation is a discrete version of the AKNS formulation and we explained only the point in Section 2.7. Thanks to the generalization, we obtain an integrable discretization of the matrix mKdV equation and the matrix NLS equation. Through a reduction of the discrete matrix mKdV equation, we obtain a discrete version of the cmKdV equations studied in Section 3.3.2. Applying the ISM, we solve the initial-value problem and construct multi-soliton solutions. Considering a further extension of the matrix Ablowitz-Ladik formulation, we obtain an integrable discretization of the cHirota equations. The discrete cHirota equations include discrete cNLS equations as a special choice of parameters. On the basis of this fact, we clarify an essential difference between the Lax matrix for the continuous cNLS equations and that for the discrete cNLS equations. Using a gauge transformation in the discrete case, we show that the ISM is also applicable to the discrete cHirota equations.

7.1 Semi-Discretization of the Matrix AKNS Hierarchy

In order to obtain a matrix generalization of the Ablowitz-Ladik formulation, we first consider the following form of the Lax pair,

$$L_n = z\begin{bmatrix} I_1 & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & Q_n \\ R_n & O \end{bmatrix} + \frac{1}{z}\begin{bmatrix} O & O \\ O & I_2 \end{bmatrix} = \begin{bmatrix} zI_1 & Q_n \\ R_n & \frac{1}{z}I_2 \end{bmatrix}, \tag{7.1}$$

$$\begin{aligned} M_n &= z^2 \begin{bmatrix} aI_1 & O \\ O & O \end{bmatrix} + z \begin{bmatrix} O & aQ_n \\ aR_{n-1} & O \end{bmatrix} + \begin{bmatrix} -aQ_nR_{n-1} + c_1 I_1 & O \\ O & bR_nQ_{n-1} + c_2 I_2 \end{bmatrix} \\ &\quad + \frac{1}{z} \begin{bmatrix} O & -bQ_{n-1} \\ -bR_n & O \end{bmatrix} + \frac{1}{z^2} \begin{bmatrix} O & O \\ O & -bI_2 \end{bmatrix} \\ &= \begin{bmatrix} z^2 aI_1 - aQ_nR_{n-1} + c_1 I_1 & zaQ_n - \frac{1}{z} bQ_{n-1} \\ zaR_{n-1} - \frac{1}{z} bR_n & -\frac{1}{z^2} bI_2 + bR_nQ_{n-1} + c_2 I_2 \end{bmatrix}, \end{aligned} \tag{7.2}$$

where z is the spectral parameter which satisfies $z_t = 0$. Here I_1 and I_2 are respectively the $p \times p$ and $q \times q$ unit matrices, Q_n is a $p \times q$ matrix, and R_n is a $q \times p$ matrix.

Substituting eqs. (7.1) and (7.2) into eq. (2.27), we obtain a set of matrix equations

$$Q_{n,t} - aQ_{n+1} - bQ_{n-1} + (c_2 - c_1)Q_n + aQ_{n+1}R_nQ_n + bQ_nR_nQ_{n-1} = O, \tag{7.3a}$$

$$R_{n,t} + bR_{n+1} + aR_{n-1} + (c_1 - c_2)R_n - bR_{n+1}Q_nR_n - aR_nQ_nR_{n-1} = O. \tag{7.3b}$$

Choosing the constants as

$$a = -b = 1, \quad c_1 = c_2 (= 0),$$

we obtain

$$Q_{n,t} - (Q_{n+1} - Q_{n-1}) + (Q_{n+1}R_nQ_n - Q_nR_nQ_{n-1}) = O, \tag{7.4a}$$

$$R_{n,t} - (R_{n+1} - R_{n-1}) + (R_{n+1}Q_nR_n - R_nQ_nR_{n-1}) = O. \tag{7.4b}$$

Since this model is interpreted as an integrable semi-discretization of eq. (3.3), we call eq. (7.4) the semi-discrete (sd-) matrix mKdV equation. If we set

$$R_n = -I, \quad Q_n = W_n - I,$$

we obtain a matrix version of the Lotka-Volterra equation,

$$W_{n,t} = W_{n+1}W_n - W_nW_{n-1},$$

where W_n is a square matrix.

When we take an alternative choice

$$a = b = \mathrm{i}, \quad c_2 - c_1 = 2\mathrm{i},$$

we obtain

$$\mathrm{i}Q_{n,t} + (Q_{n+1} + Q_{n-1} - 2Q_n) - (Q_{n+1}R_nQ_n + Q_nR_nQ_{n-1}) = O, \tag{7.5a}$$

$$\mathrm{i}R_{n,t} - (R_{n+1} + R_{n-1} - 2R_n) + (R_{n+1}Q_nR_n + R_nQ_nR_{n-1}) = O. \tag{7.5b}$$

Comparing eq. (7.5) with eq. (3.7), we call this model the sd-matrix NLS equation (cf. eq. (7.97) for a generalization). The integrable model (7.5) or, more generally, (7.3) was found and studied by Ablowitz, Ohta and Trubatch [8, 9].

Let us present a systematic method to construct local conservation laws for the system (7.3) which includes the sd-matrix mKdV equation and the sd-matrix NLS equation as its reductions. We start from a special class, $p = q = l$, of eq. (2.26),

$$\begin{bmatrix} \Psi_{1\,n+1} \\ \Psi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} F_{1\,n} & Q_n \\ R_n & F_{2\,n} \end{bmatrix} \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix}, \tag{7.6}$$

$$\begin{bmatrix} \Psi_{1n} \\ \Psi_{2n} \end{bmatrix}_t = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} \Psi_{1n} \\ \Psi_{2n} \end{bmatrix}, \tag{7.7}$$

where all the entries in eqs. (7.6) and (7.7) are assumed to be $l \times l$ square matrices. The following is a discrete version of the method for continuous systems (see Section 3.1.3). The zero-curvature condition (2.27) gives

$$F_{1n,t} + F_{1n}A_n + Q_nC_n - A_{n+1}F_{1n} - B_{n+1}R_n = O, \tag{7.8a}$$
$$F_{2n,t} + F_{2n}D_n + R_nB_n - D_{n+1}F_{2n} - C_{n+1}Q_n = O, \tag{7.8b}$$
$$Q_{n,t} + F_{1n}B_n + Q_nD_n - A_{n+1}Q_n - B_{n+1}F_{2n} = O, \tag{7.8c}$$
$$R_{n,t} + F_{2n}C_n + R_nA_n - D_{n+1}R_n - C_{n+1}F_{1n} = O. \tag{7.8d}$$

Defining an $l \times l$ square matrix Γ_n by

$$\Gamma_n \equiv \Psi_{2n}\Psi_{1n}^{-1},$$

we can prove the following relations from eqs. (7.6) and (7.7):

$$\Gamma_{n+1} = (R_n + F_{2n}\Gamma_n)(F_{1n} + Q_n\Gamma_n)^{-1}, \tag{7.9}$$

$$\Gamma_{n,t} = C_n + D_n\Gamma_n - \Gamma_nA_n - \Gamma_nB_n\Gamma_n. \tag{7.10}$$

Equation (7.10) can be interpreted as a matrix version of the Riccati equation. Using eqs. (7.8) and (7.9), we can rewrite eq. (7.10) as

$$\begin{aligned}(Q_n\Gamma_n + F_{1n})_t(Q_n\Gamma_n + F_{1n})^{-1} = {} & A_{n+1} - (Q_n\Gamma_n + F_{1n})A_n(Q_n\Gamma_n + F_{1n})^{-1} \\ & + B_{n+1}\Gamma_{n+1} - (Q_n\Gamma_n + F_{1n})B_n\Gamma_n(Q_n\Gamma_n + F_{1n})^{-1}.\end{aligned} \tag{7.11}$$

Taking the trace on both sides of eq. (7.11), we obtain

$$\mathrm{tr}\{\log(Q_n\Gamma_n + F_{1n})\}_t = \mathrm{tr}(A_{n+1} + B_{n+1}\Gamma_{n+1}) - \mathrm{tr}(A_n + B_n\Gamma_n). \tag{7.12}$$

Assuming that L_n is expressed as eq. (7.1), we have

$$F_{1n} = zI, \quad F_{2n} = \frac{1}{z}I. \tag{7.13}$$

The matrices Q_n and R_n are square matrices in this case. Then eqs. (7.12) and (7.9) are cast into

$$\left\{\mathrm{tr}\ \log\left(I + \frac{1}{z}Q_n\Gamma_n\right)\right\}_t = \mathrm{tr}(A_{n+1} + B_{n+1}\Gamma_{n+1}) - \mathrm{tr}(A_n + B_n\Gamma_n), \tag{7.14}$$

$$zQ_n\Gamma_n = Q_nR_{n-1} + \frac{1}{z}Q_nQ_{n-1}^{-1}(Q_{n-1}\Gamma_{n-1}) - (Q_n\Gamma_n)(Q_{n-1}\Gamma_{n-1}). \tag{7.15}$$

Equation (7.14) has the form of the local conservation law. This suggests that $\mathrm{tr}\{\log(I + Q_n\Gamma_n/z)\}$ is a generating function of the conserved densities for eq. (7.3). We expand $Q_n\Gamma_n$ with respect to the inverse of the spectral parameter z in the following form:

$$Q_n\Gamma_n^{(-)} = \sum_{j=1}^{\infty} \frac{1}{z^{2j-1}} f_n^{(j)}. \tag{7.16}$$

Substituting eq. (7.16) into eq. (7.15), we obtain a recursion formula for $f_n^{(j)}$:

$$f_n^{(j)} = Q_n R_{n-1}\delta_{j,1} + Q_n Q_{n-1}^{-1} f_{n-1}^{(j-1)} - \sum_{k=1}^{j-1} f_n^{(k)} f_{n-1}^{(j-k)}, \quad j = 1, 2, \ldots . \tag{7.17}$$

The formula (7.17) yields $f_n^{(j)}$; for instance,

$$\begin{aligned} f_n^{(1)} &= Q_n R_{n-1}, \\ f_n^{(2)} &= Q_n R_{n-2} - Q_n R_{n-1} Q_{n-1} R_{n-2}. \end{aligned}$$

We put eq. (7.16) into $\mathrm{tr}\{\log(I + Q_n\Gamma_n^{(-)}/z)\}$ and expand it with respect to $1/z$:

$$\mathrm{tr}\Big\{\log\Big(I + \frac{1}{z^2} f_n^{(1)} + \frac{1}{z^4} f_n^{(2)} + \frac{1}{z^6} f_n^{(3)} + \cdots\Big)\Big\} = \mathrm{tr}\Big\{\frac{1}{z^2} f_n^{(1)} + \frac{1}{z^4}\Big(f_n^{(2)} - \frac{1}{2} f_n^{(1)\,2}\Big) + \cdots\Big\}.$$

Thus, the first two conserved densities given by this expansion are

$$J_n^{(-1)} = f_n^{(1)} = Q_n R_{n-1}, \tag{7.18a}$$
$$J_n^{(-2)} = \mathrm{tr}\Big\{f_n^{(2)} - \frac{1}{2}(f_n^{(1)})^2\Big\} = \mathrm{tr}\Big\{Q_n R_{n-2} - Q_n R_{n-1} Q_{n-1} R_{n-2} - \frac{1}{2}(Q_n R_{n-1})^2\Big\}. \tag{7.18b}$$

Next we expand $Q_n\Gamma_n$ with respect to z. For this purpose, we rewrite eq. (7.15) as

$$Q_n\Gamma_n = -zQ_nR_n + zQ_nQ_{n+1}^{-1}(Q_{n+1}\Gamma_{n+1})(Q_n\Gamma_n + zI). \tag{7.19}$$

We substitute the following expansion,

$$Q_n\Gamma_n^{(+)} = \sum_{j=1}^{\infty} z^{2j-1} g_n^{(j)}, \tag{7.20}$$

into eq. (7.19) and obtain a recursion formula,

$$g_n^{(j)} = -Q_n R_n \delta_{j,1} + Q_n Q_{n+1}^{-1} g_{n+1}^{(j-1)} + Q_n Q_{n+1}^{-1} \sum_{k=1}^{j-1} g_{n+1}^{(k)} g_n^{(j-k)}, \quad j = 1, 2, \ldots . \tag{7.21}$$

By use of the formula (7.21), the first three of the coefficients $g_n^{(j)}$ are given by

$$\begin{aligned} g_n^{(1)} &= -Q_n R_n, \\ g_n^{(2)} &= -Q_n R_{n+1}(I - Q_n R_n), \\ g_n^{(3)} &= -Q_n R_{n+2}(I - Q_n R_n) + Q_n R_{n+1} Q_n R_{n+1}(I - Q_n R_n) + Q_n R_{n+2} Q_{n+1} R_{n+1}(I - Q_n R_n). \end{aligned}$$

We put eq. (7.20) into $\mathrm{tr}\{\log(I + Q_n\Gamma_n^{(+)}/z)\}$ and expand it with respect to z. Then we obtain

$$\begin{aligned} \mathrm{tr}\{\log(I + g_n^{(1)} + z^2 g_n^{(2)} + z^4 g_n^{(3)} + \cdots)\} = \mathrm{tr}\Big\{&\log(I + g_n^{(1)}) + z^2 g_n^{(2)}(I + g_n^{(1)})^{-1} \\ &+ z^4\Big[g_n^{(3)}(I + g_n^{(1)})^{-1} - \frac{1}{2}\{g_n^{(2)}(I + g_n^{(1)})^{-1}\}^2\Big] + \cdots\Big\}. \end{aligned}$$

Thus, the first three conserved densities given by this expansion are

$$J_n^{(0)} = \mathrm{tr}\{\log(I + g_n^{(1)})\} = \mathrm{tr}\{\log(I - Q_n R_n)\}, \tag{7.22a}$$

$$J_n^{(1)} = Q_n^{-1} g_n^{(2)} (I + g_n^{(1)})^{-1} Q_n = -R_{n+1} Q_n, \tag{7.22b}$$

$$\begin{aligned} J_n^{(2)} &= \mathrm{tr}\Big[g_n^{(3)}(I + g_n^{(1)})^{-1} - \frac{1}{2}\{g_n^{(2)}(I + g_n^{(1)})^{-1}\}^2\Big] \\ &= \mathrm{tr}\Big\{-Q_n R_{n+2} + Q_n R_{n+2} Q_{n+1} R_{n+1} + \frac{1}{2}(Q_n R_{n+1})^2\Big\}. \end{aligned} \tag{7.22c}$$

We should pay attention to the fact that all the entries of matrices $J_n^{(-1)}$ and $J_n^{(1)}$ are conserved densities for eq. (7.3). We can prove it by a straightforward calculation. That is the reason why we dropped the trace in eqs. (7.18a) and (7.22b). It is also noteworthy that the appearance of Q_n^{-1} in the derivation is just for a simplification of calculation and is not essential. The obtained conservation laws are valid even for a choice of irregular matrix Q_n.

The generator of the conserved densities, $\mathrm{tr}\{\log(I + Q_n \Gamma_n / z)\}$, is shown to be related with a time-independent subset of scattering data defined later (see Appendix I).

7.2 Semi-Discrete Coupled mKdV Equations

Very recently, Ohta [70, 71] obtained an N-soliton solution of semi-discrete(sd-) cmKdV equations,

$$\frac{\partial u_n^{(i)}}{\partial t} = \Big(1 + \sum_{j,k=0}^{M-1} C_{jk} u_n^{(j)} u_n^{(k)}\Big)(u_{n+1}^{(i)} - u_{n-1}^{(i)}), \quad i = 0, 1, \ldots, M-1. \tag{7.23}$$

This model reduces to the cmKdV equations (3.51) in the continuum limit by some variable transformations. Hisakado [38] revealed the connection between the sd-cmKdV equations and the 2-D Toda lattice. Hirota [35] studied a similar model,

$$\frac{\partial u_n^{(i)}}{\partial t} = \sum_{j,k=0}^{M-1} C_{jk} u_n^{(j)} u_n^{(k)} \cdot (u_{n+1}^{(i)} - u_{n-1}^{(i)}), \quad i = 0, 1, \ldots, M-1,$$

in the open-end case and obtained the so-called molecule solutions.

Let us consider a reduction of the sd-matrix mKdV equation (7.4) to the sd-cmKdV equations (7.23). As a preparation, in a similar way as in Section 3.3.2, we transform the sd-cmKdV equations into a normalized form,

$$\frac{\partial v_n^{(i)}}{\partial t} = \Big(1 - \sum_{j=0}^{M-1} \varepsilon_j v_n^{(j)\,2}\Big)(v_{n+1}^{(i)} - v_{n-1}^{(i)}), \quad \varepsilon_j = \pm 1, \quad i = 0, 1, \ldots, M-1. \tag{7.24}$$

The above expression of the sd-cmKdV equations is more convenient than eq. (7.23) to perform the ISM. Thus, we mainly deal with eq. (7.24) as the sd-cmKdV equations in what follows. Further, we assume that the dependent variables $\{v_n^{(i)}\}$ are real and $\varepsilon_j = -1$ $(j = 0, 1, \ldots, M-1)$.

In an analogous way to the continuous theory in Section 3.3.2, we assume that Q_n and R_n are $2^{m-1} \times 2^{m-1}$ $(m \geq 2)$ matrices expressed as

$$Q_n^{(m)} = v_n^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)} e_k, \qquad R_n^{(m)} = -v_n^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)} e_k. \tag{7.25}$$

Here $2^{m-1} \times 2^{m-1}$ matrices $\{e_1, \cdots, e_{2m-1}\}$ satisfy eqs. (3.55a)–(3.55c). For $Q_n^{(m)}$ and $R_n^{(m)}$, we can easily prove simple relations,

$$Q_n^{(m)} R_n^{(m)} = R_n^{(m)} Q_n^{(m)} = -\sum_{j=0}^{2m-1} v_n^{(j)\,2} \cdot I_{2^{m-1}}. \tag{7.26}$$

and

$$R_n^{(m)} = -Q_n^{(m)\,\dagger}. \tag{7.27}$$

These relations play a crucial role in performing the ISM. Substituting $Q_n^{(m)}$ and $R_n^{(m)}$ into Q_n and R_n in the sd-matrix mKdV equation (7.4), we obtain the sd-cmKdV equations,

$$\frac{\partial v_n^{(i)}}{\partial t} = \Big(1 + \sum_{j=0}^{M-1} v_n^{(j)\,2}\Big)(v_{n+1}^{(i)} - v_{n-1}^{(i)}), \qquad i = 0, 1, \ldots, M-1. \tag{7.28}$$

where we set $M = 2m$.

Because the sd-cmKdV equations are given as a reduction of the sd-matrix mKdV equation, the results in Section 7.1 assure that the sd-cmKdV equations have an infinite number of local conservation laws. Explicit forms of the first four conserved densities for the original sd-cmKdV equations (7.23) are given by

$$\mathrm{I}_1 = \log\Big(1 + \sum_{j,k} C_{jk} u_n^{(j)} u_n^{(k)}\Big), \tag{7.29a}$$

$$\mathrm{I}_2 = \begin{cases} u_{n+1}^{(j)} u_n^{(k)} - u_n^{(j)} u_{n+1}^{(k)} & \text{for all } j,\ k\ (j \neq k), \\ \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_n^{(k)}, & \end{cases} \tag{7.29b}$$

$$\begin{aligned} \mathrm{I}_3 = {} & \Big(1 + \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_{n+1}^{(k)}\Big) \cdot \sum_{j,k} C_{jk} u_{n+2}^{(j)} u_n^{(k)} + \Big(\sum_{j,k} C_{jk} u_{n+1}^{(j)} u_n^{(k)}\Big)^2 \\ & - \frac{1}{2} \sum_{j,k} C_{jk} u_n^{(j)} u_n^{(k)} \cdot \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_{n+1}^{(k)}, \end{aligned} \tag{7.29c}$$

$$\begin{aligned} \mathrm{I}_4 = {} & \Big(1 + \sum_{j,k} C_{jk} u_{n+2}^{(j)} u_{n+2}^{(k)}\Big)\Big(1 + \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_{n+1}^{(k)}\Big) \cdot \sum_{j,k} C_{jk} u_{n+3}^{(j)} u_n^{(k)} \\ & + 2 \sum_{j,k} C_{jk} u_{n+2}^{(j)} u_n^{(k)} \cdot \Big(1 + \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_{n+1}^{(k)}\Big)\Big(\sum_{j,k} C_{jk} u_{n+1}^{(j)} u_n^{(k)} + \sum_{j,k} C_{jk} u_{n+2}^{(j)} u_{n+1}^{(k)}\Big) \\ & - \Big(\sum_{j,k} C_{jk} u_{n+1}^{(j)} u_n^{(k)} \cdot \sum_{j,k} C_{jk} u_{n+2}^{(j)} u_{n+2}^{(k)} + \sum_{j,k} C_{jk} u_{n+2}^{(j)} u_{n+1}^{(k)} \cdot \sum_{j,k} C_{jk} u_n^{(j)} u_n^{(k)}\Big) \\ & \cdot \Big(1 + \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_{n+1}^{(k)}\Big) + \frac{4}{3}\Big(\sum_{j,k} C_{jk} u_{n+1}^{(j)} u_n^{(k)}\Big)^3 \\ & - \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_n^{(k)} \cdot \sum_{j,k} C_{jk} u_n^{(j)} u_n^{(k)} \cdot \sum_{j,k} C_{jk} u_{n+1}^{(j)} u_{n+1}^{(k)}. \end{aligned} \tag{7.29d}$$

7.3 ISM for the Semi-Discrete Coupled mKdV Equations

In this section we consider the scattering and inverse scattering problems associated with the $2l \times 2l$ $(l = 2^{m-1})$ matrix (7.1),

$$\begin{bmatrix} \Psi_{1\,n+1} \\ \Psi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} zI & Q_n \\ R_n & \frac{1}{z}I \end{bmatrix} \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix}, \tag{7.30}$$

where Q_n and R_n are expressed as eq. (7.25) and thus satisfy the constraints,

$$R_n = -Q_n^\dagger, \quad Q_n R_n = -\sum_{j=0}^{2m-1} v_n^{(j)\,2} \cdot I \equiv -\sigma_n I. \tag{7.31}$$

Here and hereafter the superscripts (m) of $Q_n^{(m)}$ and $R_n^{(m)}$ are often omitted for convenience. We assume the rapidly decreasing boundary conditions,

$$Q_n, R_n \to O \quad \text{as} \quad n \to \pm\infty. \tag{7.32}$$

Let $\Psi_n(z)$ and $\Phi_n(z)$ be solutions of eq. (7.30) composed of $2l(=2^m)$ rows and $l(=2^{m-1})$ columns. We introduce the following matrix function of $\Phi^{(1)}$ and $\Phi^{(2)}$:

$$W_n[\Phi^{(1)}, \Phi^{(2)}] \equiv \Phi_n^{(1)\,\dagger}(\frac{1}{z^*})\Phi_n^{(2)}(z).$$

This satisfies a recursion relation

$$W_{n+1}[\Phi^{(1)}, \Phi^{(2)}] = (I - Q_n R_n) W_n[\Phi^{(1)}, \Phi^{(2)}] = \rho_n W_n[\Phi^{(1)}, \Phi^{(2)}], \tag{7.33}$$

where ρ_n is defined by

$$\rho_n \equiv 1 + \sigma_n = 1 + \sum_{j=0}^{2m-1} v_n^{(j)\,2}. \tag{7.34}$$

Using eq. (7.33) repeatedly, we get

$$W_\infty[\Psi, \Phi] = \tau W_{-\infty}[\Psi, \Phi],$$

where τ is defined by

$$\tau \equiv \prod_{n=-\infty}^{\infty} \rho_n = \prod_{n=-\infty}^{\infty} \Big(1 + \sum_{j=0}^{2m-1} v_n^{(j)\,2}\Big), \tag{7.35}$$

and assumed to be finite. It should be noticed that τ is a conserved quantity (see eq. (7.29a)).

We introduce Jost functions ϕ_n, $\bar{\phi}_n$ and ψ_n, $\bar{\psi}_n$ which satisfy the boundary conditions,

$$\phi_n \sim \begin{bmatrix} I \\ O \end{bmatrix} z^n \qquad \text{as} \quad n \to -\infty, \tag{7.36a}$$

$$\bar{\phi}_n \sim \begin{bmatrix} O \\ -I \end{bmatrix} z^{-n} \qquad \text{as} \quad n \to -\infty, \tag{7.36b}$$

and

$$\psi_n \sim \begin{bmatrix} O \\ I \end{bmatrix} z^{-n} \qquad \text{as} \quad n \to +\infty, \tag{7.36c}$$

$$\bar{\psi}_n \sim \begin{bmatrix} I \\ O \end{bmatrix} z^{n} \qquad \text{as} \quad n \to +\infty. \tag{7.36d}$$

It can be shown that $\phi_n z^{-n}$, $\psi_n z^n$ are analytic outside the unit circle ($|z| > 1$) on the complex z plane, and that $\bar{\phi}_n z^n$, $\bar{\psi}_n z^{-n}$ are analytic inside the unit circle ($|z| < 1$) on the z plane, when Q_n, R_n go to O sufficiently rapidly as $n \to \pm\infty$. We assume the following summation representation of the Jost functions ψ_n and $\bar{\psi}_n$:

$$\psi_n = \sum_{n'=n}^{\infty} z^{-n'} K(n,n'), \qquad \bar{\psi}_n = \sum_{n'=n}^{\infty} z^{n'} \bar{K}(n,n'), \tag{7.37}$$

where $K(n,n')$ and $\bar{K}(n,n')$ are column vectors which consist of two $l \times l$ square matrices,

$$K(n,n') = \begin{bmatrix} K_1(n,n') \\ K_2(n,n') \end{bmatrix}, \qquad \bar{K}(n,n') = \begin{bmatrix} \bar{K}_1(n,n') \\ \bar{K}_2(n,n') \end{bmatrix}.$$

We substitute eq. (7.37) into eq. (7.30). Equating the terms with the same power of z, we get the relations for K_1 and K_2,

$$K_1(n,n) = O, \tag{7.38a}$$

$$Q_n K_2(n,n) = -K_1(n,n+1), \tag{7.38b}$$

$$K_1(n,n+j+1) = K_1(n+1,n+j) - Q_n K_2(n,n+j) \quad (j \geq 1), \tag{7.38c}$$

$$R_n K_1(n,n+j) = K_2(n+1,n+j) - K_2(n,n+j-1) \quad (j \geq 1), \tag{7.38d}$$

and for $\bar{K}_1$ and $\bar{K}_2$,

$$\bar{K}_2(n,n) = O, \tag{7.39a}$$

$$R_n \bar{K}_1(n,n) = -\bar{K}_2(n,n+1), \tag{7.39b}$$

$$Q_n \bar{K}_2(n,n+j) = \bar{K}_1(n+1,n+j) - \bar{K}_1(n,n+j-1) \quad (j \geq 1), \tag{7.39c}$$

$$\bar{K}_2(n,n+j+1) = \bar{K}_2(n+1,n+j) - R_n \bar{K}_1(n,n+j) \quad (j \geq 1). \tag{7.39d}$$

Because a pair of the Jost functions ϕ_n and $\bar{\phi}_n$, or ψ_n and $\bar{\psi}_n$, forms a fundamental system of the solutions of the scattering problem (7.30), we can set

$$\phi_n(z) = \bar{\psi}_n(z) A(z) + \psi_n(z) B(z), \tag{7.40a}$$

$$\bar{\phi}_n(z) = \bar{\psi}_n(z) \bar{B}(z) - \psi_n(z) \bar{A}(z). \tag{7.40b}$$

Here the coefficients $A(z)$, $\bar{A}(z)$, $B(z)$ and $\bar{B}(z)$ are n-independent $l \times l$ matrices which are called scattering data.

To derive the formula of the ISM rigorously and concisely, we assume that Q_n and R_n are on compact support. The result is, however, applicable to larger classes of potentials Q_n

and R_n. According to the asymptotic behaviors of the Jost functions (7.36a)–(7.36d), we obtain

$$
\begin{aligned}
W_{-\infty}[\phi,\phi] = W_{-\infty}[\bar{\phi},\bar{\phi}] &= W_{\infty}[\psi,\psi] = W_{\infty}[\bar{\psi},\bar{\psi}] = I, && (7.41a)\\
W_n[\phi,\bar{\phi}] &= W_n[\psi,\bar{\psi}] = O, && (7.41b)\\
A(z) &= W_{\infty}[\bar{\psi},\phi], && (7.41c)\\
\bar{A}(z) &= -W_{\infty}[\psi,\bar{\phi}], && (7.41d)\\
B(z) &= W_{\infty}[\psi,\phi], && (7.41e)\\
\bar{B}(z) &= W_{\infty}[\bar{\psi},\bar{\phi}]. && (7.41f)
\end{aligned}
$$

The expressions (7.41c) and (7.41d) show that $A(z)$ and $\bar{A}(z)$ are, respectively, analytic outside the unit circle ($|z| > 1$) and inside the unit circle ($|z| < 1$). Using the relations (7.41), we obtain the following relations among $A(z)$, $\bar{A}(z)$, $B(z)$, and $\bar{B}(z)$:

$$A^{\dagger}(\frac{1}{z^*})A(z) + B^{\dagger}(\frac{1}{z^*})B(z) = \tau I, \tag{7.42a}$$

$$\bar{A}^{\dagger}(\frac{1}{z^*})\bar{A}(z) + \bar{B}^{\dagger}(\frac{1}{z^*})\bar{B}(z) = \tau I, \tag{7.42b}$$

$$A^{\dagger}(\frac{1}{z^*})\bar{B}(z) - B^{\dagger}(\frac{1}{z^*})\bar{A}(z) = O. \tag{7.42c}$$

These relations are written in a matrix form as

$$\begin{bmatrix} A^{\dagger}(1/z^*) & B^{\dagger}(1/z^*) \\ \bar{B}^{\dagger}(1/z^*) & -\bar{A}^{\dagger}(1/z^*) \end{bmatrix} \begin{bmatrix} A(z) & \bar{B}(z) \\ B(z) & -\bar{A}(z) \end{bmatrix} = \tau \begin{bmatrix} I & O \\ O & I \end{bmatrix},$$

which leads to the inversion of eq. (7.40),

$$\tau\bar{\psi}_n(z) = \phi_n(z)A^{\dagger}(\frac{1}{z^*}) + \bar{\phi}_n(z)\bar{B}^{\dagger}(\frac{1}{z^*}), \tag{7.43a}$$

$$\tau\psi_n(z) = \phi_n(z)B^{\dagger}(\frac{1}{z^*}) - \bar{\phi}_n(z)\bar{A}^{\dagger}(\frac{1}{z^*}). \tag{7.43b}$$

Equation (7.43a) is used in Appendix F.

7.3.1 Gel'fand-Levitan-Marchenko equations

Multiplying $A(z)^{-1}$ and $\bar{A}(z)^{-1}$ from the right to eqs. (7.40a) and (7.40b), respectively, we obtain

$$\phi_n(z)A(z)^{-1} = \bar{\psi}_n(z) + \psi_n(z)B(z)A(z)^{-1}, \tag{7.44a}$$

$$\bar{\phi}_n(z)\bar{A}(z)^{-1} = -\psi_n(z) + \bar{\psi}_n(z)\bar{B}(z)\bar{A}(z)^{-1}. \tag{7.44b}$$

We substitute eq. (7.37) into the right-hand side of eq. (7.44a) and operate on both sides,

$$\frac{1}{2\pi \mathrm{i}}\oint_C \mathrm{d}z\; z^{-m-1} \quad (m \geq n),$$

where C denotes a contour along the unit circle $|z| = 1$. Then we obtain

$$J = \bar{K}(n,m) + \sum_{n'=n}^{\infty} K(n,n')F_C(n'+m),$$

where F_C and J are defined by

$$F_C(n'+m) \equiv \frac{1}{2\pi\mathrm{i}} \oint_C B(z)A(z)^{-1} z^{-(n'+m)-1} \mathrm{d}z,$$

$$J \equiv \frac{1}{2\pi\mathrm{i}} \oint_C \phi_n z^{-n} A(z)^{-1} z^{n-m-1} \mathrm{d}z. \tag{7.45}$$

We notice that $\phi_n z^{-n}$ and $A(z)$ are analytic outside the unit circle C, $|z| > 1$. The inverse of $A(z)$, i.e. $A(z)^{-1}$, is given by

$$A(z)^{-1} = \frac{1}{\det A(z)} \breve{A}(z),$$

where $\breve{A}$ denotes the cofactor matrix of A. Thus the singularities of the integrand of eq. (7.45) in $|z| > 1$ come from the zeros of $\det A(z)$. We assume that $1/\det A(z)$ has $2N$ isolated simple poles $\{z_1, z_2, \cdots, z_{2N}\}$ and is regular on the unit circle C (see eq. (7.54) for the reason why we choose the number of poles to be even). We set

$$J_{\infty,n} = \lim_{z\to\infty} \phi_n z^{-n} A(z)^{-1},$$

and use the residue theorem. Then the integral J is computed as

$$\begin{aligned} J &= -\sum_{j=1}^{2N} \psi_n(z_j) C_j z_j^{-m-1} + J_{\infty,n}\delta_{n,m} \\ &= -\sum_{j=1}^{2N} \sum_{n'=n}^{\infty} K(n,n') C_j z_j^{-(n'+m)-1} + J_{\infty,n}\delta_{n,m}, \end{aligned}$$

where C_j is the residue matrix of $B(z)A(z)^{-1}$ at $z = z_j$. Defining

$$F_D(n'+m) \equiv \sum_{j=1}^{2N} C_j z_j^{-(n'+m)-1},$$

we arrive at the discrete version of the Gel'fand-Levitan-Marchenko equation,

$$\bar{K}(n,m) + \sum_{n'=n}^{\infty} K(n,n') F(n'+m) = J_{\infty,n}\delta_{n,m} \quad (m \geq n). \tag{7.46}$$

Here $F(n)$ is defined by

$$\begin{aligned} F(n) &\equiv F_C(n) + F_D(n) \\ &= \frac{1}{2\pi\mathrm{i}} \oint_C B(z)A(z)^{-1} z^{-n-1} \mathrm{d}z + \sum_{j=1}^{2N} C_j z_j^{-n-1}. \end{aligned}$$

Similarly, we operate

$$\frac{1}{2\pi\mathrm{i}} \oint_C \mathrm{d}z\, z^{m-1} \quad (m \geq n)$$

on both sides of eq. (7.44b). Then we obtain

$$\bar{J} = -K(n,m) + \sum_{n'=n}^{\infty} \bar{K}(n,n') \bar{F}_C(n'+m),$$

where

$$\bar{F}_C(n'+m) = \frac{1}{2\pi \mathrm{i}} \oint_C \bar{B}(z)\bar{A}(z)^{-1} z^{n'+m-1} \mathrm{d}z,$$
$$\bar{J} = \frac{1}{2\pi \mathrm{i}} \oint_C \bar{\phi}_n z^n \bar{A}(z)^{-1} z^{-n+m-1} \mathrm{d}z.$$

We notice that $\bar{\phi}_n z^n$ and $\bar{A}(z)$ are analytic inside the unit circle C, $|z| < 1$. We assume that $1/\det\bar{A}(z)$ has $2\bar{N}$ isolated simple poles $\{\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{2\bar{N}}\}$ and is regular on the unit circle C. We set

$$\bar{J}_{0,n} = \lim_{z\to 0} \bar{\phi}_n z^n \bar{A}(z)^{-1},$$

and use the residue theorem. The integral $\bar{J}$ is given by

$$\begin{aligned}
\bar{J} &= \sum_{k=1}^{2\bar{N}} \bar{\psi}_n(\bar{z}_k)\bar{C}_k \bar{z}_k^{m-1} + \bar{J}_{0,n}\delta_{n,m} \\
&= \sum_{k=1}^{2\bar{N}} \sum_{n'=n}^{\infty} \bar{K}(n,n')\bar{C}_k \bar{z}_k^{n'+m-1} + \bar{J}_{0,n}\delta_{n,m},
\end{aligned}$$

where $\bar{C}_k$ is the residue matrix of $\bar{B}(z)\bar{A}(z)^{-1}$ at $z = \bar{z}_k$. Defining

$$\bar{F}_D(n'+m) \equiv -\sum_{k=1}^{2\bar{N}} \bar{C}_k \bar{z}_k^{n'+m-1},$$

we obtain the counterpart of the discrete Gel'fand-Levitan-Marchenko equation,

$$K(n,m) - \sum_{n'=n}^{\infty} \bar{K}(n,n')\bar{F}(n'+m) = -\bar{J}_{0,n}\delta_{n,m} \quad (m \geq n). \tag{7.47}$$

Here $\bar{F}(n)$ is defined by

$$\begin{aligned}
\bar{F}(n) &\equiv \bar{F}_C(n) + \bar{F}_D(n) \\
&= \frac{1}{2\pi \mathrm{i}} \oint_C \bar{B}(z)\bar{A}(z)^{-1} z^{n-1} \mathrm{d}z - \sum_{k=1}^{2\bar{N}} \bar{C}_k \bar{z}_k^{n-1}.
\end{aligned}$$

From eqs. (7.38b) and (7.38d), we obtain

$$\begin{aligned}
K_2(n,n) &= (I - R_n Q_n)^{-1} K_2(n+1,n+1) \\
&\ \ \vdots \\
&= \prod_{i=n}^{\infty} (I - R_i Q_i)^{-1} = \prod_{i=n}^{\infty} \rho_i^{-1} \cdot I,
\end{aligned}$$

where $\lim_{n\to\infty} K_2(n,n) = I$ is used. Similarly, from eqs. (7.39b) and (7.39c) with $\lim_{n\to\infty} \bar{K}_1(n,n) = I$, we obtain

$$\begin{aligned}
\bar{K}_1(n,n) &= (I - Q_n R_n)^{-1} \bar{K}_1(n+1,n+1) \\
&\ \ \vdots \\
&= \prod_{i=n}^{\infty} (I - Q_i R_i)^{-1} = \prod_{i=n}^{\infty} \rho_i^{-1} \cdot I.
\end{aligned}$$

Thus, it is natural to set

$$K(n,m) = \kappa(n,m)\prod_{i=n}^{\infty}(I - R_iQ_i)^{-1} = \kappa(n,m)\prod_{i=n}^{\infty}\rho_i^{-1} \quad (m \geq n), \tag{7.48}$$

$$\bar{K}(n,m) = \bar{\kappa}(n,m)\prod_{i=n}^{\infty}(I - Q_iR_i)^{-1} = \bar{\kappa}(n,m)\prod_{i=n}^{\infty}\rho_i^{-1} \quad (m \geq n). \tag{7.49}$$

Here $\kappa(n,m)$ and $\bar{\kappa}(n,m)$ are column vectors whose elements are $l \times l$ square matrices:

$$\kappa(n,m) = \begin{bmatrix} \kappa_1(n,m) \\ \kappa_2(n,m) \end{bmatrix}, \quad \bar{\kappa}(n,m) = \begin{bmatrix} \bar{\kappa}_1(n,m) \\ \bar{\kappa}_2(n,m) \end{bmatrix}.$$

In particular, $\kappa(n,n)$ and $\bar{\kappa}(n,n)$ are given by

$$\kappa(n,n) = \begin{bmatrix} O \\ I \end{bmatrix}, \tag{7.50}$$

$$\bar{\kappa}(n,n) = \begin{bmatrix} I \\ O \end{bmatrix}. \tag{7.51}$$

Putting eq. (7.48) with (7.50) and eq. (7.49) with (7.51) into eq. (7.38) and eq. (7.39), respectively, we obtain the relations for κ_1, κ_2, $\bar{\kappa}_1$, and $\bar{\kappa}_2$,

$$-\kappa_1(n,n+1) = Q_n,$$

$$-\bar{\kappa}_2(n,n+1) = R_n,$$

$$\kappa_1(n,n+j+1) = \rho_n\kappa_1(n+1,n+j) - Q_n\kappa_2(n,n+j) \quad (j \geq 1),$$
$$R_n\kappa_1(n,n+j) = \rho_n\kappa_2(n+1,n+j) - \kappa_2(n,n+j-1) \quad (j \geq 1),$$
$$Q_n\bar{\kappa}_2(n,n+j) = \rho_n\bar{\kappa}_1(n+1,n+j) - \bar{\kappa}_1(n,n+j-1) \quad (j \geq 1),$$
$$\bar{\kappa}_2(n,n+j+1) = \rho_n\bar{\kappa}_2(n+1,n+j) - R_n\bar{\kappa}_1(n,n+j) \quad (j \geq 1).$$

In terms of κ and $\bar{\kappa}$, the Gel'fand-Levitan-Marchenko equations (7.46) and (7.47) for $m > n$ are rewritten as

$$\bar{\kappa}(n,m) + \begin{bmatrix} O \\ I \end{bmatrix} F(n+m) + \sum_{n'=n+1}^{\infty} \kappa(n,n')F(n'+m) = \begin{bmatrix} O \\ O \end{bmatrix} \quad (m > n), \tag{7.52}$$

$$\kappa(n,m) - \begin{bmatrix} I \\ O \end{bmatrix} \bar{F}(n+m) - \sum_{n'=n+1}^{\infty} \bar{\kappa}(n,n')\bar{F}(n'+m) = \begin{bmatrix} O \\ O \end{bmatrix} \quad (m > n). \tag{7.53}$$

It should be noted that the scattering problem (7.30) gives the symmetry properties of the scattering data. Iterating eq. (7.30), we can prove that $A(z)$, $\bar{A}(z)$ are polynomials in z of even degree and $B(z)$, $\bar{B}(z)$ are polynomials in z of odd degree. This fact leads to

$$\det A(z) = \det A(-z), \tag{7.54}$$

$$\det \bar{A}(z) = \det \bar{A}(-z), \tag{7.55}$$

which mean the eigenvalues z_j, $\bar{z}_k$ appear as positive-negative pairs. Further, we have

$$B(z)A(z)^{-1} = -B(-z)A(-z)^{-1}, \tag{7.56}$$

$$\bar{B}(z)\bar{A}(z)^{-1} = -\bar{B}(-z)\bar{A}(-z)^{-1}. \tag{7.57}$$

Therefore, we can simplify the forms of F and $\bar{F}$ as

$$F(n+m) = \begin{cases} 2F_R(n+m), & m = n+2j-1, \\ O, & m = n+2j, \end{cases} \quad j \geq 1,$$

$$F_R(n+m) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} B(z)A(z)^{-1} z^{-(n+m)-1} \mathrm{d}z + \sum_{j=1}^{N} C_j z_j^{-(n+m)-1},$$

and

$$\bar{F}(n+m) = \begin{cases} 2\bar{F}_R(n+m), & m = n+2j-1, \\ O, & m = n+2j, \end{cases} \quad j \geq 1,$$

$$\bar{F}_R(n+m) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} \bar{B}(z)\bar{A}(z)^{-1} z^{n+m-1} \mathrm{d}z - \sum_{k=1}^{\bar{N}} \bar{C}_k \bar{z}_k^{n+m-1}.$$

Here C_R denotes a contour along the right-half portion of the unit circle C.

The symmetry properties of F and $\bar{F}$ give rise to those of κ and $\bar{\kappa}$. From eqs. (7.52) and (7.53), we obtain

$$\kappa_1(n,m) = \begin{cases} \kappa_{1R}(n,m), & m = n+2j-1, \\ O, & m = n+2j, \end{cases} \quad j \geq 1,$$

$$\bar{\kappa}_2(n,m) = \begin{cases} \bar{\kappa}_{2R}(n,m), & m = n+2j-1, \\ O, & m = n+2j, \end{cases} \quad j \geq 1.$$

Considering the above symmetry properties, we obtain the simplified Gel'fand-Levitan-Marchenko equations for κ_{1R} and $\bar{\kappa}_{2R}$:

$$\kappa_{1R}(n,m) = 2\bar{F}_R(n+m) - 4 \sum_{\substack{n'=n+2 \\ n'-n=\text{even}}}^{\infty} \sum_{\substack{n''=n+1 \\ n''-n=\text{odd}}}^{\infty} \kappa_{1R}(n,n'')F_R(n''+n')\bar{F}_R(n'+m), \tag{7.58}$$

$$\bar{\kappa}_{2R}(n,m) = -2F_R(n+m) - 4 \sum_{\substack{n'=n+2 \\ n'-n=\text{even}}}^{\infty} \sum_{\substack{n''=n+1 \\ n''-n=\text{odd}}}^{\infty} \bar{\kappa}_{2R}(n,n'')\bar{F}_R(n''+n')F_R(n'+m), \tag{7.59}$$

where $m-n$ is a positive odd number.

7.3.2 Time dependence of the scattering data

Under the rapidly decreasing boundary conditions (7.32), the asymptotic form of the Lax matrix M_n for the sd-matrix mKdV equation (7.4) is given by

$$M_n \to \begin{bmatrix} z^2 I & O \\ O & \frac{1}{z^2} I \end{bmatrix} \quad \text{as} \quad n \to \pm\infty.$$

We define time-dependent Jost functions by

$$\phi_n^{(t)} \equiv \phi_n \mathrm{e}^{z^2 t} \sim \begin{bmatrix} I \\ O \end{bmatrix} z^n \mathrm{e}^{z^2 t} \qquad \text{as} \quad n \to -\infty,$$

$$\bar{\phi}_n^{(t)} \equiv \bar{\phi}_n \mathrm{e}^{\frac{1}{z^2} t} \sim \begin{bmatrix} O \\ -I \end{bmatrix} z^{-n} \mathrm{e}^{\frac{1}{z^2} t} \qquad \text{as} \quad n \to -\infty.$$

From the relations

$$\phi_{n,t}^{(t)} = M_n \phi_n^{(t)}, \quad \bar{\phi}_{n,t}^{(t)} = M_n \bar{\phi}_n^{(t)},$$

we obtain

$$\phi_{n,t} = (M_n - z^2 I)\phi_n, \quad \bar{\phi}_{n,t} = \Big(M_n - \frac{1}{z^2} I\Big)\bar{\phi}_n. \tag{7.60}$$

We put the definitions of the scattering data,

$$\phi_n(z) = \bar{\psi}_n(z) A(z,t) + \psi_n(z) B(z,t),$$

$$\bar{\phi}_n(z) = \bar{\psi}_n(z) \bar{B}(z,t) - \psi_n(z) \bar{A}(z,t),$$

into eq. (7.60). Then taking the limit $n \to +\infty$, we obtain the time dependences of BA^{-1}, C_j and $\bar{B}\bar{A}^{-1}$, $\bar{C}_k$. They are respectively calculated as

$$A(z,t) = A(z,0), \tag{7.61a}$$

$$B(z,t)A(z,t)^{-1} = B(z,0)A(z,0)^{-1} \mathrm{e}^{-\left(z^2 - \frac{1}{z^2}\right)t}, \tag{7.61b}$$

$$C_j(t) = C_j(0) \mathrm{e}^{-\left(z_j^2 - \frac{1}{z_j^2}\right)t}, \tag{7.61c}$$

and

$$\bar{A}(z,t) = \bar{A}(z,0),$$

$$\bar{B}(z,t)\bar{A}(z,t)^{-1} = \bar{B}(z,0)\bar{A}(z,0)^{-1} \mathrm{e}^{\left(z^2 - \frac{1}{z^2}\right)t},$$

$$\bar{C}_k(t) = \bar{C}_k(0) \mathrm{e}^{\left(\bar{z}_k^2 - \frac{1}{\bar{z}_k^2}\right)t}.$$

The above result gives explicitly time-dependent forms of $F_R(n,t)$ and $\bar{F}_R(n,t)$ for odd n,

$$F_R(n,t) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} B(z,0)A(z,0)^{-1} z^{-n-1} \mathrm{e}^{-\left(z^2 - \frac{1}{z^2}\right)t} \mathrm{d}z + \sum_{j=1}^{N} C_j(0) z_j^{-n-1} \mathrm{e}^{-\left(z_j^2 - \frac{1}{z_j^2}\right)t},$$

$$\bar{F}_R(n,t) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} \bar{B}(z,0)\bar{A}(z,0)^{-1} z^{n-1} \mathrm{e}^{\left(z^2 - \frac{1}{z^2}\right)t} \mathrm{d}z - \sum_{k=1}^{\bar{N}} \bar{C}_k(0) \bar{z}_k^{n-1} \mathrm{e}^{\left(\bar{z}_k^2 - \frac{1}{\bar{z}_k^2}\right)t}.$$

7.3.3 Initial-value problem

Thanks to the constraints $R_n = -Q_n^\dagger$ and $Q_n R_n = -\sigma_n I$, we have some additional relations besides eqs. (7.54)–(7.57). The first additional relation is

$$\det\bar{A}(z) = \{\det A(\frac{1}{z^*})\}^*, \tag{7.62}$$

which is proved in Appendix F. This relation restricts the numbers and the positions of the poles of $1/\det A(z)$ and $1/\det\bar{A}(z)$, i.e.

$$\bar{N} = N, \quad \bar{z}_k = \frac{1}{z_k^*}. \tag{7.63}$$

Due to eq. (7.42c), we have the second additional relation,

$$\bar{B}(z)\bar{A}(z)^{-1} = \{B(\frac{1}{z^*})A(\frac{1}{z^*})^{-1}\}^\dagger,$$

which leads to

$$\bar{B}(z)\bar{A}(z)^{-1} = \{B(z)A(z)^{-1}\}^\dagger \text{ (on } |z| = 1), \tag{7.64a}$$

$$\bar{C}_k = -\frac{1}{z_k^{*\,2}} C_k^\dagger. \tag{7.64b}$$

The relations (7.63) and (7.64) give a relation between $\bar{F}_R(n,t)$ and $F_R(n,t)$,

$$\bar{F}_R(n,t) = F_R(n,t)^\dagger. \tag{7.65}$$

So as to make the ISM applicable to the sd-cmKdV equations, we have to take account of the internal symmetries of Q_n and R_n defined by eq. (7.25). Considering the scattering problem (7.30) with the potentials $Q_n^{(m)}$ and $R_n^{(m)}$ for $m \geq 2$, we can show the following properties of the scattering data.

Proposition 3. (1) The determinant of $A(z)$ satisfies

$$\det A(z) = \{\det A(z^*)\}^*,$$

as a function of z. Thus the poles of $1/\det A(z)$ outside the unit circle appear as pairs which are situated symmetric with respect to the real axis. Therefore, we replace N in Section 7.3.1 with $2N$ and set the values of $2N$ poles in the right portion outside the unit circle C as

$$\begin{aligned} z_{2j-1} &= \xi_j + \mathrm{i}\eta_j = a_j \mathrm{e}^{\mathrm{i}\theta_j}, \\ z_{2j} &= z_{2j-1}^* = \xi_j - \mathrm{i}\eta_j = a_j \mathrm{e}^{-\mathrm{i}\theta_j}, \end{aligned} \qquad j = 1, 2, \ldots, N, \tag{7.66}$$

where

$$a_j > 1, \quad 0 < \theta_j \leq \frac{\pi}{2},$$

for $\theta_j \neq 0$. The condition (7.66) should be interpreted as follows. If $\theta_j = 0$, the corresponding pole does not need its counterpart. The values of the remaining $2N$ poles in the left portion outside the unit circle are given by

$$z_{2N+k} = -z_k, \quad k = 1, 2, \ldots, 2N.$$

(2) The reflection coefficient $B(z)A(z)^{-1}$ on $|z|=1$ is expressed as

$$B(z)A(z)^{-1} = r^{(0)}\mathbb{1} + \sum_{k=1}^{2m-1} r^{(k)} e_k.$$

Here $r^{(0)}$ and $r^{(k)}$ are complex functions of z and t which satisfy

$$r^{(0)}(z^*) = r^{(0)}(z)^*, \quad r^{(k)}(z^*) = r^{(k)}(z)^*.$$

(3) The residue matrices $\{C_1, C_2, \cdots, C_{2N-1}, C_{2N}\}$ are expressed as

$$\left.\begin{aligned} C_{2j-1} &= c_j^{(0)}\mathbb{1} + \sum_{k=1}^{2m-1} c_j^{(k)} e_k, \\ C_{2j} &= c_j^{(0)*}\mathbb{1} + \sum_{k=1}^{2m-1} c_j^{(k)*} e_k, \end{aligned}\right. \qquad j = 1, 2, \ldots, N,$$

where $c_j^{(0)}$ and $c_j^{(k)}$ are complex functions of t.

A proof of the statements is given in Appendix G.

Taking account of the above conditions, we obtain explicit expressions of $F_R(n,t)$ and $\bar{F}_R(n,t)$ for odd n,

$$\begin{aligned} F_R(n,t) &= \frac{1}{2\pi\mathrm{i}} \int_{C_R} B(z)A(z)^{-1} z^{-n-1} \mathrm{d}z + \sum_{j=1}^{2N} C_j z_j^{-n-1} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{C_{UR}} \Big\{ (r^{(0)} z^{-n-1} + r^{(0)*} z^{n-1}) \mathbb{1} + \sum_{k=1}^{2m-1} (r^{(k)} z^{-n-1} + r^{(k)*} z^{n-1}) e_k \Big\} \mathrm{d}z \\ &\quad + \sum_{j=1}^{N} \Big\{ (c_j^{(0)} z_j^{-n-1} + c_j^{(0)*} z_j^{*\,-n-1}) \mathbb{1} + \sum_{k=1}^{2m-1} (c_j^{(k)} z_j^{-n-1} + c_j^{(k)*} z_j^{*\,-n-1}) e_k \Big\}, \end{aligned} \tag{7.67}$$

$$\begin{aligned} \bar{F}_R(n,t) &= F_R(n,t)^\dagger \\ &= \frac{1}{2\pi\mathrm{i}} \int_{C_{UR}} \Big\{ (r^{(0)} z^{-n-1} + r^{(0)*} z^{n-1}) \mathbb{1} - \sum_{k=1}^{2m-1} (r^{(k)} z^{-n-1} + r^{(k)*} z^{n-1}) e_k \Big\} \mathrm{d}z \\ &\quad + \sum_{j=1}^{N} \Big\{ (c_j^{(0)} z_j^{-n-1} + c_j^{(0)*} z_j^{*\,-n-1}) \mathbb{1} - \sum_{k=1}^{2m-1} (c_j^{(k)} z_j^{-n-1} + c_j^{(k)*} z_j^{*\,-n-1}) e_k \Big\}, \end{aligned} \tag{7.68}$$

where C_{UR} denotes the quadrant (upper-right portion) of the unit circle contour C. We see that the coefficients of $\mathbb{1}$ and $\{e_k\}$ in eqs. (7.67) and (7.68) are real. Thus, a pair of $\bar{F}_R$ and $-F_R$ is expressed in the same form as eq. (7.25), as is expected from the viewpoint of successive approximations for the Gel'fand-Levitan-Marchenko equations (see Appendix H).

Because $B(z)A(z)^{-1}$ and C_j depend on t as eqs. (7.61b) and (7.61c), the time dependences of $r^{(0)}$, $r^{(k)}$ and $c_j^{(0)}$, $c_j^{(k)}$ are given by

$$r^{(0)}(z,t) = r^{(0)}(z,0)\mathrm{e}^{-\left(z^2 - \frac{1}{z^2}\right)t}, \qquad r^{(k)}(z,t) = r^{(k)}(z,0)\mathrm{e}^{-\left(z^2 - \frac{1}{z^2}\right)t},$$

$$c_j^{(0)}(t) = c_j^{(0)}(0)\mathrm{e}^{-\left(z_j^2 - \frac{1}{z_j^2}\right)t}, \quad c_j^{(k)}(t) = c_j^{(k)}(0)\mathrm{e}^{-\left(z_j^2 - \frac{1}{z_j^2}\right)t}.$$

It should be noted that $F_R(n,t)$ and $\bar{F}_R(n,t)$ for odd n are expressed as

$$F_R(n,t) = f^{(0)}(n,t)\mathbb{1} + \sum_{k=1}^{2m-1} f^{(k)}(n,t)e_k,$$

$$\bar{F}_R(n,t) = f^{(0)}(n,t)\mathbb{1} - \sum_{k=1}^{2m-1} f^{(k)}(n,t)e_k,$$

where the real functions $f^{(0)}(n,t)$ and $f^{(k)}(n,t)$ satisfy the linearized dispersion relation,

$$\partial_t f^{(0)}(2n+1,t) - f^{(0)}(2n+3,t) + f^{(0)}(2n-1,t) = 0,$$

$$\partial_t f^{(k)}(2n+1,t) - f^{(k)}(2n+3,t) + f^{(k)}(2n-1,t) = 0.$$

Combining eqs. (7.58) and (7.59) with the relation (7.65), we arrive at

$$\begin{aligned}&\kappa_{1R}(n,m;t)\\ &= 2F_R(n+m,t)^\dagger - 4\sum_{\substack{n_1=n+2\\ n_1-n=\text{even}}}^{\infty}\ \sum_{\substack{n_2=n+1\\ n_2-n=\text{odd}}}^{\infty} \kappa_{1R}(n,n_2;t)F_R(n_2+n_1,t)F_R(n_1+m,t)^\dagger,\end{aligned} \tag{7.69}$$

$$\begin{aligned}&\bar{\kappa}_{2R}(n,m;t)\\ &= -2F_R(n+m,t) - 4\sum_{\substack{n_1=n+2\\ n_1-n=\text{even}}}^{\infty}\ \sum_{\substack{n_2=n+1\\ n_2-n=\text{odd}}}^{\infty} \bar{\kappa}_{2R}(n,n_2;t)F_R(n_2+n_1,t)^\dagger F_R(n_1+m,t),\end{aligned} \tag{7.70}$$

for $m > n$, $m - n = \text{odd}$, where $F_R(n,t)$ for odd n is given by eq. (7.67).

Now the initial-value problem of the sd-cmKdV equations can be solved in the same steps as in section 2.6. The same procedure is applicable to solve the initial-value problem of the higher flows of the sd-cmKdV hierarchy. For instance, we can consider the Lax equation (2.27) with the L_n-matrix (7.30) whose time derivative is replaced with the second flow of the sd-cmKdV hierarchy. As a solution of such a Lax equation, we get an M_n-matrix which contains from z^4 to $1/z^4$ terms. Correspondingly, the time dependences of the scattering data (7.61b)–(7.61c) should be replaced by, for instance,

$$B(z,t)A(z,t)^{-1} = B(z,0)A(z,0)^{-1}\mathrm{e}^{-\left(z^4 - \frac{1}{z^4}\right)t},$$

$$C_j(t) = C_j(0)\mathrm{e}^{-\left(z_j^4 - \frac{1}{z_j^4}\right)t}.$$

Employing the above time dependences, we can solve the initial-value problem of the second flow of the sd-cmKdV hierarchy:

$$\begin{aligned}\frac{\partial v_n^{(i)}}{\partial t} &= \Big(1 + \sum_{j=0}^{M-1} v_n^{(j)\,2}\Big)\Big\{\Big(1 + \sum_{k=0}^{M-1} v_{n+1}^{(k)\,2}\Big)v_{n+2}^{(i)} - \Big(1 + \sum_{k=0}^{M-1} v_{n-1}^{(k)\,2}\Big)v_{n-2}^{(i)}\\ &\quad - \Big(\sum_{k=0}^{M-1} v_{n+1}^{(k)\,2} - \sum_{k=0}^{M-1} v_{n-1}^{(k)\,2}\Big)v_n^{(i)}\\ &\quad + 2\Big(\sum_{k=0}^{M-1} v_{n+1}^{(k)}v_n^{(k)} + \sum_{k=0}^{M-1} v_n^{(k)}v_{n-1}^{(k)}\Big)\big(v_{n+1}^{(i)} - v_{n-1}^{(i)}\big)\Big\}, \quad i = 0,1,\ldots,M-1.\end{aligned}$$

7.3.4 Soliton solutions

To construct soliton solutions of the sd-cmKdV equations, we assume the reflection-free condition, i.e. $B(z) = \bar{B}(z) = O$ on $|z| = 1$. Then, $F_R(n,t)$ and $\bar{F}_R(n,t)$ for odd n are given by

$$F_R(n,t) = \sum_{j=1}^{2N} C_j(t) z_j^{-n-1}, \quad C_j(t) = C_j(0) \mathrm{e}^{-\left(z_j^2 - \frac{1}{z_j^2}\right)t}, \tag{7.71}$$

$$\bar{F}_R(n,t) = -\sum_{k=1}^{2N} \bar{C}_k(t) \bar{z}_k^{n-1}, \quad \bar{C}_k(t) = \bar{C}_k(0) \mathrm{e}^{\left(\bar{z}_k^2 - \frac{1}{\bar{z}_k^2}\right)t}. \tag{7.72}$$

To solve eq. (7.58) (or eq. (7.69)) with eqs. (7.71) and (7.72), we set

$$\kappa_{1R}(n,m) = \sum_{k=1}^{2N} P_k(n) \bar{C}_k(t) \bar{z}_k^{n+m-1} \quad (m-n = \text{odd}). \tag{7.73}$$

Substituting eq. (7.73) into eq. (7.58), we have

$$P_k(n) - 4\sum_{l=1}^{2N}\sum_{j=1}^{2N} \left(\frac{\bar{z}_l}{z_j}\right)^{2n} \frac{\bar{z}_k^2}{(z_j^2 - \bar{z}_k^2)(z_j^2 - \bar{z}_l^2)} P_l(n) \bar{C}_l(t) C_j(t) = -2I. \tag{7.74}$$

In terms of a matrix S whose elements are defined by

$$\begin{aligned} S_{lk} &\equiv \delta_{lk} I - 4\sum_{j=1}^{2N} \left(\frac{\bar{z}_l}{z_j}\right)^{2n} \frac{\bar{z}_k^2}{(z_j^2 - \bar{z}_k^2)(z_j^2 - \bar{z}_l^2)} \bar{C}_l(t) C_j(t) \\ &= \delta_{lk} I + 4\sum_{j=1}^{2N} \frac{1}{z_j^{2n} z_l^{*\,2n} (z_j^2 z_k^{*\,2} - 1)(z_j^2 z_l^{*\,2} - 1)} C_l(t)^\dagger C_j(t), \quad 1 \le l,k \le 2N, \end{aligned}$$

eq. (7.74) is expressed by

$$\left(P_1 \; P_2 \; \cdots \; P_{2N} \right) \begin{pmatrix} S_{11} & \cdots & S_{1\,2N} \\ \vdots & \ddots & \vdots \\ S_{2N\,1} & \cdots & S_{2N\,2N} \end{pmatrix} = -2(\underbrace{I \; I \; \cdots \; I}_{2N}). \tag{7.75}$$

Similarly, to solve eq. (7.59) (or eq. (7.70)) with eqs. (7.71) and (7.72), we set

$$\bar{\kappa}_{2R}(n,m) = \sum_{j=1}^{2N} \bar{P}_j(n) C_j(t) z_j^{-(n+m)-1} \quad (m-n = \text{odd}). \tag{7.76}$$

Substituting eq. (7.76) into eq. (7.59), we have

$$\bar{P}_j(n) - 4\sum_{l=1}^{2N}\sum_{k=1}^{2N} \frac{\bar{z}_k^{2n+2}}{z_l^{2n}} \frac{1}{(z_l^2 - \bar{z}_k^2)(z_j^2 - \bar{z}_k^2)} \bar{P}_l(n) C_l(t) \bar{C}_k(t) = -2I. \tag{7.77}$$

Using a matrix $\bar{S}$,

$$\begin{aligned} \bar{S}_{lk} &\equiv \delta_{lk} I - 4\sum_{j=1}^{2N} \frac{\bar{z}_j^{2n+2}}{z_l^{2n}} \frac{1}{(z_l^2 - \bar{z}_j^2)(z_k^2 - \bar{z}_j^2)} C_l(t) \bar{C}_j(t) \\ &= \delta_{lk} I + 4\sum_{j=1}^{2N} \frac{1}{z_l^{2n} z_j^{*\,2n} (z_l^2 z_j^{*\,2} - 1)(z_k^2 z_j^{*\,2} - 1)} C_l(t) C_j(t)^\dagger, \quad 1 \le l,k \le 2N, \end{aligned}$$

we rewrite eq. (7.77) as

$$(\bar{P}_1 \; \bar{P}_2 \; \cdots \; \bar{P}_{2N}) \begin{pmatrix} \bar{S}_{11} & \cdots & \bar{S}_{1\,2N} \\ \vdots & \ddots & \vdots \\ \bar{S}_{2N\,1} & \cdots & \bar{S}_{2N\,2N} \end{pmatrix} = -2(\underbrace{I \; I \; \cdots \; I}_{2N}). \tag{7.78}$$

Equations (7.75) and (7.78) are readily solved. Thus the N-soliton solution of the sd-cmKdV equations (7.28) is given by

$$\begin{aligned} Q_n^{(m)}(t) &= -\kappa_{1R}(n, n+1; t) = -\sum_{k=1}^{2N} P_k(n)\bar{C}_k(t)\frac{1}{z_k^{*\,2n}} \\ &= -2\,(\underbrace{I \; I \; \cdots \; I}_{2N})\; S^{-1} \begin{pmatrix} C_1(t)^\dagger \dfrac{1}{z_1^{*\,2n+2}} \\ C_2(t)^\dagger \dfrac{1}{z_2^{*\,2n+2}} \\ \vdots \\ C_{2N}(t)^\dagger \dfrac{1}{z_{2N}^{*\,2n+2}} \end{pmatrix}, \end{aligned} \tag{7.79a}$$

$$\begin{aligned} R_n^{(m)}(t) &= -\bar{\kappa}_{2R}(n, n+1; t) = -\sum_{j=1}^{2N} \bar{P}_j(n)C_j(t)\frac{1}{z_j^{2n+2}} \\ &= 2\,(\underbrace{I \; I \; \cdots \; I}_{2N})\; \bar{S}^{-1} \begin{pmatrix} C_1(t) \dfrac{1}{z_1^{2n+2}} \\ C_2(t) \dfrac{1}{z_2^{2n+2}} \\ \vdots \\ C_{2N}(t) \dfrac{1}{z_{2N}^{2n+2}} \end{pmatrix}. \end{aligned} \tag{7.79b}$$

Strictly speaking, eq. (7.79) includes solitons, breathers, and their coexistence. To extract soliton solutions, we should impose appropriate conditions on the residue matrices $\{C_j\}$, $\{\bar{C}_k\}$. As a criterion of soliton, we employ the condition that each solitary wave observed in the first conserved density,

$$\log\Big\{1 + \sum_{j=0}^{2m-1} (v_n^{(j)}(t))^2\Big\}, \tag{7.80}$$

has a time-independent shape. Calculating an asymptotic form of eq. (7.80) at $n \to +\infty$, we find that the criterion gives necessary conditions

$$C_{2j-1}\bar{C}_{2j} = \bar{C}_{2j}C_{2j-1} = C_{2j}\bar{C}_{2j-1} = \bar{C}_{2j-1}C_{2j} = O, \quad j = 1, 2, \ldots, N. \tag{7.81}$$

Equation (7.81) is simplified to

$$\sum_{i=0}^{2m-1} (c_j^{(i)})^2 = 0, \quad j = 1, 2, \ldots, N. \tag{7.82}$$

To check the sufficiency of the conditions (7.81) or (7.82), we consider the $N = 1$ case. For this purpose, we set

$$
\begin{aligned}
\bar{C}_1 &= -\frac{1}{z_1^{*\,2}} C_1^\dagger = -\frac{1}{z_1^{*\,2}}\Big(c_1^{(0)\,*}\mathbb{I} - \sum_{k=1}^{2m-1} c_1^{(k)\,*} e_k\Big) \equiv \bar{c}_1^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} \bar{c}_1^{(k)} e_k, \\
\bar{C}_2 &= \bar{c}_1^{(0)\,*}\mathbb{I} + \sum_{k=1}^{2m-1} \bar{c}_1^{(k)\,*} e_k, \\
\bar{z}_1 &= \frac{1}{z_1^*} = \mathrm{e}^{-W+\mathrm{i}\theta}, \quad W > 0, \\
\mathrm{e}^{\phi_0} &= \frac{\sinh 2W}{\sqrt{2\sum_{j=0}^{2m-1} |\bar{c}_1^{(j)}(0)|^2}}.
\end{aligned}
$$

Here eq. (7.82) for $N = 1$ gives

$$
\sum_{j=0}^{2m-1} (\bar{c}_1^{(j)}(0))^2 = 0. \tag{7.83}
$$

With the help of the conditions (7.81), eq. (7.79) for $N = 1$ yields

$$
\begin{aligned}
Q_n^{(m)}(t) &= \operatorname{sech}\{2nW + 2(\sinh 2W \cos 2\theta)t + \phi_0\} \frac{\sinh 2W}{\sqrt{2\sum_{j=0}^{2m-1} |\bar{c}_1^{(j)}(0)|^2}} \\
&\quad \cdot \{\bar{C}_1(0)\mathrm{e}^{2\mathrm{i}\{n\theta + (\cosh 2W \sin 2\theta)t\}} + \bar{C}_2(0)\mathrm{e}^{-2\mathrm{i}\{n\theta + (\cosh 2W \sin 2\theta)t\}}\}, \qquad (7.84\mathrm{a}) \\
R_n^{(m)}(t) &= -Q_n^{(m)}(t)^\dagger. \qquad (7.84\mathrm{b})
\end{aligned}
$$

It is straightforward to show that eq. (7.84) can be expressed as eq. (7.25) with real coefficients $v_n^{(i)}(t)$ of $\mathbb{I}$ and $\{e_k\}$. Thus we have checked in terms of the inverse problem of scattering that the conditions (7.25) are satisfied under Proposition 3 and the conditions (7.81), in the case of the one-soliton solution.

Equation (7.84) can be expressed explicitly for each component as

$$
\begin{aligned}
v_n^{(i)}(t) &= \operatorname{sech}\{2nW + 2(\sinh 2W \cos 2\theta)t + \phi_0\} \frac{\sinh 2W}{\sqrt{2\sum_{j=0}^{2m-1} |\bar{c}_1^{(j)}(0)|^2}} \\
&\quad \cdot \{\bar{c}_1^{(i)}(0)\mathrm{e}^{2\mathrm{i}\{n\theta + (\cosh 2W \sin 2\theta)t\}} + \bar{c}_1^{(i)\,*}(0)\mathrm{e}^{-2\mathrm{i}\{n\theta + (\cosh 2W \sin 2\theta)t\}}\}, \\
&\qquad i = 0, 1, \ldots, 2m-1. \qquad (7.85)
\end{aligned}
$$

The soliton solution (7.85) exhibits an interesting property. Because there are two carrier waves in one envelope soliton, the shape of soliton observed in $(v_n^{(i)}(t))^2$ periodically oscillates in time. It is observed for (7.85) that the summation of $(v_n^{(i)}(t))^2$ with respect to components, $i(= 0, 1, \ldots, 2m-1)$

$$
\sum_{i=0}^{2m-1} (v_n^{(i)}(t))^2 = \sinh^2 2W \operatorname{sech}^2\{2nW + 2(\sinh 2W \cos 2\theta)t + \phi_0\},
$$

has a time-independent shape, as is expected. Thus, eq. (7.85) with eq. (7.83) is interpreted as the one-soliton solution of the sd-cmKdV equations (7.28) with $M = 2m$. The structure of the one-soliton solution is essentially the same as that for the continuous cmKdV equations (3.57) (see Section 3.3.2). This result shows that the conditions (7.81) are necessary and sufficient for eq. (7.79) to be the pure N-soliton solution.

A comment is in order. Ohta [70, 71] obtained an N-soliton solution for sd-cmKdV equations. We conjecture that our result coincides with Ohta's result by a particular choice of those parameters.

It is not evident whether eqs. (7.79a) and (7.79b) can be expressed as

$$Q_n^{(m)}(t) = v_n^{(0)}(t)\mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)}(t)e_k, \tag{7.86a}$$

$$R_n^{(m)}(t) = -v_n^{(0)}(t)\mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)}(t)e_k, \tag{7.86b}$$

without using the products of e_i such as e_ie_j, $e_ie_je_k$. Further, $v_n^{(0)}$, $v_n^{(k)}$ should be real in eqs. (7.86a) and (7.86b). Noting the fact that summations and products of real quaternions are real quaternions, either eq. (7.86a) or eq. (7.86b) can be proved for $m = 2$ (four-component sd-cmKdV equations) by using the Neumann-Liouville expansion (see Appendix H). It is left for a future problem to prove both eqs. (7.86a) and (7.86b) for $M = 2m \geq 4$ rigorously.

7.4 Full-Discretization

We can obtain an integrable full-discretization of the matrix AKNS formulation and, as a reduction, an integrable full-discretization of the cmKdV equations by considering a time discretization of the sd-cmKdV equations. For this purpose, we choose the form of the Lax pair in the full-discrete case, L_n and V_n, as

$$L_n = \begin{bmatrix} zI_1 & Q_n \\ R_n & \frac{1}{z}I_2 \end{bmatrix}, \tag{7.87}$$

$$\begin{aligned} V_n = {} & \begin{bmatrix} I_1 & \\ & I_2 \end{bmatrix} + \delta t \Big\{ z^2 \begin{bmatrix} cI_1 & \\ & D_n \end{bmatrix} + z \begin{bmatrix} O & -\tilde{Q}_n D_n + cQ_n \\ c\tilde{R}_{n-1} - D_n R_{n-1} & O \end{bmatrix} \\ & + \begin{bmatrix} \tilde{Q}_n D_n R_{n-1} + X_n & \\ & \tilde{R}_n A_n Q_{n-1} + Y_n \end{bmatrix} \\ & + \frac{1}{z} \begin{bmatrix} O & -A_n Q_{n-1} - d\tilde{Q}_{n-1} \\ -\tilde{R}_n A_n - dR_n & O \end{bmatrix} + \frac{1}{z^2} \begin{bmatrix} A_n & \\ & -dI_2 \end{bmatrix} \Big\}. \end{aligned} \tag{7.88}$$

Here a and d are constants. A_n and X_n are $p \times p$ matrices. D_n and Y_n are $q \times q$ matrices. As was mentioned in Section 2.7, the tilde ˜ denotes the time shift in discrete time $l \in \mathbb{Z}$. δt is a difference interval of time. It is remarkable that we do not change the form of L_n-matrix from the semi-discrete case (see eq. (7.1)). Substitution of eqs. (7.87) and (7.88) into eq. (2.33) gives

$$\frac{1}{\delta t}(\tilde{Q}_n - Q_n) - cQ_{n+1} - d\tilde{Q}_{n-1} - X_{n+1}Q_n + \tilde{Q}_n Y_n - (I_1 - \tilde{Q}_n \tilde{R}_n) A_n Q_{n-1}$$

$$+ \tilde{Q}_{n+1} D_{n+1} (I_2 - R_n Q_n) = O, \tag{7.89a}$$

$$\frac{1}{\delta t}(\tilde{R}_n - R_n) + dR_{n+1} + c\tilde{R}_{n-1} - Y_{n+1} R_n + \tilde{R}_n X_n - (I_2 - \tilde{R}_n \tilde{Q}_n) D_n R_{n-1}$$
$$+ \tilde{R}_{n+1} A_{n+1} (I_1 - Q_n R_n) = O, \tag{7.89b}$$

$$X_{n+1} - X_n = c(\tilde{Q}_n \tilde{R}_{n-1} - Q_{n+1} R_n), \tag{7.89c}$$

$$Y_{n+1} - Y_n = d(R_{n+1} Q_n - \tilde{R}_n \tilde{Q}_{n-1}), \tag{7.89d}$$

$$(I_1 - \tilde{Q}_n \tilde{R}_n) A_n = A_{n+1} (I_1 - Q_n R_n). \tag{7.89e}$$

$$(I_2 - \tilde{R}_n \tilde{Q}_n) D_n = D_{n+1} (I_2 - R_n Q_n), \tag{7.89f}$$

The above set of matrix equations are interpreted as an integrable time discretization of eq. (7.3). To obtain full-discrete (fd-) cmKdV equations, we choose

$$c = d = 0, \quad X_n = O, \quad Y_n = O.$$

Further we assume that Q_n and R_n are square matrices expressed as eq. (7.25) and

$$A_n = D_n = -\Gamma_n I,$$

where Γ_n is a scalar variable. Then eq. (7.89) reduces to fd-cmKdV equations,

$$\begin{aligned} \frac{1}{\delta t}(\tilde{v}_n^{(j)} - v_n^{(j)}) &= \Big(1 + \sum_{k=0}^{2m-1} v_n^{(k)\,2}\Big) \Gamma_{n+1} (\tilde{v}_{n+1}^{(j)} - v_{n-1}^{(j)}), \quad j = 0, 1, \dots, 2m-1. \\ \Gamma_{n+1}\Big(1 + \sum_{k=0}^{2m-1} v_n^{(k)\,2}\Big) &= \Gamma_n \Big(1 + \sum_{k=0}^{2m-1} \tilde{v}_n^{(k)\,2}\Big), \end{aligned} \tag{7.90}$$

Since the spatial part of the Lax formulation for the fd-cmKdV equations is common with that for the sd-cmKdV equations, we can apply the ISM to the fd-cmKdV equations in the same way as in Section 7.3. The only essential difference lies in time dependences of the scattering data. Assuming the boundary conditions,

$$v_n^{(j)} \to 0 \quad \text{as} \quad n \to \pm\infty, \qquad j = 1, 2, \dots, m,$$
$$\Gamma_n \to 1 \quad \text{as} \quad n \to \pm\infty,$$

we obtain the time dependences of the scattering data in the full-discrete case:

$$A(z, l) = A(z, 0),$$

$$B(z, l) A(z, l)^{-1} = B(z, 0) A(z, 0)^{-1} \left(\frac{1 - \delta t\, z^2}{1 - \delta t / z^2} \right)^l,$$

$$C_j(l) = C_j(0) \left(\frac{1 - \delta t\, z_j^2}{1 - \delta t / z_j^2} \right)^l.$$

By use of the above time dependence, the one-soliton solution of eq. (7.90) is computed as

$$v_n^{(i)}(l) = \operatorname{sech}\{2nW + 2al + \phi_0\} \frac{\sinh 2W}{\sqrt{2 \sum_{j=0}^{2m-1} |\bar{c}_1^{(j)}(0)|^2}} \cdot \{\bar{c}_1^{(i)}(0) \mathrm{e}^{2\mathrm{i}\{n\theta + bl\}} + \bar{c}_1^{(i)\,*}(0) \mathrm{e}^{-2\mathrm{i}\{n\theta + bl\}}\},$$
$$i = 0, 1, \dots, 2m-1,$$

where $W > 0$,

$$\sum_{j=0}^{2m-1} (\bar{c}_1^{(j)})^2 = 0,$$

$$\begin{aligned}
\mathrm{e}^{\phi_0} &= \frac{\sinh 2W}{\sqrt{\sum_{j=1}^{m} |\bar{c}_1^{(j)}(0)|^2}}, \\
\mathrm{e}^{4a} &= \frac{(1-\delta t\,\mathrm{e}^{-2W-2\mathrm{i}\theta})(1-\delta t\,\mathrm{e}^{-2W+2\mathrm{i}\theta})}{(1-\delta t\,\mathrm{e}^{2W+2\mathrm{i}\theta})(1-\delta t\,\mathrm{e}^{2W-2\mathrm{i}\theta})}, \\
\mathrm{e}^{4\mathrm{i}b} &= \frac{(1-\delta t\,\mathrm{e}^{2W-2\mathrm{i}\theta})(1-\delta t\,\mathrm{e}^{-2W-2\mathrm{i}\theta})}{(1-\delta t\,\mathrm{e}^{2W+2\mathrm{i}\theta})(1-\delta t\,\mathrm{e}^{-2W+2\mathrm{i}\theta})}.
\end{aligned}$$

In the continuum limit of time ($\delta t \to 0$), the one-soliton solution reduces to that of the sd-cmKdV equations due to the following relations,

$$\begin{aligned}
a &= \delta t \sinh 2W \cos 2\theta + O(\delta t^2), \\
b &= \delta t \cosh 2W \sin 2\theta + O(\delta t^2).
\end{aligned}$$

The fd-cmKdV equations possesses an infinite series of conserved quantities in common with the sd-cmKdV equations (see eqs. (7.29a)–(7.29d)).

7.5 Discrete Coupled Hirota Equations

We can transform the sd-cmKdV equations and the fd-cmKdV equations into other discrete integrable systems of physical interest. For instance, if we take new dependent variables as

$$v_{2n}^{(i)} = V_n^{(i)}, \qquad v_{2n-1}^{(i)} = -I_n^{(i)},$$

the sd-cmKdV equations (7.24) are cast into a new coupled version of the self-dual network equations [5, 101],

$$\begin{aligned}
\frac{1}{1-\sum_{j=0}^{M-1} \varepsilon_j V_n^{(j)\,2}} \frac{\partial V_n^{(i)}}{\partial t} &= I_n^{(i)} - I_{n+1}^{(i)}, \\
\frac{1}{1-\sum_{j=0}^{M-1} \varepsilon_j I_n^{(j)\,2}} \frac{\partial I_n^{(i)}}{\partial t} &= V_{n-1}^{(i)} - V_n^{(i)},
\end{aligned}$$

with $\varepsilon_j = \pm 1$. Setting one variable, *e.g.* $v_n^{(0)}$, to be a constant in eq. (7.24) and shifting the variables $v_n^{(i)}$ by $v_n^{(i)} \to a_i v_n^{(i)} + b_i$, we obtain a coupled version of the generalized Volterra model [101]:

$$\frac{\partial v_n^{(i)}}{\partial t} = \Big(\alpha + \sum_j \beta_j v_n^{(j)} + \sum_j \gamma_j v_n^{(j)\,2}\Big)(v_{n+1}^{(i)} - v_{n-1}^{(i)}).$$

In what follows we consider another interesting transformation for the sd-cmKdV equations and the fd-cmKdV equations.

7.5.1 Semi-discrete coupled Hirota equations

Let us consider a transformation for the following form of sd-cmKdV equations,

$$\frac{\partial v_n^{(i)}}{\partial t} = a\Big(1 + \sum_{j=0}^{M-1} v_n^{(j)\,2}\Big)\big(v_{n+1}^{(i)} - v_{n-1}^{(i)}\big), \qquad i = 0, 1, \ldots, 2m-1. \tag{7.91}$$

In an analogous way to the continuous case (see Section 3.3.3), we introduce a transformation of dependent variables to obtain sd-cHirota equations,

$$\begin{aligned} v_n^{(2j-2)} + \mathrm{i} v_n^{(2j-1)} &= \mathrm{e}^{2\mathrm{i}(\theta n + a \sin 2\theta \cdot t)} q_n^{(j)}, \\ -v_n^{(2j-2)} + \mathrm{i} v_n^{(2j-1)} &= \mathrm{e}^{-2\mathrm{i}(\theta n + a \sin 2\theta \cdot t)} r_n^{(j)}, \end{aligned} \qquad j = 1, 2, \ldots, m. \tag{7.92}$$

Substitution of eq. (7.92) into eq. (7.91) yields the sd-cHirota equations in a general form,

$$\begin{aligned} & \mathrm{i} q_{n,t}^{(j)} - \mathrm{i} a \cos 2\theta \Big(1 - \sum_{k=1}^{m} q_n^{(k)} r_n^{(k)}\Big)\big(q_{n+1}^{(j)} - q_{n-1}^{(j)}\big) \\ & \quad + a \sin 2\theta \Big\{\big(q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}\big) - \sum_{k=1}^{m} q_n^{(k)} r_n^{(k)} \big(q_{n+1}^{(j)} + q_{n-1}^{(j)}\big)\Big\} = 0, \\ & \mathrm{i} r_{n,t}^{(j)} - \mathrm{i} a \cos 2\theta \Big(1 - \sum_{k=1}^{m} r_n^{(k)} q_n^{(k)}\Big)\big(r_{n+1}^{(j)} - r_{n-1}^{(j)}\big) \\ & \quad - a \sin 2\theta \Big\{\big(r_{n+1}^{(j)} + r_{n-1}^{(j)} - 2r_n^{(j)}\big) - \sum_{k=1}^{m} r_n^{(k)} q_n^{(k)} \big(r_{n+1}^{(j)} + r_{n-1}^{(j)}\big)\Big\} = 0, \\ & \qquad\qquad j = 1, 2, \ldots, m. \end{aligned} \tag{7.93}$$

Assuming the reduction

$$r_n^{(j)} = -\sigma_j q_n^{(j)*}, \qquad \sigma_j = \pm 1,$$

we obtain the sd-cHirota equations,

$$\begin{aligned} & \mathrm{i} q_{n,t}^{(j)} - \mathrm{i} a \cos 2\theta \Big(1 + \sum_{k=1}^{m} \sigma_k |q_n^{(k)}|^2\Big)\big(q_{n+1}^{(j)} - q_{n-1}^{(j)}\big) \\ & + a \sin 2\theta \Big\{\big(q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}\big) + \sum_{k=1}^{m} \sigma_k |q_n^{(k)}|^2 \big(q_{n+1}^{(j)} + q_{n-1}^{(j)}\big)\Big\} = 0, \\ & \qquad\qquad j = 1, 2, \ldots, m. \end{aligned} \tag{7.94}$$

Pulling back the transformation (7.92) to the level of the Lax representation, we obtain an explicit Lax pair for the sd-cHirota equations (7.93). The obtained Lax formulation is summarized as follows.

We introduce the following form of the Lax pair,

$$L_n = z \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta H_1} & \\ & O \end{bmatrix} + \begin{bmatrix} O & \mathrm{e}^{\mathrm{i}\theta H_1} \mathcal{Q}_n \\ \mathrm{e}^{\mathrm{i}\theta H_2} \mathcal{R}_n & O \end{bmatrix} + \frac{1}{z} \begin{bmatrix} O & \\ & \mathrm{e}^{\mathrm{i}\theta H_2} \end{bmatrix}, \tag{7.95}$$

$$\begin{aligned} M_n = a\Big\{ & z^2 \begin{bmatrix} I & \\ & O \end{bmatrix} + z \begin{bmatrix} O & \mathcal{Q}_n \\ \mathrm{e}^{\mathrm{i}\theta H_2} \mathcal{R}_{n-1} \mathrm{e}^{-\mathrm{i}\theta H_1} & O \end{bmatrix} \\ & + \begin{bmatrix} -\mathcal{Q}_n \mathrm{e}^{\mathrm{i}\theta H_2} \mathcal{R}_{n-1} \mathrm{e}^{-\mathrm{i}\theta H_1} & \\ O & -\mathcal{R}_n \mathrm{e}^{\mathrm{i}\theta H_1} \mathcal{Q}_{n-1} \mathrm{e}^{-\mathrm{i}\theta H_2} \end{bmatrix} + \begin{bmatrix} \mathrm{i} H_1 \sin 2\theta & O \\ O & \mathrm{i} H_2 \sin 2\theta \end{bmatrix} \\ & + \frac{1}{z} \begin{bmatrix} O & \mathrm{e}^{\mathrm{i}\theta H_1} \mathcal{Q}_{n-1} \mathrm{e}^{-\mathrm{i}\theta H_2} \\ \mathcal{R}_n & O \end{bmatrix} + \frac{1}{z^2} \begin{bmatrix} O & \\ & I \end{bmatrix} \Big\}, \end{aligned} \tag{7.96}$$

where I is the $p \times p$ unit matrix, $\mathcal{Q}_n$ and $\mathcal{R}_n$ are $p \times p$ matrices. The constant matrices H_1 and H_2 are assumed to satisfy

$$\begin{aligned} H_1\mathcal{Q}_n - \mathcal{Q}_n H_2 &= -2F_1\mathcal{Q}_n F_2, \\ H_2\mathcal{R}_n - \mathcal{R}_n H_1 &= 2F_2\mathcal{R}_n F_1. \end{aligned}$$

Here F_1 and F_2 satisfy eq. (3.70). We notice that the following relations hold:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}yH_1}\mathcal{Q}_n\mathrm{e}^{\mathrm{i}yH_2} &= \cos(2y)\mathcal{Q}_n + \mathrm{i}\sin(2y)F_1\mathcal{Q}_n F_2, \\ \mathrm{e}^{\mathrm{i}yH_2}\mathcal{R}_n\mathrm{e}^{-\mathrm{i}yH_1} &= \cos(2y)\mathcal{R}_n + \mathrm{i}\sin(2y)F_2\mathcal{R}_n F_1. \end{aligned}$$

Substituting eqs. (7.95) and (7.96) into eq. (2.27), we get a set of matrix equations

$$\begin{aligned} &\mathrm{i}\mathcal{Q}_{n,t} - \mathrm{i}a\cos 2\theta(\mathcal{Q}_{n+1} - \mathcal{Q}_{n-1} - \mathcal{Q}_{n+1}\mathcal{R}_n\mathcal{Q}_n + \mathcal{Q}_n\mathcal{R}_n\mathcal{Q}_{n-1}) \\ &\quad + a\sin 2\theta\{F_1(\mathcal{Q}_{n+1} + \mathcal{Q}_{n-1} - 2\mathcal{Q}_n)F_2 - F_1\mathcal{Q}_{n+1}F_2\mathcal{R}_n\mathcal{Q}_n - \mathcal{Q}_n\mathcal{R}_n F_1\mathcal{Q}_{n-1}F_2\} = O \end{aligned} \tag{7.97a}$$

$$\begin{aligned} &\mathrm{i}\mathcal{R}_{n,t} - \mathrm{i}a\cos 2\theta(\mathcal{R}_{n+1} - \mathcal{R}_{n-1} - \mathcal{R}_{n+1}\mathcal{Q}_n\mathcal{R}_n + \mathcal{R}_n\mathcal{Q}_n\mathcal{R}_{n-1}) \\ &\quad - a\sin 2\theta\{F_2(\mathcal{R}_{n+1} + \mathcal{R}_{n-1} - 2\mathcal{R}_n)F_1 - F_2\mathcal{R}_{n+1}F_1\mathcal{Q}_n\mathcal{R}_n - \mathcal{R}_n\mathcal{Q}_n F_2\mathcal{R}_{n-1}F_1\} = O. \end{aligned} \tag{7.97b}$$

We recursively define $2^{m-1} \times 2^{m-1}$ matrices $H_1^{(m)}$, $H_2^{(m)}$, $F_1^{(m)}$, $F_2^{(m)}$, $\mathcal{Q}_n^{(m)}$ and $\mathcal{R}_n^{(m)}$ by eqs. (3.72)–(3.74) and

$$\mathcal{Q}_n^{(1)} = q_n^{(1)}, \quad \mathcal{R}_n^{(1)} = r_n^{(1)},$$

$$\mathcal{Q}_n^{(m+1)} = \begin{bmatrix} \mathcal{Q}_n^{(m)} & q_n^{(m+1)}I_{2^{m-1}} \\ r_n^{(m+1)}I_{2^{m-1}} & -\mathcal{R}_n^{(m)} \end{bmatrix}, \quad \mathcal{R}_n^{(m+1)} = \begin{bmatrix} \mathcal{R}_n^{(m)} & q_n^{(m+1)}I_{2^{m-1}} \\ r_n^{(m+1)}I_{2^{m-1}} & -\mathcal{Q}_n^{(m)} \end{bmatrix}.$$

Then putting $\mathcal{Q}_n^{(m)}$, $\mathcal{R}_n^{(m)}$, etc. into $\mathcal{Q}_n$, $\mathcal{R}_n$, etc. in the matrix equations (7.97), we obtain the non-reduced sd-cHirota equations (7.93).

7.5.2 Full-discrete coupled Hirota equations

Let us introduce a transformation of dependent variables,

$$\begin{aligned} v_n^{(2j-2)} + \mathrm{i}v_n^{(2j-1)} &= \mathrm{e}^{2\mathrm{i}(n\alpha+l\beta)}q_n^{(j)}, \\ -v_n^{(2j-2)} + \mathrm{i}v_n^{(2j-1)} &= \mathrm{e}^{-2\mathrm{i}(n\alpha+l\beta)}r_n^{(j)}, \end{aligned} \qquad j = 1, 2, \ldots, m, \tag{7.98}$$

$$\Gamma_n = \frac{\sin\beta}{\delta t\sin(2\alpha+\beta)}j_n. \tag{7.99}$$

Substitution of eqs. (7.98) and (7.99) into the fd-cmKdV equations (7.90) gives

$$\begin{aligned}
&\mathrm{i}\frac{1}{\delta t}(\tilde{q}_n^{(j)}-q_n^{(j)})-\mathrm{i}\frac{\tan\beta}{\delta t\tan(2\alpha+\beta)}j_{n+1}\Big(1-\sum_{k=1}^{m}q_n^{(k)}r_n^{(k)}\Big)(\tilde{q}_{n+1}^{(j)}-q_{n-1}^{(j)})\\
&\qquad+\frac{\tan\beta}{\delta t}\Big\{j_{n+1}\Big(1-\sum_{k=1}^{m}q_n^{(k)}r_n^{(k)}\Big)(\tilde{q}_{n+1}^{(j)}+q_{n-1}^{(j)})-(\tilde{q}_n^{(j)}+q_n^{(j)})\Big\}=0,\\
&\mathrm{i}\frac{1}{\delta t}(\tilde{r}_n^{(j)}-r_n^{(j)})-\mathrm{i}\frac{\tan\beta}{\delta t\tan(2\alpha+\beta)}j_{n+1}\Big(1-\sum_{k=1}^{m}r_n^{(k)}q_n^{(k)}\Big)(\tilde{r}_{n+1}^{(j)}-r_{n-1}^{(j)})\\
&\qquad-\frac{\tan\beta}{\delta t}\Big\{j_{n+1}\Big(1-\sum_{k=1}^{m}r_n^{(k)}q_n^{(k)}\Big)(\tilde{r}_{n+1}^{(j)}+r_{n-1}^{(j)})-(\tilde{r}_n^{(j)}+r_n^{(j)})\Big\}=0,\\
&\qquad\qquad j_{n+1}\Big(1-\sum_{k=1}^{m}q_n^{(k)}r_n^{(k)}\Big)=j_n\Big(1-\sum_{k=1}^{m}\tilde{q}_n^{(k)}\tilde{r}_n^{(k)}\Big).
\end{aligned}$$

Choosing

$$\begin{aligned}
&\frac{\tan\beta}{\delta t\tan(2\alpha+\beta)}=a\cos 2\theta,\qquad \frac{\tan\beta}{\delta t}=a\sin 2\theta,\\
\Longleftrightarrow\quad &\alpha=\theta-\frac{1}{2}\tan^{-1}(a\,\delta t\sin 2\theta),\qquad \beta=\tan^{-1}(a\,\delta t\sin 2\theta),
\end{aligned}$$

we obtain the fd-cHirota equations,

$$\begin{aligned}
&\mathrm{i}\frac{1}{\delta t}(\tilde{q}_n^{(j)}-q_n^{(j)})-\mathrm{i}a\cos 2\theta\cdot j_{n+1}\Big(1+\sum_{k=1}^{m}\sigma_k|q_n^{(k)}|^2\Big)(\tilde{q}_{n+1}^{(j)}-q_{n-1}^{(j)})\\
&+a\sin 2\theta\Big\{j_{n+1}\Big(1+\sum_{k=1}^{m}\sigma_k|q_n^{(k)}|^2\Big)(\tilde{q}_{n+1}^{(j)}+q_{n-1}^{(j)})-(\tilde{q}_n^{(j)}+q_n^{(j)})\Big\}=0,\qquad j=1,2,\ldots,m,\\
&\qquad\qquad j_{n+1}\Big(1+\sum_{k=1}^{m}\sigma_k|q_n^{(k)}|^2\Big)=j_n\Big(1+\sum_{k=1}^{m}\sigma_k|\tilde{q}_n^{(k)}|^2\Big),
\end{aligned}\tag{7.100}$$

where we have assumed $r_n^{(j)}=-\sigma_j q_n^{(j)*}$, $\sigma_j=\pm 1$.

A Lax pair for the fd-cHirota equations is given by

$$L_n=z\begin{bmatrix}\mathrm{e}^{\mathrm{i}\alpha H_1} & \\ & O\end{bmatrix}+\begin{bmatrix}O & \mathrm{e}^{\mathrm{i}\alpha H_1}\mathcal{Q}_n\\ \mathrm{e}^{\mathrm{i}\alpha H_2}\mathcal{R}_n & O\end{bmatrix}+\frac{1}{z}\begin{bmatrix}O & \\ & \mathrm{e}^{\mathrm{i}\alpha H_2}\end{bmatrix},$$

$$\begin{aligned}
V_n=&\begin{bmatrix}\mathrm{e}^{\mathrm{i}\beta H_1} & \\ & \mathrm{e}^{\mathrm{i}\beta H_2}\end{bmatrix}+\frac{\sin\beta}{\sin(2\alpha+\beta)}j_n\Big\{z^2\begin{bmatrix}O & \\ & -\mathrm{e}^{\mathrm{i}\beta H_2}\end{bmatrix}+z\begin{bmatrix}O & \tilde{\mathcal{Q}}_n\mathrm{e}^{\mathrm{i}\beta H_2}\\ \mathrm{e}^{\mathrm{i}(\alpha+\beta)H_2}\mathcal{R}_{n-1}\mathrm{e}^{-\mathrm{i}\alpha H_1} & O\end{bmatrix}\\
&+\begin{bmatrix}-\tilde{\mathcal{Q}}_n\mathrm{e}^{\mathrm{i}(\alpha+\beta)H_2}\mathcal{R}_{n-1}\mathrm{e}^{-\mathrm{i}\alpha H_1} & \\ & -\tilde{\mathcal{R}}_n\mathrm{e}^{\mathrm{i}(\alpha+\beta)H_1}\mathcal{Q}_{n-1}\mathrm{e}^{-\mathrm{i}\alpha H_2}\end{bmatrix}\\
&+\frac{1}{z}\begin{bmatrix}O & \mathrm{e}^{\mathrm{i}(\alpha+\beta)H_1}\mathcal{Q}_{n-1}\mathrm{e}^{-\mathrm{i}\alpha H_2}\\ \tilde{\mathcal{R}}_n\mathrm{e}^{\mathrm{i}\beta H_1} & O\end{bmatrix}+\frac{1}{z^2}\begin{bmatrix}-\mathrm{e}^{\mathrm{i}\beta H_1} & \\ & O\end{bmatrix}\Big\}.
\end{aligned}$$

Here H_1, H_2, $\mathcal{Q}_n$ and $\mathcal{R}_n$ are defined in the same way as in the continuous and the semi-discrete case.

7.5.3 ISM and conserved quantities

As we have explained, the scattering problem both for the sd-cHirota equations (7.94) and the fd-cHirota equations (7.100) is given in terms of a $2^m \times 2^m$ matrix,

$$\begin{bmatrix} \Psi_{1\,n+1} \\ \Psi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} z\mathrm{e}^{\mathrm{i}\alpha H_1} & \mathrm{e}^{\mathrm{i}\alpha H_1}\mathcal{Q}_n \\ \mathrm{e}^{\mathrm{i}\alpha H_2}\mathcal{R}_n & \frac{1}{z}\mathrm{e}^{\mathrm{i}\alpha H_2} \end{bmatrix} \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix}. \tag{7.101}$$

Here

$$\alpha = \begin{cases} \theta & \text{for the semi-discretization} \\ \theta - \frac{1}{2}\tan^{-1}(a\,\delta t \sin 2\theta) & \text{for the full-discretization.} \end{cases} \tag{7.102}$$

To simplify the analysis, we consider a gauge transformation,

$$\Phi_n = g_n \Psi_n.$$

Here g_n is given by

$$g_n = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(n\theta + ta\sin 2\theta)H_1} & \\ & \mathrm{e}^{-\mathrm{i}(n\theta + ta\sin 2\theta)H_2} \end{bmatrix},$$

for the semi-discrete case and

$$g_n = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(n\alpha + l\beta)H_1} & \\ & \mathrm{e}^{-\mathrm{i}(n\alpha + l\beta)H_2} \end{bmatrix},$$

for the full-discrete case (l: discrete time). Then the scattering problem (7.101) is changed into the standard form,

$$\begin{bmatrix} \Phi_{1\,n+1} \\ \Phi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} zI & Q_n \\ R_n & \frac{1}{z}I \end{bmatrix} \begin{bmatrix} \Phi_{1\,n} \\ \Phi_{2\,n} \end{bmatrix},$$

where the transformed potentials are

$$\begin{aligned} Q_n &= \mathrm{e}^{-\mathrm{i}(n\theta + ta\sin 2\theta)H_1}\mathcal{Q}_n \mathrm{e}^{\mathrm{i}(n\theta + ta\sin 2\theta)H_2}, \\ R_n &= \mathrm{e}^{-\mathrm{i}(n\theta + ta\sin 2\theta)H_2}\mathcal{R}_n \mathrm{e}^{\mathrm{i}(n\theta + ta\sin 2\theta)H_1}, \end{aligned}$$

for the semi-discrete case and

$$\begin{aligned} Q_n &= \mathrm{e}^{-\mathrm{i}(n\alpha + l\beta)H_1}\mathcal{Q}_n \mathrm{e}^{\mathrm{i}(n\alpha + l\beta)H_2}, \\ R_n &= \mathrm{e}^{-\mathrm{i}(n\alpha + l\beta)H_2}\mathcal{R}_n \mathrm{e}^{\mathrm{i}(n\alpha + l\beta)H_1}, \end{aligned}$$

for the full-discrete case. These are, respectively, Lax formulations for the sd-cmKdV equations and the fd-cmKdV equations. Thus, we can solve the initial-value problem and construct multi-soliton solutions and conservation laws for the sd-cHirota equations and the fd-cHirota equations. In fact, the N-soliton solution of the sd-cHirota equations (7.94) with $a = 1$, $\sigma_k = 1$ is obtained by combining eq. (7.79) and eq. (7.92). The N-soliton solution exhibits remarkable behaviors as well as the N-soliton solution of the continuous cNLS equations (see Section 3.3.1). In Fig. 7.1 and Fig. 7.2, we have two pictures of the two-soliton solution ($N = 2$) in the two-component case with assuming the following form of the residue matrices:

$$\bar{C}_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & 0 \end{bmatrix}, \quad \bar{C}_2 = \begin{bmatrix} 0 & 0 \\ -\beta_1^* & \alpha_1^* \end{bmatrix}, \tag{7.103a}$$

$$\bar{C}_3 = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & 0 \end{bmatrix}, \quad \bar{C}_4 = \begin{bmatrix} 0 & 0 \\ -\beta_2^* & \alpha_2^* \end{bmatrix}. \tag{7.103b}$$

We remark that $|q_n^{(1)}|$ and $|q_n^{(2)}|$ in Fig. 7.1 and Fig. 7.2 are independent of the value of θ (cf. eq. (7.92)).

The first three conserved densities for the sd-cHirota equations and the fd-cHirota equations are given by

$$\mathrm{I}_1 = \log\Big(1 - \sum_j q_n^{(j)} r_n^{(j)}\Big),$$

$$\mathrm{I}_2 = \cos 2\alpha \sum_j (q_{n+1}^{(j)} r_n^{(j)} + q_n^{(j)} r_{n+1}^{(j)}) + \mathrm{i} \sin 2\alpha \sum_j (q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)}),$$

$$\begin{aligned} \mathrm{I}_3 = {} & \Big(1 - \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)}\Big)\Big\{\cos 4\alpha \sum_j (q_{n+2}^{(j)} r_n^{(j)} + q_n^{(j)} r_{n+2}^{(j)}) \\ & + \mathrm{i}\sin 4\alpha \sum_j (q_{n+2}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+2}^{(j)})\Big\} - \frac{1}{2}\Big\{\cos 2\alpha \sum_j (q_{n+1}^{(j)} r_n^{(j)} + q_n^{(j)} r_{n+1}^{(j)}) \\ & + \mathrm{i}\sin 2\alpha \sum_j (q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)})\Big\}^2 + \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)} \sum_j q_n^{(j)} r_n^{(j)}, \end{aligned}$$

where α is given by eq. (7.102).

There are plural ways of full-discretization of the one-component Hirota equation. In fact we have some freedom to determine the linearized dispersion relation of fd-Hirota equation (see [7]). However, as far as we have considered, there is no such freedom in the multi-component case.

7.5.4 Continuum limit

In the continuum limit of time ($\delta t \to 0$), we can equate j_n to 1. Thus, we can check that the fd-cHirota equations and the Lax pair reduce to the sd-cHirota equations and the Lax pair. Now we consider the continuum limit of space. We denote by δx the lattice spacing of the sd-cHirota equations. We rescale t by

$$t \to \frac{1}{(\delta x)^2} t,$$

and set

$$q_n^{(i)}(t) = \delta x \cdot u_i(x,t), \quad x = n\,\delta x,$$
$$\theta = \mu\,\delta x, \quad a = \frac{\gamma}{\delta x}.$$

Then, we can rewrite the sd-cHirota equations (7.94) as

$$\begin{aligned} & \mathrm{i}u_{j,t} - \mathrm{i}\gamma\Big\{\frac{2\cos(2\mu\,\delta x)}{(\delta x)^2} u_{j,x} + \frac{1}{3} u_{j,xxx} + 2\sum_{k=1}^m \sigma_k |u_k|^2 \cdot u_{j,x} + O(\delta x^2)\Big\} \\ & + 2\gamma\mu\Big\{u_{j,xx} + 2\sum_{k=1}^m \sigma_k |u_k|^2 \cdot u_j + O(\delta x^2)\Big\} = 0. \end{aligned}$$

Therefore, with the help of a Galilei transformation, we obtain the cHirota equations (3.77) in the continuum limit of space (when $\sigma_j = 1$). We remark that the restriction $\theta = O(\delta x)$ is necessary for taking the continuum limit of both evolution equations and the Lax pair (see eqs. (2.28) and (7.95)). In the case $a = 1,\ \theta = \pi/4$, the sd-cHirota equations (7.94) reduce to the sd-cNLS equations,

$$\mathrm{i}q_{n,t}^{(j)} + (q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}) + \sum_{k=1}^{m} \sigma_k |q_n^{(k)}|^2 (q_{n+1}^{(j)} + q_{n-1}^{(j)}) = 0, \quad j = 1, 2, \ldots, m. \tag{7.104}$$

Obviously, this system is a natural multi-field generalization of the sd-NLS equation (eq. (2.32) with $r_n = \pm q_n^*$) and reduces to the cNLS equations (3.49) in the continuum limit of space. However, the Lax pair for eq. (7.104) in terms of $2^m \times 2^m$ matrices cannot be reduced to the Lax pair for eq. (3.49) in terms of $(m+1) \times (m+1)$ matrices if $m \geq 2$. This case gives a counter-example for eq. (2.28) (cf. remark (b) in Section 2.7).

7.6 Summary

In this chapter, we have presented a new multi-component extension of the discrete version of the ISM formulation proposed by Ablowitz and Ladik. We have obtained an integrable discretization of the matrix AKNS formulation and have given a method to construct conservation laws. Considering a reduction similar to the continuous case, we have obtained an integrable discretization of the cmKdV equations. By applying the ISM to the model, we have solved the initial-value problem and have obtained multi-soliton solutions. We remark that both of the constraints (7.31) play an important role in the process of the ISM, while only the former constraint is necessary for the continuous theory. This has its origin in the fact that eq. (7.3) does not generally allow us to assume the reduction $R_n = \pm Q_n^\dagger$, because of the order of the products. To consider the reduction $R_n = \pm Q_n^\dagger$, we should impose the additional restriction $Q_n R_n = R_n Q_n =$ scalar on Q_n and R_n.

By means of a transformation of variables for the discrete cmKdV equations, we have obtained a discrete version of the cHirota equations. The discrete cHirota equations include discrete cNLS equations as a special case. It has been pointed out that there is a difference in sizes between the Lax pair for the discrete cNLS equations and that for the continuous cNLS equations. Thus, the continuum limit of space cannot cast the discrete Lax pair into the continuous one.

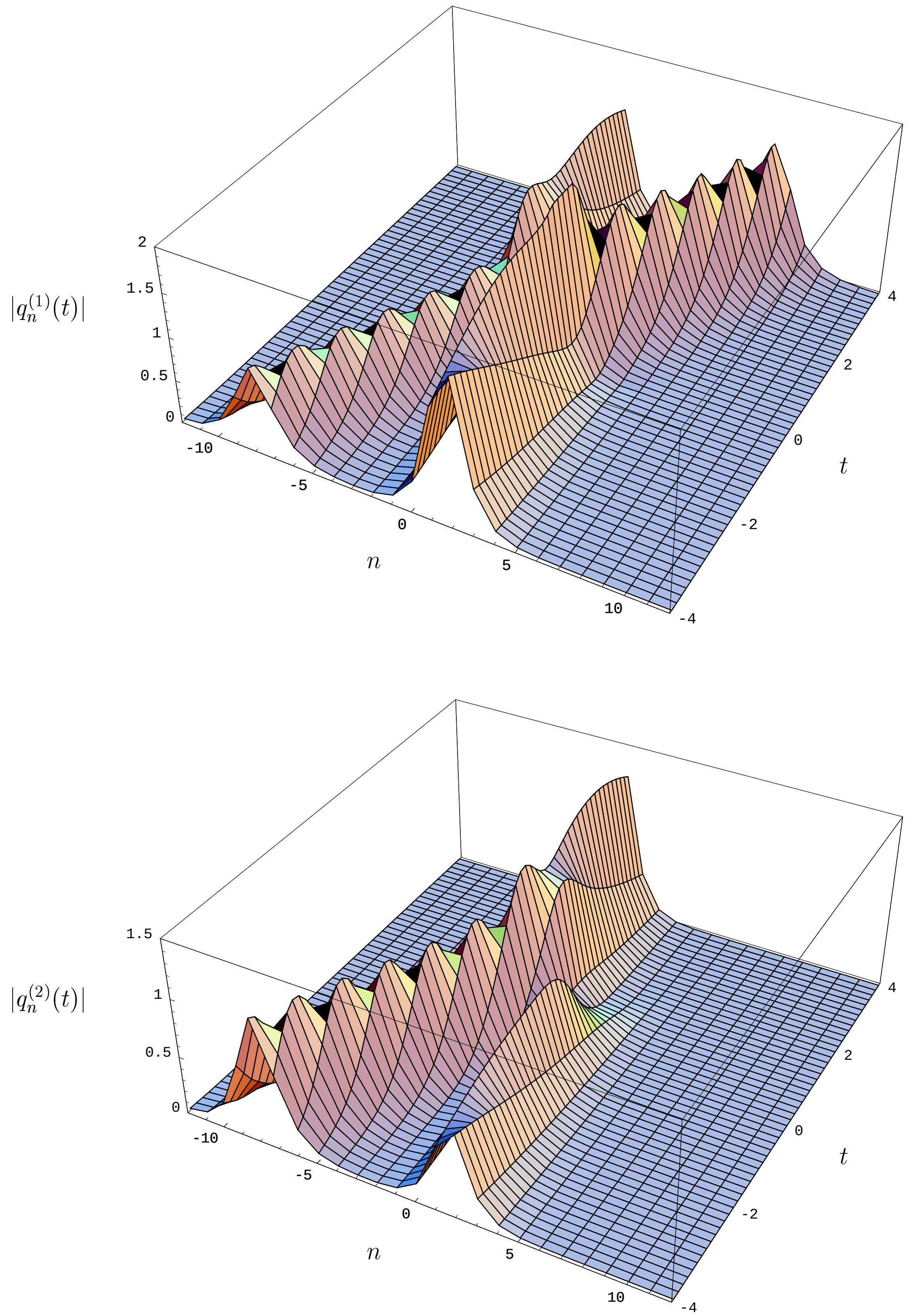
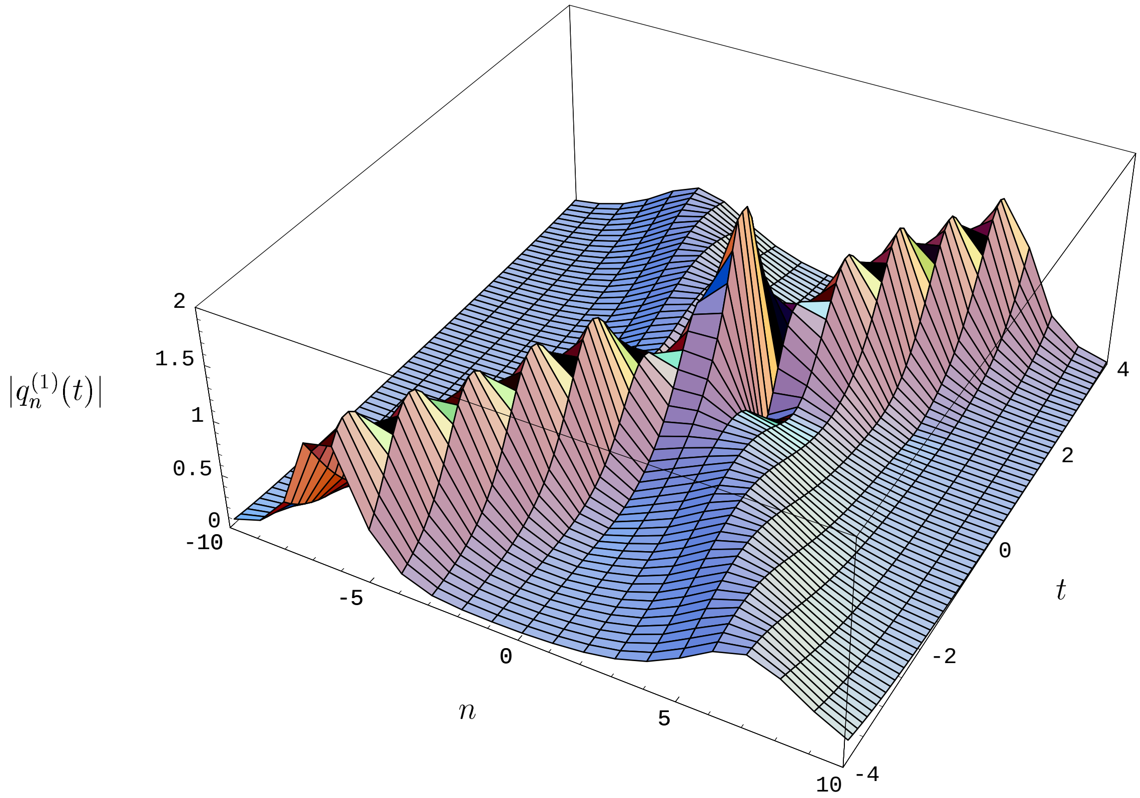


Figure 7.1: Two-soliton solution of the sd-cHirota equations (7.94) with $a = 1$, $\sigma_k = 1$, $m = 2$. The parameters in eq. (7.79) and eq. (7.103) are chosen as $z_1 = 77/50 + 7\mathrm{i}/5$, $z_3 = 4/5 + 8\mathrm{i}/5$, $(\alpha_1(0), \beta_1(0)) = (60/13, 25/13)$, $(\alpha_2(0), \beta_2(0)) = (2, 0)$.



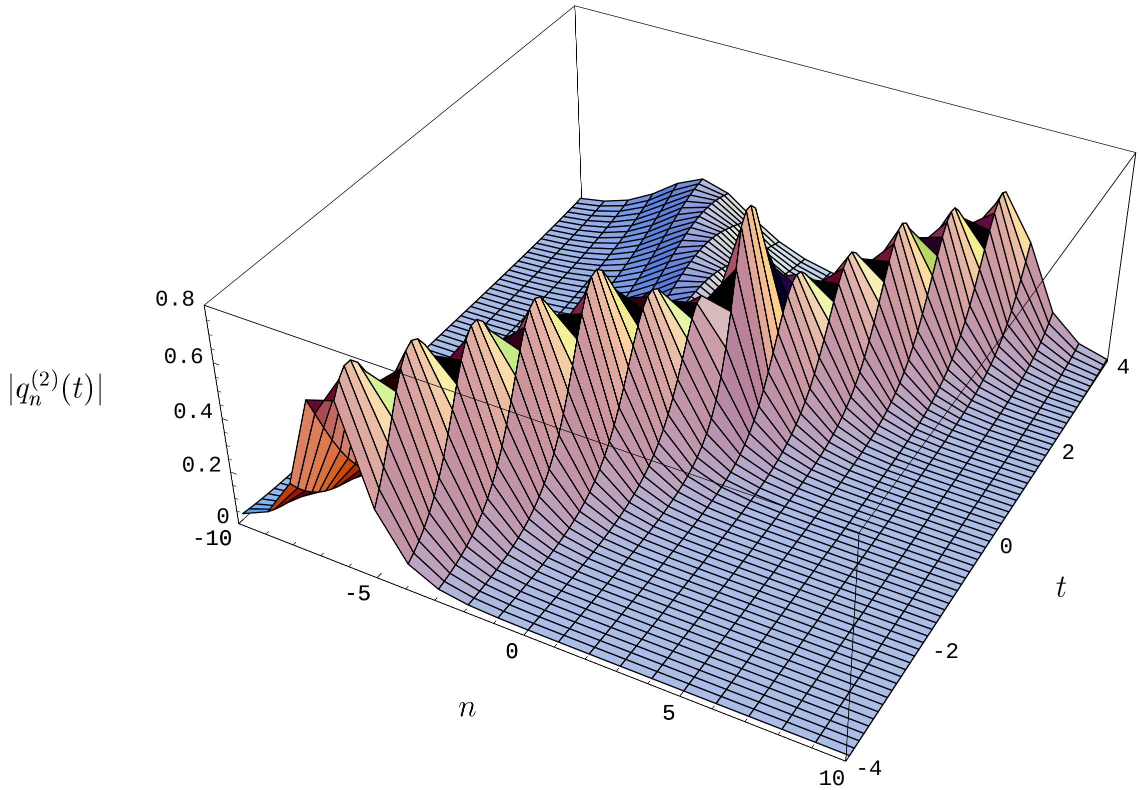


Figure 7.2: Two-soliton solution of the sd-cHirota equations (7.94) with $a = 1$, $\sigma_k = 1$, $m = 2$. The parameters in eq. (7.79) and eq. (7.103) are chosen as $z_1 = 6/5 + 3\mathrm{i}/5$, $z_3 = 4/5 + 8\mathrm{i}/5$, $(\alpha_1(0), \beta_1(0)) = (2, 0)$, $(\alpha_2(0), \beta_2(0)) = (24/13, 10/13)$.

Chapter 8

Summary and Concluding Remarks

In this thesis, we have studied a variety of soliton equations with multiple components from a point of view of the inverse scattering method (ISM). The ISM is one of the approaches in soliton theory and each approach has its own advantages. For example, the symmetry approach is effective in an exhaustive search for some classes of integrable systems (see, *e.g.* [64, 72, 73]). Among various approaches, we have employed the ISM as the main method of studying multi-component soliton equations in this thesis because of its wide applicability and universality. In fact, by using the ISM formulation, we can construct conservation laws, obtain most general soliton solutions, solve the initial-value problem, have superposition principle of solutions despite the nonlinearity and clarify interrelations among soliton equations. In spite of its effectiveness, as far as the author knows, the applications of the ISM for multi-component systems in both the continuous case and the discrete case have not been well-developed so far. One of the main reasons for this fact lies in the difficulty of finding Lax pairs, which is indispensable to the ISM. On the other hand, alternative techniques such as the Hirota method [35, 36, 43, 69–71], the symmetry approach with using computers [72, 73] and the approach based on Jordan algebras and Jordan pairs [79–81] have been effectively used in studying multi-component systems. In this thesis, with all sorts of devices, we have found a number of novel Lax pairs for multi-component systems. Some of the systems have not been studied by any other method and seem to be new multi-component soliton equations. On the basis of the Lax pairs, we have advanced the analysis of the multi-component soliton equations and have obtained rigorous results such as conservation laws, soliton solutions, solution of the initial-value problem, etc. Let us compactly itemize the fruits of study which we have developed in this thesis.

(a) We have proposed a matrix generalization of the AKNS formulation and have applied the ISM to the matrix AKNS hierarchy. Considering a reduction, we have shown that the coupled mKdV (cmKdV) equations can be solved by the ISM [85]. By superposing the cmKdV equations on the coupled NLS (cNLS) equations, we have obtained a new system of coupled Hirota (cHirota) equations.

(b) We have found a Lax pair for a matrix generalization of a derivative NLS (DNLS) equations proposed by Chen, Lee and Liu [88, 90]. Through vector reductions, we have obtained two types of coupled Chen-Lee-Liu equations. With the help of gauge transformations and changes of variables, we have clarified the properties and the differences between the two types. As a secondary product, we have found a new multi-field extension of a DNLS equation studied by Kaup and Newell.

(c) We have studied matrix-valued systems of DNLS type which Olver and Sokolov [73] showed to have a higher symmetry. Introducing a transformation of matrix-valued variables for the matrix generalization of Chen-Lee-Liu equation, we have obtained Lax pairs for all systems but two in [73]. For the remaining two systems, we have derived the general solution [90].

(d) We have discussed a multi-field generalization of the second flows in the Heisenberg ferromagnet hierarchy and the Wadati-Konno-Ichikawa hierarchy [89]. For the obtained two systems, we have clarified their correspondence via a change of independent and dependent variables.

(e) We have proposed an integrable discretization of the matrix AKNS hierarchy. By an analogous reduction to the continuous case, we have found an integrable discretization of the cmKdV equations. We have applied the ISM to the discrete cmKdV equations and have obtained some new results [86, 91]. By a somewhat tricky transformation to the discrete cmKdV equations, we have obtained an integrable discretization of the cHirota equations. Assuming a special choice of parameters in the discrete cHirota equations, we have found that the discrete cNLS equations have a rather different property from the continuous cNLS equations [87, 92].

Through the thesis, we have often considered reductions of matrix-valued evolution equations to obtain multi-component soliton equations of physical significance. It is quite important to reflect the reductions in the symmetry of scattering data for the completeness of the ISM. Multi-component soliton equations are known to be fundamental in various fields such as fluid dynamics, nonlinear optics and plasma physics in describing physical phenomena with internal freedoms. A typical example is the cNLS equations which describe propagation of two polarized electromagnetic waves in a plane.

We believe that the thesis has developed the study of multi-component soliton equations from an original aspect and will provide a new basis for further study.

Appendices

A Proof of Eq. (3.38)

In this appendix, we provide a proof of eq. (3.38). We begin with a counterpart of the linear equations (2.1),

$$\Psi_x = U\Psi,$$

where Ψ is assumed to be a square matrix. Using this equation, we get a chain of identities,

$$\begin{aligned}
&\Psi_x\Psi^{-1} = U, \quad \mathrm{tr}(\log\Psi)_x = \mathrm{tr}\,U, \\
&(\log \det\Psi)_x = \mathrm{tr}\,U, \\
&\det\Psi = \det\Psi(x_0)\cdot \mathrm{e}^{\mathrm{tr}\int_{x_0}^{x} U\mathrm{d}x}.
\end{aligned} \tag{A.1}$$

In case U is given by

$$U = \begin{bmatrix} -\mathrm{i}\zeta I & Q \\ R & \mathrm{i}\zeta I \end{bmatrix},$$

with square matrices I, Q and R, eq. (A.1) leads to

$$\det\Psi = \text{const}.$$

If we take $[\,\bar{\phi}\ \ \bar{\psi}\,]$ as Ψ, we get

$$\det[\,\bar{\phi}\ \ \bar{\psi}\,] = \det A^{\dagger}(\zeta^*) = \det\bar{A}(\zeta),$$

which means,

$$\det\bar{A}(\zeta) = \{\det A(\zeta^*)\}^*.$$

B Matrix Representation of $\{e_j\}$

In this appendix, we give a matrix representation of the anti-commutative matrices $\{e_1, \cdots, e_{2m-1}\}$ which satisfy the relations (3.55). The representation is recursively constructed in the following manner:

(i) In case of $m = 2$, set

$$e_1 = \begin{bmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{bmatrix} \equiv \Lambda_1, \quad e_2 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \equiv \Lambda_2, \quad e_3 = \begin{bmatrix} & \mathrm{i} \\ \mathrm{i} & \end{bmatrix} \equiv \Lambda_3.$$

(ii) Let $\{e_1, e_2, \cdots, e_{2m-1}\}$ have a matrix representation which satisfies eq. (3.55). Then the following set of matrices,

$$-\mathrm{i}\Lambda_1 \otimes e_1,\ -\mathrm{i}\Lambda_1 \otimes e_2,\ \cdots,\ -\mathrm{i}\Lambda_1 \otimes e_{2m-1},\ \Lambda_2 \otimes \mathbb{I},\ \Lambda_3 \otimes \mathbb{I},$$

gives a matrix representation of $\{e_1, e_2, \cdots, e_{2m+1}\}$.

As is clear from the above construction, the representation of $\{e_1, \cdots, e_{2m-1}\}$ is given in terms of $2^{m-1} \times 2^{m-1}$ matrices. It should be noted that there is a freedom of some similarity transformations for the matrix representation of $\{e_j\}$.

C Continuous Neumann-Liouville Expansion

In this appendix, we explain the Neumann-Liouville expansion for the continuous Gel'fand-Levitan-Marchenko equations. Due to the Gel'fand-Levitan-Marchenko equations (3.32) and (3.33), we get closed integral equations for K_1 and $\bar{K}_2$,

$$K_1(x,y) = \bar{F}(x+y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 K_1(x,s_2) F(s_2+s_1)\bar{F}(s_1+y),$$

$$\bar{K}_2(x,y) = -F(x+y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \bar{K}_2(x,s_2) \bar{F}(s_2+s_1)F(s_1+y).$$

By successive approximations, we obtain the Neumann-Liouville expansions for K_1 and $\bar{K}_2$,

$$\begin{aligned} K_1(x,x) \ =\ & \bar{F}(2x) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \bar{F}(x+s_2)F(s_2+s_1)\bar{F}(s_1+x) + \cdots \\ & + (-1)^n \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \cdots \int_x^\infty \mathrm{d}s_{2n} \bar{F}(x+s_{2n})F(s_{2n}+s_{2n-1})\bar{F}(s_{2n-1}+s_{2n-2}) \\ & \qquad \cdots F(s_2+s_1)\bar{F}(s_1+x) \\ & + \cdots, \end{aligned}$$

$$\begin{aligned} \bar{K}_2(x,x) \ =\ & -F(2x) + \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 F(x+s_2)\bar{F}(s_2+s_1)F(s_1+x) + \cdots \\ & + (-1)^{n-1} \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \cdots \int_x^\infty \mathrm{d}s_{2n} F(x+s_{2n})\bar{F}(s_{2n}+s_{2n-1})F(s_{2n-1}+s_{2n-2}) \\ & \qquad \cdots \bar{F}(s_2+s_1)F(s_1+x) \\ & + \cdots. \end{aligned}$$

D Proof of Proposition 2

In this appendix, we give a proof of Proposition 2 in Section 3.3.2. The relation (3.12) with eq. (3.13) becomes

$$\Gamma_x = 2\mathrm{i}\zeta\Gamma + R^{(m)} - \Gamma Q^{(m)}\Gamma, \tag{D.1}$$

for the cmKdV equations. We assume

$$\lim_{|\zeta|\to\infty} \Gamma = O \ \ (\mathrm{Im}\,\zeta > 0),$$

and expand Γ as

$$\Gamma = \sum_{l=1}^{\infty} \frac{1}{(2\mathrm{i}\zeta)^l} G_l, \tag{D.2}$$

instead of eq. (3.16). Substituting eq. (D.2) into eq. (D.1), we obtain a recursion formula,

$$G_{l+1} = -\delta_{l,0} R^{(m)} + (G_l)_x + \sum_{j=1}^{l-1} G_j Q^{(m)} G_{l-j}, \quad l = 0, 1, \ldots,$$

where $Q^{(m)}$ and $R^{(m)}$ are given by

$$Q^{(m)} = v_0 \mathbb{1} + \sum_{k=1}^{2m-1} v_k e_k, \qquad R^{(m)} = -v_0 \mathbb{1} + \sum_{k=1}^{2m-1} v_k e_k.$$

We first show the following theorem.

Theorem. Let X, Y, Z be $2^{m-1} \times 2^{m-1}$ matrices expressed as

$$X = x_0 \mathbb{1} + \sum_{k=1}^{2m-1} x_k e_k,$$

$$Y = y_0 \mathbb{1} + \sum_{k=1}^{2m-1} y_k e_k,$$

$$Z = z_0 \mathbb{1} + \sum_{k=1}^{2m-1} z_k e_k,$$

where the coefficients x_0, x_k; y_0, y_k; z_0, z_k are real. Then there exist real numbers w_0, w_k that satisfy

$$W \equiv XYZ + ZYX = w_0 \mathbb{1} + \sum_{k=1}^{2m-1} w_k e_k. \tag{D.3}$$

Proof. By use of eq. (3.55a), a direct calculation gives

$$\begin{aligned}
W &= \Big(x_0 \mathbb{1} + \sum_{i=1}^{2m-1} x_i e_i\Big)\Big(y_0 \mathbb{1} + \sum_{j=1}^{2m-1} y_j e_j\Big)\Big(z_0 \mathbb{1} + \sum_{k=1}^{2m-1} z_k e_k\Big) \\
&\quad + \Big(z_0 \mathbb{1} + \sum_{k=1}^{2m-1} z_k e_k\Big)\Big(y_0 \mathbb{1} + \sum_{j=1}^{2m-1} y_j e_j\Big)\Big(x_0 \mathbb{1} + \sum_{i=1}^{2m-1} x_i e_i\Big) \\
&= 2x_0 y_0 z_0 \mathbb{1} + 2x_0 y_0 \sum_{k=1}^{2m-1} z_k e_k + 2x_0 z_0 \sum_{j=1}^{2m-1} y_j e_j + 2y_0 z_0 \sum_{i=1}^{2m-1} x_i e_i \\
&\quad + x_0 \Big\{\sum_{j,k=1}^{2m-1} y_j z_k (e_j e_k + e_k e_j)\Big\} + y_0 \Big\{\sum_{i,k=1}^{2m-1} x_i z_k (e_i e_k + e_k e_i)\Big\} + z_0 \Big\{\sum_{i,j=1}^{2m-1} x_i y_j (e_i e_j + e_j e_i)\Big\} \\
&\quad + \sum_{i,j,k=1}^{2m-1} x_i y_j z_k (e_i e_j e_k + e_k e_j e_i) \\
&= 2\Big[\Big(x_0 y_0 z_0 - x_0 \sum_{i=1}^{2m-1} y_i z_i - y_0 \sum_{i=1}^{2m-1} x_i z_i - z_0 \sum_{i=1}^{2m-1} x_i y_i\Big)\mathbb{1} + \sum_{i=1}^{2m-1} \Big\{\Big(x_0 y_0 - \sum_{k=1}^{2m-1} x_k y_k\Big) z_i \\
&\quad + \Big(y_0 z_0 - \sum_{k=1}^{2m-1} y_k z_k\Big) x_i + \Big(z_0 x_0 + \sum_{k=1}^{2m-1} z_k x_k\Big) y_i\Big\} e_i\Big].
\end{aligned}$$

Thus W does not include terms such as $e_i e_j$, $e_i e_j e_k$ and can be expressed in the form of eq. (D.3). □

Using the theorem, we can prove by the inductive method that $\{G_l\}$ are expressed as

$$G_l = g_l^{(0)}\mathbb{1} + \sum_{k=1}^{2m-1} g_l^{(k)} e_k,$$

where $g_l^{(0)}$ and $g_l^{(k)}$ are real coefficients. Therefore, Γ is written as

$$\begin{aligned}\Gamma &= \sum_{l=1}^{\infty} \frac{1}{(2\mathrm{i}\zeta)^l}\Big(g_l^{(0)}\mathbb{1} + \sum_{k=1}^{2m-1} g_l^{(k)} e_k\Big) \\ &= = \sum_{l=1}^{\infty} \frac{g_l^{(0)}}{(2\mathrm{i}\zeta)^l}\mathbb{1} + \sum_{k=1}^{2m-1}\Big(\sum_{l=1}^{\infty} \frac{g_l^{(k)}}{(2\mathrm{i}\zeta)^l}\Big) e_k \\ &= \gamma^{(0)}(\zeta)\mathbb{1} + \sum_{k=1}^{2m-1} \gamma^{(k)}(\zeta) e_k, \end{aligned} \tag{D.4}$$

where $\gamma^{(0)}(\zeta)$ and $\gamma^{(k)}(\zeta)$ satisfy

$$\gamma^{(0)}(\zeta) = \{\gamma^{(0)}(-\zeta^*)\}^*, \quad \gamma^{(k)}(\zeta) = \{\gamma^{(k)}(-\zeta^*)\}^*. \tag{D.5}$$

We recall the asymptotic behavior of the Jost function ϕ at $x \to \pm\infty$,

$$\begin{aligned}\phi &\sim \begin{bmatrix} I \\ O \end{bmatrix} \mathrm{e}^{-\mathrm{i}\zeta x} \quad \text{as} \quad x \to -\infty, \\ &\sim \begin{bmatrix} A(\zeta)\mathrm{e}^{-\mathrm{i}\zeta x} \\ B(\zeta)\mathrm{e}^{\mathrm{i}\zeta x} \end{bmatrix} \quad \text{as} \quad x \to +\infty. \end{aligned}$$

These relations yield

$$\lim_{x\to-\infty} \phi_1 \mathrm{e}^{\mathrm{i}\zeta x} = I, \quad \lim_{x\to+\infty} \phi_1 \mathrm{e}^{\mathrm{i}\zeta x} = A(\zeta),$$

where we have defined ϕ_1 and ϕ_2 by

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

Since we easily see that

$$\lim_{|\zeta|\to\infty} \phi_2 \phi_1^{-1} = O \quad (\mathrm{Im}\,\zeta > 0),$$

we can replace Γ in eq. (D.4) with $\phi_2\phi_1^{-1}$. Thus $\det A(\zeta)$ is expressed as

$$\begin{aligned}\det A(\zeta) &= \exp\{\mathrm{tr}\log A(\zeta)\} \\ &= \exp\Big\{\mathrm{tr}\int_{-\infty}^{\infty} (\log \phi_1 \mathrm{e}^{\mathrm{i}\zeta x})_x \mathrm{d}x\Big\} \\ &= \exp\Big\{\mathrm{tr}\int_{-\infty}^{\infty} (\phi_1 \mathrm{e}^{\mathrm{i}\zeta x})_x (\phi_1 \mathrm{e}^{\mathrm{i}\zeta x})^{-1} \mathrm{d}x\Big\} \\ &= \exp\Big\{\mathrm{tr}\int_{-\infty}^{\infty} Q^{(m)} \phi_2 \phi_1^{-1} \mathrm{d}x\Big\} \\ &= \exp\Big\{\mathrm{tr}\int_{-\infty}^{\infty} \Big(v_0\gamma^{(0)}(\zeta) - \sum_{k=1}^{2m-1} v_k \gamma^{(k)}(\zeta)\Big)\mathbb{1}\,\mathrm{d}x\Big\} \\ &= \exp\Big\{2^{m-1}\int_{-\infty}^{\infty} \Big(v_0\gamma^{(0)}(\zeta) - \sum_{k=1}^{2m-1} v_k \gamma^{(k)}(\zeta)\Big)\mathrm{d}x\Big\}, \end{aligned}$$

where we have used eqs. (3.55c) and (3.56). Due to eq. (D.5), $\det A(\zeta)$, as a function of ζ, satisfies

$$\det A(\zeta) = \{\det A(-\zeta^*)\}^*.$$

This is the proof of Proposition 2 (1). Further, we obtain

$$\begin{aligned} B(\zeta)A(\zeta)^{-1} &= \lim_{x\to+\infty} \phi_2\phi_1^{-1}\mathrm{e}^{-2\mathrm{i}\zeta x} \\ &= \lim_{x\to+\infty}\left[\gamma^{(0)}(\zeta)\mathrm{e}^{-2\mathrm{i}\zeta x}\mathbb{I} + \sum_{k=1}^{2m-1}\gamma^{(k)}(\zeta)\mathrm{e}^{-2\mathrm{i}\zeta x}e_k\right] \\ &= r^{(0)}(\zeta)\mathbb{I} + \sum_{k=1}^{2m-1} r^{(k)}(\zeta)e_k, \end{aligned} \tag{D.6}$$

with the conditions

$$r^{(0)}(\zeta) = \{r^{(0)}(-\zeta^*)\}^*, \;\; r^{(k)}(\zeta) = \{r^{(k)}(-\zeta^*)\}^*. \tag{D.7}$$

By use of eqs. (D.6) and (D.7), it is straightforward to prove Proposition 2 (2)(3).

E Massive Thirring Model

In this appendix, we show a list of matrix generalizations of the massive Thirring model, which are respectively a member of the DNLS-type hierarchies studied in Section 5.2. For this purpose, we consider Lax pairs with the following dependence on the spectral parameter ζ,

$$U = \mathrm{i}\zeta^2 \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \zeta \begin{bmatrix} & Q \\ R & \end{bmatrix} + \mathrm{i}\begin{bmatrix} U_{11} & \\ & U_{22} \end{bmatrix},$$

$$V = \mathrm{i}\frac{m^2}{4\zeta^2} \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \frac{m}{2\zeta} \begin{bmatrix} & \mathcal{P} \\ \mathcal{S} & \end{bmatrix} + \mathrm{i}\begin{bmatrix} V_{11} & \\ & V_{22} \end{bmatrix}.$$

Here U_{11}, V_{11} and U_{22}, V_{22} are ζ-independent square matrices whose size are respectively $n_1 \times n_1$ and $n_2 \times n_2$. Q and $\mathcal{P}$ are $n_1 \times n_2$ matrices. R and $\mathcal{S}$ are $n_2 \times n_1$ matrices. m is a nonzero constant. We have derived new Lax pairs for several matrix generalizations of the DNLS-type equation (5.3) in Section 5.2. To obtain matrix versions of the massive Thirring model, we have only to change the time part of the Lax pairs as above. In this formulation, the new pair of potentials $\mathcal{P}$ and $\mathcal{S}$ appears. In correspondence with the choices of U_{jj} $(j = 1, 2)$, we can determine V_{jj} $(j = 1, 2)$ so that the zero-curvature condition (2.2) yields a consistent set of evolution equations.

We can obtain four matrix generalizations of the massive Thirring model by the above-mentioned method. The choices of U_{jj}, V_{jj} $(j = 1, 2)$ and the corresponding evolution equations are listed as follows.

(a)

$$U_{11} = O, \quad U_{22} = \frac{1}{2}RQ, \quad V_{11} = -\frac{1}{2}\mathcal{P}\mathcal{S}, \quad V_{22} = O,$$

$$
\begin{aligned}
&Q_t - \mathrm{i}m\mathcal{P} + \mathrm{i}\frac{1}{2}\mathcal{P}\mathcal{S}Q = O,\\
&R_t + \mathrm{i}m\mathcal{S} - \mathrm{i}\frac{1}{2}R\mathcal{P}\mathcal{S} = O,\\
&\mathcal{P}_x - \mathrm{i}mQ + \mathrm{i}\frac{1}{2}\mathcal{P}RQ = O,\\
&\mathcal{S}_x + \mathrm{i}mR - \mathrm{i}\frac{1}{2}RQ\mathcal{S} = O.
\end{aligned}
\tag{E.1}
$$

($\mathfrak{b}$)

$$U_{11} = O, \quad U_{22} = O, \quad V_{11} = -\frac{1}{2}\mathcal{P}\mathcal{S}, \quad V_{22} = \frac{1}{2}\mathcal{S}\mathcal{P},$$

$$
\begin{aligned}
&Q_t - \mathrm{i}m\mathcal{P} + \mathrm{i}\frac{1}{2}(Q\mathcal{S}\mathcal{P} + \mathcal{P}\mathcal{S}Q) = O,\\
&R_t + \mathrm{i}m\mathcal{S} - \mathrm{i}\frac{1}{2}(\mathcal{S}\mathcal{P}R + R\mathcal{P}\mathcal{S}) = O,\\
&\mathcal{P}_x - \mathrm{i}mQ = O,\\
&\mathcal{S}_x + \mathrm{i}mR = O.
\end{aligned}
\tag{E.2}
$$

($\mathfrak{c}$)

$$U_{11} = O, \quad U_{22} = \frac{1}{2}(RQ + QR), \quad V_{11} = -\frac{1}{2}\mathcal{P}\mathcal{S}, \quad V_{22} = -\frac{1}{2}\mathcal{P}\mathcal{S},$$

$$
\begin{aligned}
&Q_t - \mathrm{i}m\mathcal{P} + \mathrm{i}\frac{1}{2}(\mathcal{P}\mathcal{S}Q - Q\mathcal{P}\mathcal{S}) = O,\\
&R_t + \mathrm{i}m\mathcal{S} - \mathrm{i}\frac{1}{2}(R\mathcal{P}\mathcal{S} - \mathcal{P}\mathcal{S}R) = O,\\
&\mathcal{P}_x - \mathrm{i}mQ + \mathrm{i}\frac{1}{2}\mathcal{P}(RQ + QR) = O,\\
&\mathcal{S}_x + \mathrm{i}mR - \mathrm{i}\frac{1}{2}(RQ + QR)\mathcal{S} = O.
\end{aligned}
\tag{E.3}
$$

($\mathfrak{d}$)

$$U_{11} = -\frac{1}{2}RQ, \quad U_{22} = O, \quad V_{11} = \frac{1}{2}(\mathcal{S}\mathcal{P} - \mathcal{P}\mathcal{S}), \quad V_{22} = \frac{1}{2}\mathcal{S}\mathcal{P},$$

$$
\begin{aligned}
&Q_t - \mathrm{i}m\mathcal{P} + \mathrm{i}\frac{1}{2}(Q\mathcal{S}\mathcal{P} + \mathcal{P}\mathcal{S}Q - \mathcal{S}\mathcal{P}Q) = O,\\
&R_t + \mathrm{i}m\mathcal{S} - \mathrm{i}\frac{1}{2}(\mathcal{S}\mathcal{P}R + R\mathcal{P}\mathcal{S} - R\mathcal{S}\mathcal{P}) = O,\\
&\mathcal{P}_x - \mathrm{i}mQ + \mathrm{i}\frac{1}{2}RQ\mathcal{P} = O,\\
&\mathcal{S}_x + \mathrm{i}mR - \mathrm{i}\frac{1}{2}\mathcal{S}RQ = O.
\end{aligned}
\tag{E.4}
$$

In view of the Lax matrix U the cases ($\mathfrak{a}$)–($\mathfrak{d}$) respectively correspond to eq. (5.4), (a), (c) and (e) in Section 5.2. Other choices of U obtained in Section 5.2 lead to systems which coincide with one of ($\mathfrak{a}$)–($\mathfrak{d}$) up to the exchange of t and x, etc. All of the systems ($\mathfrak{a}$)–($\mathfrak{d}$) are shown to be connected with the others through the gauge transformations utilized in Section 5.2.

The system (E.1) has been obtained in [84]. The others seem to be new integrable systems. For the existence of products of the matrices in the evolution equations, Q, R, $\mathcal{P}$ and $\mathcal{S}$ must be square matrices of the same size in ($\mathfrak{c}$) and ($\mathfrak{d}$).

F Proof of Eq. (7.62)

In this appendix, we prove eq. (7.62). We start from a counterpart of the linear equations (2.26),

$$\Psi_{n+1} = L_n \Psi_n,$$

where Ψ_n is assumed to be a square matrix. Then, we get a chain of identities,

$$\Psi_{n+1}\Psi_n^{-1} = L_n, \qquad \frac{\det\Psi_{n+1}}{\det\Psi_n} = \det L_n,$$
$$\det\Psi_\infty = \det\Psi_{-\infty} \cdot \prod_{n=-\infty}^{\infty} \det L_n. \tag{F.1}$$

We consider the case that L_n is given by

$$L_n = \begin{bmatrix} zI & Q_n \\ R_n & \frac{1}{z}I \end{bmatrix},$$

with $l \times l$ ($l = 2^{m-1}$) square matrices I, Q_n and R_n, which satisfy the constraints (7.31). Equation (F.1) gives

$$\det\Psi_\infty = \det\Psi_{-\infty} \cdot \Big(\prod_{n=-\infty}^{\infty} \rho_n \Big)^l = \tau^l \cdot \det\Psi_{-\infty},$$

where ρ_n and τ are defined by eqs. (7.34) and (7.35),

$$\rho_n = (\det L_n)^{\frac{1}{l}} = 1 + \sigma_n = 1 + \sum_{j=0}^{2m-1} v_n^{(j)\,2}, \qquad \tau = \prod_{n=-\infty}^{\infty} \rho_n.$$

It should be noted that τ is a conserved quantity, which is proved by the zero-curvature condition (2.27) (see also eq. (7.29a)).

If we take $[\,\bar{\phi}_n \;\; \bar{\psi}_n\,]$ as Ψ_n, we obtain

$$\lim_{n\to-\infty} \det[\,\bar{\phi}_n \;\; \bar{\psi}_n\,] = \frac{1}{\tau^l} \cdot \det A^\dagger(\frac{1}{z^*}),$$

$$\lim_{n\to+\infty} \det[\,\bar{\phi}_n \;\; \bar{\psi}_n\,] = \det\bar{A}(z),$$

which lead to the formula to be proved,

$$\det\bar{A}(z) = \{\det A(\frac{1}{z^*})\}^*.$$

G Proof of Proposition 3

In this appendix, we give a proof of Proposition 3 in Section 7.3.3. Let us recall $\Gamma_n = \Psi_{2n}\Psi_{1n}^{-1}$ defined in Section 7.1. We assume

$$\lim_{|z|\to\infty} \Gamma_n = O.$$

Then the relation (7.9) with eq. (7.13) becomes

$$\begin{aligned}
\Gamma_{n+1} &= \Big(R_n + \frac{1}{z}\Gamma_n\Big)(zI + Q_n\Gamma_n)^{-1} \\
&= \Big(R_n + \frac{1}{z}\Gamma_n\Big)z^{-1}\sum_{j=0}^{\infty}(-1)^j\Big(\frac{1}{z}Q_n\Gamma_n\Big)^j \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j}{z^{j+1}}R_nQ_n\cdot\Gamma_n(Q_n\Gamma_n)^{j-1} + \sum_{j=0}^{\infty}\frac{(-1)^j}{z^{j+2}}\Gamma_n(Q_n\Gamma_n)^j.
\end{aligned}$$

Here Q_n and R_n for the $2m$-component sd-cmKdV equations (7.28) ($m \geq 2$) are given by

$$Q_n^{(m)} = v_n^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} v_n^{(k)}e_k, \qquad R_n^{(m)} = -v_n^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} v_n^{(k)}e_k.$$

From the theorem in D, we have the following corollary.

Corollary. Let X, Y be $2^{m-1}\times 2^{m-1}$ matrices given by

$$X = x^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} x^{(k)}e_k,$$

$$Y = y^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} y^{(k)}e_k,$$

where the coefficients $x^{(0)}$, $x^{(k)}$; $y^{(0)}$, $y^{(k)}$ are real. Then there exist real numbers $s_j^{(0)}$, $s_j^{(k)}$ that satisfy

$$\begin{aligned}
S_j &\equiv (XY)^{j-1}X = X(YX)^{j-1} \\
&= s_j^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1} s_j^{(k)}e_k.
\end{aligned}$$

The corollary can be proved from the theorem by induction.

Using the corollary, we can prove by induction that Γ_n is expressed by

$$\Gamma_n = \gamma_n^{(0)}\mathbb{I} + \sum_{k=1}^{2m-1}\gamma_n^{(k)}e_k, \tag{G.1}$$

where the coefficients $\gamma_n^{(0)}$ and $\gamma_n^{(k)}$ satisfy

$$\gamma_n^{(0)}(z) = \{\gamma_n^{(0)}(z^*)\}^*, \qquad \gamma_n^{(k)}(z) = \{\gamma_n^{(k)}(z^*)\}^*. \tag{G.2}$$

In other words, $\gamma_n^{(0)}(z)$ and $\gamma_n^{(k)}(z)$ are polynomials in $1/z$ with real coefficients. This fact is particularly easy to understand when the potentials Q_n and R_n are on compact support.

We recall the asymptotic behavior of the Jost function ϕ_n at $n \to \pm\infty$,

$$\begin{aligned}\phi_n \equiv \begin{bmatrix} \phi_{1\,n} \\ \phi_{2\,n} \end{bmatrix} &\sim \begin{bmatrix} I \\ O \end{bmatrix} z^n \qquad \text{as} \quad n \to -\infty, \\ &\sim \begin{bmatrix} A(z) z^n \\ B(z) z^{-n} \end{bmatrix} \qquad \text{as} \quad n \to +\infty.\end{aligned}$$

These relations yield

$$\lim_{n\to-\infty} \phi_{1\,n} z^{-n} = I, \quad \lim_{n\to+\infty} \phi_{1\,n} z^{-n} = A(z).$$

We can show that

$$\lim_{|z|\to\infty} \phi_{2\,n}\phi_{1\,n}^{-1} = O.$$

Then we can replace Γ_n in eq. (G.1) with $\phi_{2\,n}\phi_{1\,n}^{-1}$. Thus $\det A(z)$ is expressed as

$$\begin{aligned}\det A(z) &= \exp\{\operatorname{tr}\log A(z)\} \\ &= \exp\Big\{\operatorname{tr} \sum_{n=-\infty}^{\infty} [\log(\phi_{1\,n+1} z^{-n-1}) - \log(\phi_{1\,n} z^{-n})]\Big\} \\ &= \exp\Big\{\operatorname{tr} \sum_{n=-\infty}^{\infty} \log[\phi_{1\,n+1}\phi_{1\,n}^{-1} z^{-1}]\Big\} \\ &= \exp\Big\{\operatorname{tr} \sum_{n=-\infty}^{\infty} \log\Big[I + \frac{1}{z} Q_n^{(m)} \phi_{2\,n}\phi_{1\,n}^{-1}\Big]\Big\} \\ &= \exp\Big\{\operatorname{tr} \sum_{n=-\infty}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\Big(\frac{1}{z} Q_n^{(m)}\Gamma_n\Big)^j\Big\} \\ &= \exp\Big\{\operatorname{tr} \sum_{n=-\infty}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j\cdot z^j} Q_n^{(m)} \cdot (\Gamma_n Q_n^{(m)})^{j-1}\Gamma_n\Big\} \\ &= \exp\Big\{\operatorname{tr} \sum_{n=-\infty}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j\cdot z^j} Q_n^{(m)} \Big(s_{j,n}^{(0)}\mathbb{1} + \sum_{k=1}^{2m-1} s_{j,n}^{(k)} e_k\Big)\Big\} \\ &= \exp\Big\{2^{m-1} \sum_{n=-\infty}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j\cdot z^j} \Big(v_n^{(0)} s_{j,n}^{(0)} - \sum_{k=1}^{2m-1} v_n^{(k)} s_{j,n}^{(k)}\Big)\Big\}.\end{aligned}$$

At the last equality, we have used eqs. (3.55c) and (3.56). Due to eq. (G.2), $\det A(z)$, as a function of z, satisfies

$$\det A(z) = \{\det A(z^*)\}^*.$$

This is a proof of Proposition 3 (1).

Further, we obtain

$$\begin{aligned}B(z)A(z)^{-1} &= \lim_{n\to+\infty} \phi_{2\,n}\phi_{1\,n}^{-1} \cdot z^{2n} \\ &= \lim_{n\to+\infty} \Big[\gamma_n^{(0)}(z) z^{2n}\mathbb{1} + \sum_{k=1}^{2m-1} \gamma_n^{(k)}(z) z^{2n} e_k\Big] \\ &= r^{(0)}(z)\mathbb{1} + \sum_{k=1}^{2m-1} r^{(k)}(z) e_k, \qquad \text{(G.3)}\end{aligned}$$

with conditions

$$r^{(0)}(z) = \{r^{(0)}(z^*)\}^*, \ \ r^{(k)}(z) = \{r^{(k)}(z^*)\}^*. \tag{G.4}$$

Using eqs. (G.3) and (G.4), we can straightforwardly prove Proposition 3 (2)(3).

H Discrete Neumann-Liouville Expansion

In this appendix, we explain the Neumann-Liouville expansion for the discrete Gel'fand-Levitan-Marchenko equations. We begin with the Gel'fand-Levitan-Marchenko equations (7.58) and (7.59),

$$\begin{aligned}
&\kappa_{1R}(n,m) \quad (m>n,\, m-n=\text{odd})\\
&= 2\bar{F}_R(n+m) - 4\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\kappa_{1R}(n,n+2l_2-1)F_R(2n+2l_2+2l_1-1)\bar{F}_R(n+2l_1+m),
\end{aligned}$$

$$\begin{aligned}
&\bar{\kappa}_{2R}(n,m) \quad (m>n,\, m-n=\text{odd})\\
&= -2F_R(n+m) - 4\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\bar{\kappa}_{2R}(n,n+2l_2-1)\bar{F}_R(2n+2l_2+2l_1-1)F_R(n+2l_1+m).
\end{aligned}$$

By successive approximations, we obtain the Neumann-Liouville expansion for κ_{1R} and $\bar{\kappa}_{2R}$,

$$\begin{aligned}
&\kappa_{1R}(n,n+1)\\
&= 2\bar{F}_R(2n+1) - 8\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\bar{F}_R(2n+2l_2-1)F_R(2n+2l_2+2l_1-1)\bar{F}_R(2n+2l_1+1)\\
&\quad+\cdots\\
&\quad+2\cdot(-4)^j\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\cdots\sum_{l_{2j}=1}^{\infty}\bar{F}_R(2n+2l_{2j}-1)F_R(2n+2l_{2j}+2l_{2j-1}-1)\\
&\qquad\quad\bar{F}_R(2n+2l_{2j-1}+2l_{2j-2}-1)\cdots\bar{F}_R(2n+2l_3+2l_2-1)\\
&\qquad\quad F_R(2n+2l_2+2l_1-1)\bar{F}_R(2n+2l_1+1)\\
&\quad+\cdots,
\end{aligned}$$

$$\begin{aligned}
&\bar{\kappa}_{2R}(n,n+1)\\
&= -2F_R(2n+1) + 8\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}F_R(2n+2l_2-1)\bar{F}_R(2n+2l_2+2l_1-1)F_R(2n+2l_1+1)\\
&\quad+\cdots\\
&\quad-2\cdot(-4)^j\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\cdots\sum_{l_{2j}=1}^{\infty}F_R(2n+2l_{2j}-1)\bar{F}_R(2n+2l_{2j}+2l_{2j-1}-1)\\
&\qquad\quad F_R(2n+2l_{2j-1}+2l_{2j-2}-1)\cdots F_R(2n+2l_3+2l_2-1)\\
&\qquad\quad \bar{F}_R(2n+2l_2+2l_1-1)F_R(2n+2l_1+1)\\
&\quad+\cdots.
\end{aligned}$$

I Trace Formulae

In this appendix, we show interrelations between the generator of the conserved densities $\mathrm{tr}\{\log(I + Q_n\Gamma_n/z)\}$ for eq. (7.3) in Section 7.1 and the scattering data in Section 7.3.

For the moment, we do not impose restrictions such as eqs. (7.26) and (7.27) on square matrices Q_n and R_n and consider the sd-matrix equation (7.3). We supplement some definitions and relations. First, let us define the inverse of (7.40) by

$$\begin{aligned}\bar{\psi}_n(z) &= \phi_n(z)\mathcal{A}(z) + \bar{\phi}_n(z)\mathcal{B}(z),\\ \psi_n(z) &= \phi_n(z)\bar{\mathcal{B}}(z) - \bar{\phi}_n(z)\bar{\mathcal{A}}(z).\end{aligned}$$

Secondly, the asymptotic behaviors of the Jost functions ϕ_n and $\bar{\psi}_n$ are given by

$$\begin{aligned}\phi_n \equiv \begin{bmatrix}\phi_{1\,n}\\ \phi_{2\,n}\end{bmatrix} &\sim \begin{bmatrix} I \\ O\end{bmatrix} z^n \qquad \text{as} \quad n \to -\infty,\\ &\sim \begin{bmatrix} A(z)z^n \\ B(z)z^{-n}\end{bmatrix} \qquad \text{as} \quad n \to +\infty,\end{aligned}$$

$$\begin{aligned}\bar{\psi}_n \equiv \begin{bmatrix}\bar{\psi}_{1\,n}\\ \bar{\psi}_{2\,n}\end{bmatrix} &\sim \begin{bmatrix} \mathcal{A}(z)z^n \\ -\mathcal{B}(z)z^{-n}\end{bmatrix} \qquad \text{as} \quad n \to -\infty,\\ &\sim \begin{bmatrix} I \\ O\end{bmatrix} z^n \qquad \text{as} \quad n \to +\infty.\end{aligned}$$

Further, we can prove that $\phi_{2\,n}\phi_{1\,n}^{-1}$ is a polynomial in $1/z$ and $\bar{\psi}_{2\,n}\bar{\psi}_{1\,n}^{-1}$ is a polynomial in z. Therefore, we can replace $\phi_{2\,n}\phi_{1\,n}^{-1}$ and $\bar{\psi}_{2\,n}\bar{\psi}_{1\,n}^{-1}$ by $\Gamma_n^{(-)}$ and $\Gamma_n^{(+)}$ in Section 7.1 respectively. It is important that the ratios of two components, $\phi_{2\,n}\phi_{1\,n}^{-1}$ and $\bar{\psi}_{2\,n}\bar{\psi}_{1\,n}^{-1}$, are invariant when we consider the time-dependent Jost functions $\phi_n^{(t)} \equiv \phi_n \mathrm{e}^{z^2 t}$ and $\bar{\psi}_n^{(t)} \equiv \bar{\psi}_n \mathrm{e}^{z^2 t}$.

Now, we can relate the scattering data with the generator of the conserved densities. The determinant of $A(z)$, $\det A(z)$, is expressed as

$$\begin{aligned}&\log\det A(z) = \mathrm{tr}\log A(z)\\ &= \mathrm{tr}\sum_{n=-\infty}^{\infty}[\log(\phi_{1\,n+1}z^{-n-1}) - \log(\phi_{1\,n}z^{-n})]\\ &= \mathrm{tr}\sum_{n=-\infty}^{\infty}\log[\phi_{1\,n+1}\phi_{1\,n}^{-1}z^{-1}]\\ &= \mathrm{tr}\sum_{n=-\infty}^{\infty}\log\Big[I + \frac{1}{z}Q_n\phi_{2\,n}\phi_{1\,n}^{-1}\Big]\\ &= \mathrm{tr}\sum_{n=-\infty}^{\infty}\log\Big[I + \frac{1}{z}Q_n\Gamma_n^{(-)}\Big]\\ &= \mathrm{tr}\sum_{n=-\infty}^{\infty}\Big[\frac{1}{z^2}Q_nR_{n-1} + \frac{1}{z^4}\{Q_nR_{n-2} - Q_nR_{n-1}Q_{n-1}R_{n-2} - \frac{1}{2}(Q_nR_{n-1})^2\} + \cdots\Big].\end{aligned}$$

Similarly, $\det\mathcal{A}(z)$ is rewritten as

$$\log\det\mathcal{A}(z)$$

$$
\begin{aligned}
&= -\mathrm{tr} \sum_{n=-\infty}^{\infty} [\log(\bar{\psi}_{1\,n+1} z^{-n-1}) - \log(\bar{\psi}_{1\,n} z^{-n})] \\
&= -\mathrm{tr} \sum_{n=-\infty}^{\infty} \log\Big[I + \frac{1}{z} Q_n \bar{\psi}_{2\,n} \bar{\psi}_{1\,n}^{-1}\Big] \\
&= -\mathrm{tr} \sum_{n=-\infty}^{\infty} \log\Big[I + \frac{1}{z} Q_n \Gamma_n^{(+)}\Big] \\
&= \mathrm{tr} \sum_{n=-\infty}^{\infty} \Big[-\log(I - Q_n R_n) + z^2 Q_n R_{n+1} \\
&\qquad + z^4 \{Q_n R_{n+2} - Q_n R_{n+2} Q_{n+1} R_{n+1} - \frac{1}{2}(Q_n R_{n+1})^2\} + \cdots\Big].
\end{aligned}
$$

Here the time independence of $\mathcal{A}(z)$, $\mathcal{A}_t(z) = O$, is proved in the same manner as in Section 7.3.2. It is now clear how the scattering data are expressed in terms of the integrals of motion for the sd-matrix equation (7.3). However, it should be stressed that the above expansions do not yield local conservation laws. The method presented in Section 7.1 is useful because it gives not only the densities but also the corresponding fluxes.

Conversely, the integrals of motion can be expressed in terms of the scattering data. For simplicity, we assume that

(a) Q_n and R_n are expressed as eq. (7.25). Thus, Proposition 3 holds.

(b) $\det A(z)$ and $\det \bar{A}(z)$ have $4N$ simple zeros outside and inside the unit circle C, respectively. None of them lies on the unit circle C.

(c) $\det A(z)$ and $\det \bar{A}(z)$ approach 1 rapidly as $|z| \to \infty$ and $z \to 0$, respectively.

Then, we can derive the following expansion for the sd-cmKdV equations (7.28):

$$
\begin{aligned}
&\log \det A(z) \\
&= \sum_{n=1}^{\infty} \frac{1}{z^n} \Big[\frac{1}{n} \sum_{j=1}^{4N} \Big\{\Big(\frac{1}{z_j^*}\Big)^n - z_j^n\Big\} + \frac{1}{2\pi \mathrm{i}} \oint_C w^{n-1} \log\det(A(w)\bar{A}(w))\mathrm{d}w\Big] \\
&= \sum_{k=1}^{\infty} \frac{1}{z^{2k}} \Big[\frac{1}{k} \sum_{j=1}^{2N} \Big\{\Big(\frac{1}{z_j^*}\Big)^{2k} - z_j^{2k}\Big\} + \frac{1}{\pi \mathrm{i}} \int_{C_R} w^{2k-1} \log\det(A(w)\bar{A}(w))\mathrm{d}w\Big] \\
&= \sum_{k=1}^{\infty} \frac{1}{z^{2k}} \Big[\frac{1}{k} \sum_{j=1}^{N} \Big\{\Big(\frac{1}{z_j}\Big)^{2k} + \Big(\frac{1}{z_j^*}\Big)^{2k} - z_j^{2k} - z_j^{*\,2k}\Big\} + \frac{1}{\pi \mathrm{i}} \int_{C_{UR}} (w^{2k-1} + w^{-2k-1}) \log|\det A(w)|^2 \mathrm{d}w\Big].
\end{aligned}
\tag{I.1}
$$

The coefficients of $1/z^{2k}$ $(k = 1, 2, \ldots)$ give an infinite number of the integrals of motion, which we call the *trace formulae*. Here, C_R and C_{UR} denote the right-half portion and the upper-right portion of the unit circle C, respectively, as is mentioned in Section 7.3. The determinant of $\bar{A}(z)$ is related to the determinant of $A(z)$ by eq. (7.62). It can also be shown that $\det \mathcal{A}(z)$ and $\det \bar{A}(z)$ are connected by

$$
\log \det \mathcal{A}(z) = -\sum_{n=-\infty}^{\infty} \log\det(I - Q_n R_n) + \log\det \bar{A}(z).
$$

Thus, we can directly obtain expansions of $\log\det \bar{A}(z)$ and $\log\det \mathcal{A}(z)$ with respect to z from eq. (I.1).

A derivation of eq. (I.1) is omitted because it is analogous to that in the continuous theory [1, 23, 107]. Related results were also obtained by Kodama [52].

Bibliography

[1] M. J. Ablowitz and H. Segur: *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[2] M. J. Ablowitz and P. A. Clarkson: *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[3] M. J. Ablowitz and R. Haberman: *Resonantly Coupled Nonlinear Evolution Equations*, J. Math. Phys. **16** (1975) 2301–2305.

[4] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: *Nonlinear-Evolution Equations of Physical Significance*, Phys. Rev. Lett. **31** (1973) 125–127.

[5] M. J. Ablowitz and J. F. Ladik: *Nonlinear Differential-Difference Equations*, J. Math. Phys. **16** (1975) 598–603.

[6] M. J. Ablowitz and J. F. Ladik: *Nonlinear Differential-Difference Equations and Fourier Analysis*, J. Math. Phys. **17** (1976) 1011–1018.

[7] M. J. Ablowitz and J. F. Ladik: *On the Solution of a Class of Nonlinear Partial Difference Equations*, Stud. Appl. Math. **57** (1977) 1–12.

[8] M. J. Ablowitz, Y. Ohta and A. D. Trubatch: *On Discretizations of the Vector Nonlinear Schrödinger Equation*, Phys. Lett. A **253** (1999) 287–304.

[9] M. J. Ablowitz, Y. Ohta and A. D. Trubatch: *On Integrability and Chaos in Discrete Systems*, Chaos, Solitons & Fractals **11** (2000) 159–169.

[10] C. Athorne and A. Fordy: *Generalised KdV and MKdV Equations Associated with Symmetric Spaces*, J. Phys. A: Math. Gen. **20** (1987) 1377–1386.

[11] R. Beals and R. R. Coifman: *Linear Spectral Problems, Non-Linear Equations and the $\bar{\partial}$-Method*, Inverse Problems **5** (1989) 87–130.

[12] A. L. Berkhoer and V. E. Zakharov: *Self Excitation of Waves with Different Polarizations in Nonlinear Media*, Sov. Phys.–JETP **31** (1970) 486–490.

[13] F. Beukers F, J. A. Sanders and J. P. Wang: *One Symmetry Does Not Imply Integrability*, J. Diff. Eq. **146** (1998) 251–260.

[14] A. H. Bilge: *A System with a Recursion Operator but One Higher Local Symmetry*, Lie Groups Appl. **1** (1994) 132–139.

[15] F. Calogero and W. Eckhaus: *Nonlinear Evolution Equations, Rescalings, Model PDEs and Their Integrability: I*, Inverse Problems **3** (1987) 229–262.

[16] F. Calogero and S. De Lillo: *The Eckhaus PDE* $\mathrm{i}\psi_t + \psi_{xx} + 2(|\psi|^2)_x\psi + |\psi|^4\psi = 0$, Inverse Problems **3** (1987) 633–681.

[17] K. M. Case and M. Kac: *A Discrete Version of the Inverse Scattering Problem*, J. Math. Phys. **14** (1973) 594–603.

[18] P. J. Caudrey: *The Inverse Problem for a General $N \times N$ Spectral Equation*, Physica D **6** (1982) 51–66.

[19] P. L. Christiansen, J. C. Eilbeck and R. D. Parmentier (ed.): *Future Directions of Nonlinear Dynamics in Physical and Biological Systems* (Plenum, New York, 1993).

[20] H. H. Chen, Y. C. Lee and C. S. Liu: *Integrability of Nonlinear Hamiltonian Systems by Inverse Scattering Method*, Phys. Scr. **20** (1979) 490–492.

[21] H. Eichenherr and K. Pohlmeyer: *Lax Pairs for Certain Generalizations of the Sine-Gordon Equation*, Phys. Lett. B **89** (1979) 76–78.

[22] L. D. Faddeev and L. A. Takhtajan: *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin, 1987).

[23] H. Flaschka and A. C. Newell: *Integrable Systems of Nonlinear Evolution Equations*, Dynamical Systems, Theory and Applications (Lecture Notes in Physics **38**) ed. J. Moser (Springer, New York, 1975) 355–440.

[24] H. Flaschka: *On the Toda Lattice. II —Inverse-Scattering Solution—*, Prog. Theor. Phys. **51** (1974) 703–716.

[25] A. S. Fokas: *Symmetries and Integrability*, Stud. Appl. Math. **77** (1987) 253–299.

[26] A. P. Fordy and P. P. Kulish: *Nonlinear Schrödinger Equations and Simple Lie Algebras*, Commun. Math. Phys. **89** (1983) 427–443.

[27] A. P. Fordy: *Derivative Nonlinear Schrödinger Equations and Hermitian Symmetric Spaces*, J. Phys. A: Math. Gen. **17** (1984) 1235–1245.

[28] A. P. Fordy (ed.): *Soliton Theory: a Survey of Results* (Manchester University Press, Manchester, 1990).

[29] A. P. Fordy and M. N. Ozer: *A New Integrable Reduction of the Matrix NLS Equation*, Proceedings of the Conference on Nonlinear Structures in Physics and Biology, ed. D. B. Duncan and J. C. Eilbeck (published on WWW: http://www.ma.hw.ac.uk/solitons/procs/).

[30] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura: *Method for Solving the Korteweg-de Vries Equation*, Phys. Rev. Lett. **19** (1967) 1095–1097.

[31] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura: *Korteweg-de Vries Equation and Generalizations. VI. Methods for Exact Solution*, Commun. Pure Appl. Math. **27** (1974) 97–133.

[32] I. T. Habibullin, V. V. Sokolov and R. I. Yamilov: *Multi-Component Integrable Systems and Nonassociative Structures*, Nonlinear Physics: Theory and Experiment, ed. E. Alfinito et al. (World Scientific, Singapore, 1996) 139–168.

[33] R. Hirota: *Exact Solution of the Modified Korteweg-de Vries Equation for Multiple Collisions of Solitons*, J. Phys. Soc. Jpn. **33** (1972) 1456–1458.

[34] R. Hirota: *Exact Envelope-Soliton Solutions of a Nonlinear Wave Equation*, J. Math. Phys. **14** (1973) 805–809.

[35] R. Hirota: *"Molecule Solutions" of Coupled Modified KdV Equations*, J. Phys. Soc. Jpn. **66** (1997) 2530–2532.

[36] R. Hirota: *Discretization of Coupled Modified KdV Equations*, RIMS kokyuroku **1020** (1997) 143–159.

[37] M. Hisakado, T. Iizuka and M. Wadati: *Coupled Hybrid Nonlinear Schrödinger Equation and Optical Solitons*, J. Phys. Soc. Jpn. **63** (1994) 2887–2894.

[38] M. Hisakado: *Coupled Nonlinear Schrödinger Equation and Toda Equation (The Root of Integrability)*, J. Phys. Soc. Jpn. **66** (1997) 1939–1942.

[39] M. Hisakado: *Chiral solitons from dimensional reduction of Chern-Simons gauged coupled non-linear Schrödinger model*, hep-th/9712255.

[40] Y. Inoue: *Nonlinear Coupling of Polarized Plasma Waves*, J. Plasma Phys. **16** (1976) 439–459.

[41] Y. Ishimori: *A Relationship between the Ablowitz-Kaup-Newell-Segur and Wadati-Konno-Ichikawa Schemes of the Inverse Scattering Method*, J. Phys. Soc. Jpn. **51** (1982) 3036–3041.

[42] Y. Ishimori: *An Integrable Classical Spin Chain*, J. Phys. Soc. Jpn. **51** (1982) 3417–3418.

[43] M. Iwao and R. Hirota: *Soliton Solutions of a Coupled Modified KdV Equations*, J. Phys. Soc. Jpn. **66** (1997) 577–588.

[44] S. Kakei, N. Sasa and J. Satsuma: *Bilinearization of a Generalized Derivative Nonlinear Schrödinger Equation*, J. Phys. Soc. Jpn. **64** (1995) 1519–1523.

[45] H. Kakuhata and K. Konno: *A Generalization of Coupled Integrable, Dispersionless System*, J. Phys. Soc. Jpn. **65** (1996) 340–341.

[46] M. Karlsson, D. J. Kaup and B. A. Malomed: *Interactions between Polarized Soliton Pulses in Optical Fibers: Exact Solutions*, Phys. Rev. E **54** (1996) 5802–5808.

[47] D. J. Kaup and A. C. Newell: *On the Coleman Correspondence and the Solution of the Massive Thirring Model*, Lett. Nuovo Cimento **20** (1977) 325–331.

[48] D. J. Kaup and A. C. Newell: *An Exact Solution for a Derivative Nonlinear Schrödinger Equation*, J. Math. Phys. **19** (1978) 798–801.

[49] D. J. Kaup: *The Three-Wave Interaction—A Nondispersive Phenomenon*, Stud. Appl. Math. **55** (1976) 9–44.

[50] T. Kawata and H. Inoue: *Exact Solutions of the Derivative Nonlinear Schrödinger Equation under the Nonvanishing Conditions*, J. Phys. Soc. Jpn. **44** (1978) 1968.

[51] Y. S. Kivshar and S. K. Turitsyn: *Vector Dark Solitons*, Opt. Lett. **18** (1993) 337–339.

[52] Y. Kodama: *Complete Integrability of Nonlinear Evolution Equations*, Prog. Theor. Phys. **54** (1975) 669–686.

[53] K. Konno and A. Jeffrey: *The Loop Soliton*, Advances in Nonlinear Waves Vol. 1, L. Debnath ed. (1984, Pitman Publishing, Boston) 162–183.

[54] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin: *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).

[55] A. Kundu: *Landau-Lifshitz and Higher-Order Nonlinear Systems Gauge Generated from Nonlinear Schrödinger Type Equations*, J. Math. Phys. **25** (1984) 3433–3438.

[56] E. A. Kuznetsov and A. V. Mikhailov: *On the Complete Integrability of the Two-Dimensional Classical Thirring Model*, Theor. Math. Phys. **30** (1977) 193–200.

[57] P. D. Lax: *Integrals of Nonlinear Equations of Evolution and Solitary Waves*, Commun. Pure Appl. Math. **21** (1968) 467–490.

[58] D. Levi, O. Ragnisco and M. Bruschi: *Extension of the Zakharov-Shabat Generalized Inverse Method to Solve Differential-Difference and Difference-Difference Equations*, Nuovo Cimento A **58** (1980) 56–66.

[59] F. Magri: *A Simple Model of the Integrable Hamiltonian Equation*, J. Math. Phys. bf 19 (1978) 1156–1162.

[60] S. V. Manakov: *On the Theory of Two-Dimensional Stationary Self-Focusing of Electromagnetic Waves*, Sov. Phys.–JETP **38** (1974) 248–253.

[61] S. V. Manakov: *Complete Integrability and Stochastization of Discrete Dynamical Systems*, Sov. Phys.–JETP **40** (1975) 269–274.

[62] I. Merola, O. Ragnisco and T. Gui-Zhang: *A Novel Hierarchy of Integrable Lattices*, Inverse Problems **10** (1994) 1315–1334.

[63] C. R. Menyuk: *Nonlinear Pulse Propagation in Birefringent Optical Fibers*, IEEE J. Quantum Electron. **23** (1987) 174–176.

[64] A. V. Mikhailov, A. B. Shabat and R. I. Yamilov: *The Symmetry Approach to the Classification of Non-Linear Equations. Complete Lists of Integrable Systems*, Russ. Math. Surv. **42:4** (1987) 1–63.

[65] R. M. Miura, C. S. Gardner and M. D. Kruskal: *Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion*, J. Math. Phys. **9** (1968) 1204–1209.

[66] H. C. Morris and R. K. Dodd: *The Two Component Derivative Nonlinear Schrödinger Equation*, Phys. Scr. **20** (1979) 505–508.

[67] A. Nagai, T. Tokihiro and J. Satsuma: *Ultra-Discrete Toda Molecule Equation*, Phys. Lett. A **244** (1998) 383–388.

[68] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov: *Theory of Solitons: The Inverse Scattering Method* (Consultants Bureau, New York, 1984).

[69] Y. Ohta: *Wronskian Solutions of Soliton Equations*, RIMS kokyuroku **684** (1989) 1–17 [in Japanese].

[70] Y. Ohta: *Pfaffian Solution for Coupled Discrete Nonlinear Schrödinger Equation* Chaos, Solitons & Fractals **11** (2000) 91–95.

[71] Y. Ohta: *Determinant and Pfaffian Solutions for Discrete Soliton Equations*, Proceedings of the 3rd International Interdisciplinary Meeting on Symmetries and Integrability of Difference Equations, in press.

[72] P. J. Olver and V. V. Sokolov: *Integrable Evolution Equations on Associative Algebras*, Commun. Math. Phys. **193** (1998) 245–268.

[73] P. J. Olver and V. V. Sokolov: *Non-Abelian Integrable Systems of the Derivative Nonlinear Schrödinger Type*, Inverse Problems **14** (1998) L5-L8.

[74] P. J. Olver: *Applications of Lie Groups to Differential Equations*, 2nd edn (Springer-Verlag, New York, 1993).

[75] K. Porsezian and M. Lakshmanan: *Discretised Hirota Equation, Equivalent Spin Chain and Bäcklund Transformations*, Inverse Problems **5** (1989) L15–L19.

[76] R. Radhakrishnan, M. Lakshmanan and J. Hietarinta: *Inelastic Collision and Switching of Coupled Bright Solitons in Optical Fibers*, Phys. Rev. E **56** (1997) 2213–2216.

[77] N. Sasa and J. Satsuma: *New-Type of Soliton Solutions for a Higher-Order Nonlinear Schrödinger Equation*, J. Phys. Soc. Jpn. **60** (1991) 409–417.

[78] T. Shimizu and M. Wadati: *A New Integrable Nonlinear Evolution Equation*, Prog. Theor. Phys. **63** (1980) 808–820.

[79] S. I. Svinolupov: *Generalized Schrödinger Equations and Jordan Pairs*, Commun. Math. Phys. **143** (1992) 559–575.

[80] S. I. Svinolupov: *Jordan Algebras and Integrable Systems*, Func. Anal. Appl. **27** (1993) 257–265.

[81] S. I. Svinolupov and V. V. Sokolov: *Vector-Matrix Generalizations of Classical Integrable Equations*, Theor. Math. Phys. **100** (1994) 959–962.

[82] L. A. Takhtajan: *Integration of the Continuous Heisenberg Spin Chain through the Inverse Scattering Method*, Phys. Lett. A **64** (1977) 235–237.

[83] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma: *From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure* Phys. Rev. Lett. **76** (1996) 3247–3250.

[84] T. Tsuchida and M. Wadati: *Lax Pairs for Four-Wave Interaction Systems*, J. Phys. Soc. Jpn. **65** (1996) 3153–3156.

[85] T. Tsuchida and M. Wadati: *The Coupled Modified Korteweg-de Vries Equations*, J. Phys. Soc. Jpn. **67** (1998) 1175–1187.

[86] T. Tsuchida, H. Ujino and M. Wadati: *Integrable Semi-Discretization of the Coupled Modified KdV Equations*, J. Math. Phys. **39** (1998) 4785–4813.

[87] T. Tsuchida, H. Ujino and M. Wadati: *Integrable Semi-Discretization of the Coupled Nonlinear Schrödinger Equations*, J. Phys. A: Math. Gen. **32** (1999) 2239–2262.

[88] T. Tsuchida and M. Wadati: *New Integrable Systems of Derivative Nonlinear Schrödinger Equations with Multiple Components*, Phys. Lett. A **257** (1999) 53–64.

[89] T. Tsuchida and M. Wadati: *Multi-Field Integrable Systems Related to WKI-Type Eigenvalue Problems*, J. Phys. Soc. Jpn. **68** (1999) 2241–2245.

[90] T. Tsuchida and M. Wadati: *Complete Integrability of Derivative Nonlinear Schrödinger-Type Equations*, Inverse Problems **15** (1999) 1363–1373.

[91] T. Tsuchida: *Inverse Scattering Method for the Coupled Modified KdV Equations*, Proceedings of Kyushu Symposium 1998, Recent Advances in Soliton Theory, **10ME-S1** (RIAM Kyushu University, Kyushu, 1999) 190–196 [in Japanese].

[92] T. Tsuchida: *Integrable Discretization of the Coupled Nonlinear Schrödinger Equations*, Proceedings of 31st Symposium on Mathematical Physics "Solitons and Nonlinear Phenomena", Torun, Poland (1999), to appear.

[93] M. Wadati and M. Toda: *The Exact N-Soliton Solution of the Korteweg-de Vries Equation*, J. Phys. Soc. Jpn. **32** (1972) 1403–1411.

[94] M. Wadati, H. Sanuki and K. Konno: *Relationships among Inverse Method, Bäcklund Transformations and an Infinite Number of Conservation Laws*, Prog. Theor. Phys. **53** (1975) 419–436.

[95] M. Wadati, K. Konno and Y. Ichikawa: *A Generalization of Inverse Scattering Method*, J. Phys. Soc. Jpn. **46** (1979) 1965–1966.

[96] M. Wadati, K. Konno and Y. Ichikawa: *New Integrable Nonlinear Evolution Equations*, J. Phys. Soc. Jpn. **47** (1979) 1698–1700.

[97] M. Wadati and K. Sogo: *Gauge Transformations in Soliton Theory*, J. Phys. Soc. Jpn. **52** (1983) 394–398.

[98] M. Wadati: *The Exact Solution of the Modified Korteweg-de Vries Equation*, J. Phys. Soc. Jpn. **32** (1972) 1681.

[99] M. Wadati: *The Modified Korteweg-de Vries Equation*, J. Phys. Soc. Jpn. **34** (1973) 1289–1296.

[100] M. Wadati and T. Kamijo: *On the Extension of Inverse Scattering Method*, Prog. Theor. Phys. **52** (1974) 397–414.

[101] M. Wadati: *Transformation Theories for Nonlinear Discrete Systems*, Prog. Theor. Phys. Suppl. **59** (1976) 36–63.

[102] M. Wadati: *Generalized Matrix Form of the Inverse Scattering Method*, Solitons, ed. R. K. Bullough and P. J. Caudrey (Springer, Berlin, 1980) 287–299.

[103] M. Wadati, T. Iizuka and M. Hisakado: *A Coupled Nonlinear Schrödinger Equation and Optical Solitons*, J. Phys. Soc. Jpn. **61** (1992) 2241–2245.

[104] N. Yajima and M. Oikawa: *A Class of Exactly Solvable Nonlinear Evolution Equations*, Prog. Theor. Phys. **54** (1975) 1576–1577.

[105] T. Yajima: *Derivative Nonlinear Schrödinger Type Equations with Multiple Components and Their Solutions*, J. Phys. Soc. Jpn. **64** (1995) 1901–1909.

[106] N. J. Zabusky and M. D. Kruskal: *Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States*, Phys. Rev. Lett. **15** (1965) 240–243.

[107] V. E. Zakharov and S. V. Manakov: *On the Complete Integrability of a Nonlinear Schrödinger Equation*, Theor. Math. Phys. **19** (1974) 551–559.

[108] V. E. Zakharov and A. B. Shabat: *Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media*, Sov. Phys.–JETP **34** (1972) 62–69.

[109] V. E. Zakharov and A. B. Shabat: *Interaction between Solitons in a Stable Medium*, Sov. Phys.–JETP **37** (1973) 823–828.

[110] V. E. Zakharov and A. B. Shabat: *A Scheme for Integrating the Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Problem. I*, Func. Anal. Appl. **8** (1974) 226–235.

[111] V. E. Zakharov and L. A. Takhtadzhyan: *Equivalence of the Nonlinear Schrödinger Equation and the Equation of a Heisenberg Ferromagnet*, Theor. Math. Phys. **38** (1979) 17–23.